TRANSIENT FLUID MOTIONS IN SATURATED POROUS MEDIA

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Summary

The transition from rest to steady motion on the sudden application of a potential gradient to the fluid contained in a saturated porous medium is investigated. An approximate analysis gives the result that the time of the effective establishment of the steady motion is proportional to the permeability and inversely proportional to the kinematic viscosity. Two exact solutions (one of them new) for simple cases suggest that the approximate analysis is remarkably accurate. An analogy between this problem and one in heat conduction makes the relevant results in that field immediately applicable here.

The analysis is extended to motion with the time variation of the applied potential gradient quite general. Certain properties of the motion when the potential varies periodically are determined, and the simple harmonic case is studied in some detail.

It is found that the error in using Darcy's law (which neglects the transient phase) will usually be unimportant for the case of a suddenly applied potential gradient. However, significant deviations from Darcy's law may occur when the applied potential gradient is periodic, even for systems of quite low frequency.

The equations derived from the approximate analysis may be regarded as generalizations of Darcy's law which take into account time variation of the applied potential gradient.

I. INTRODUCTION

This communication reports part of an attempt to interpret Darcy's law in terms of classical hydrodynamics. Darcy's law has been expressed at a number of levels of generality. Darcy (1856) himself gave the equation

$$q=KA(h+L)/L, \qquad \text{(1.1)}$$

where q is the discharge through a filter in unit time, A is the area of the filter, L is the thickness of the sand layer, h the water depth over the sand, and K " un coefficient dependant de la nature du sable ". Later investigators have applied the equation to the flow of water in homogeneous isotropic porous media in forms such as

$$q/A=\bar{U}=KS, \qquad \text{(1.2)}$$

where $\bar{U}$ is the macroscopic flow velocity and S the magnitude of the potential gradient. An extension of (**1.2**) to any fluid may be made, provided the medium remains stable and does not react with the fluid,

$$\bar{U}=(K/\nu)S, \qquad \text{(1.3)}$$

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where ν is the kinematic viscosity. Here K, the permeability, is a characteristic of the geometry of the medium and is independent of the fluid.

A vector extension of (**1.3**) is

$$\bar{\mathbf{U}} = -(K/\nu)\nabla\Phi, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1.4})$$

where $\bar{\mathbf{U}}$ is the vector macroscopic velocity and Φ is the potential defined by (**1.5**)

$$\Phi = P/\rho + \Omega, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1.5})$$

where P is the pressure, ρ the density, and Ω the potential of the external or " body " forces. (**1.4**) holds for homogeneous isotropic porous media. The extension of (**1.4**) to anisotropic media is

$$\bar{\mathbf{U}} = -(1/\nu)\mathbf{K}\nabla\Phi, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1.6})$$

where $\mathbf{K}$ is the permeability tensor. In this general case, the direction of the vectors $\bar{\mathbf{U}}$ and $-\nabla\Phi$ differ except for certain special directions in the medium.

All these forms of Darcy's law are macroscopic approximations to the result given exactly by the Stokes-Navier equation for an incompressible fluid,

$$\partial\mathbf{U}/\partial t - \mathbf{U}\times(\nabla\times\mathbf{U}) + \tfrac{1}{2}\nabla\mathbf{U}^2 = -\nabla\Phi + \nu\nabla^2\mathbf{U}, \quad \ldots\ldots \quad (\mathbf{1.7})$$

where $\mathbf{U}$ is the vector (microscopic) velocity and t the time, subject to the appropriate conditions.*

For

$$\mathbf{U}\times(\nabla\times\mathbf{U}) - \tfrac{1}{2}\nabla\mathbf{U}^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1.8})$$

(**1.7**) becomes the " complete " (Sommerfeld 1950) Stokes-Navier equation

$$\partial\mathbf{U}/\partial t = -\nabla\Phi + \nu\nabla^2\mathbf{U}. \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1.9})$$

For certain special motions and media (e.g. flow parallel to the generators of the generalized tube) (**1.8**) will hold exactly and we designate such motions " complete ". However, in most media, the validity of (**1.9**) depends on the reduction of the Reynolds number of the motion, R, to sufficiently low values for the inertia terms (i.e. the terms of the second degree in $\mathbf{U}$) to be negligible. When such a restriction is necessary we term the motion " approximately complete ". Both analysis (Lamb 1924) and experiment (Muskat 1937) lead to an upper limit of R of order of magnitude unity, below which (**1.9**) adequately describes approximately complete motions. Pipe flow experiments (e.g. Goldstein 1938) suggest the existence of a " lower critical Reynolds number " of order of magnitude 10^3, above which (**1.9**) fails, even for complete motions, due to the onset of instability.

* A distinction exists between the significance of $\nabla\Phi$ in macroscopic equations such as (**1.4**) and (**1.6**) and in microscopic equations such as (**1.7**). This does not lead to difficulties in the present work, but it should be borne in mind that in macroscopic equations $\nabla\Phi$ denotes a smoothed potential gradient, whereas in microscopic equations it denotes a vector point function which can be expected to vary in magnitude and direction from point to point. This note is intended as a warning and not as an adequate discussion of the dualism of the macroscopic and microscopic pictures.

Darcy's law predicts a motion dependent only on the potential distribution and quite independent of time. Most studies of fluid motion in porous media, relying as they do on Darcy's law, imply that the steady motion corresponding to any potential distribution suddenly applied to the system is set up within a negligibly short time. Apparently the validity of this assumption has never been examined.

The hydrodynamic problem to be considered here is not to be confused with the *elastic* problem of the time of propagation of sudden changes in potential, treated by Muskat (1937). Here both the medium and the fluid are taken as inelastic, i.e. the velocity of propagation of disturbances is infinite. It must also be stated that the treatment is confined to saturated media.

II. The Transition from Rest to Steady Motion

We investigate here the motion described by (**1.9**) subject to the continuity condition (**2.1**) and the boundary conditions (**2.2**)

$$\nabla\cdot\mathbf{U}=0, \qquad \text{(2.1)}$$

$$t=0,\ \mathbf{U}=0 \text{ in } B;\quad t\geqslant 0,\ \mathbf{U}=0 \text{ at } C,\ \Phi=\Phi_D \text{ at } D. \qquad \text{(2.2)}$$

B is the region of fluid occupation of the medium, C is the surface of fluid-solid contact, and D is the external boundary of B (i.e. that part of the boundary of B not contained in C).

In employing (**1.9**) here instead of (**1.7**), we exclude motions for which $R>1$. For such motions it is known, in any case, that Darcy's law fails (Muskat 1937). Further, nearly all fluid motions in the porous media of nature and technology possess Reynolds numbers less than unity.

It follows simply from (**1.9**) and (**2.1**) that

$$\nabla^2\Phi=0, \qquad \text{(2.3)}$$

subject to the conditions (*vide* (**2.2**))

$$\Phi=\Phi_D \text{ at } D;\quad \partial\Phi/\partial\eta=0 \text{ at } C, \qquad \text{(2.4)}$$

where length η is normal to the surface C.

Clearly Φ at all points in the fluid depends only on Φ_D, the imposed potential distribution at the external boundary of B, i.e. Φ is independent of t and the distribution of Φ at all t is the same as that at the steady state. Denoting the velocity at the steady state by $\mathbf{U}_\infty$, we have for the steady motion

$$\nabla\Phi=\nu\nabla^2\mathbf{U}_\infty \qquad \text{(2.5)}$$

subject to conditions (**2.6**) and (**2.7**):

$$\nabla\cdot\mathbf{U}_\infty=0, \qquad \text{(2.6)}$$

$$\mathbf{U}_\infty=0 \text{ at } C;\quad \Phi=\Phi_D \text{ at } D. \qquad \text{(2.7)}$$

Subtracting (**1.9**) from (**2.5**) and denoting ($\mathbf{U}_\infty - \mathbf{U}$) by $\mathbf{W}$, we have (**2.8**) subject to conditions (**2.9**):

$$\partial \mathbf{W}/\partial t = \nu \nabla^2 \mathbf{W}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{2.8})$$

$$\left.\begin{array}{lll} t=0\,; & \mathbf{W}=\mathbf{U}_\infty & \text{in } B, \\ t\geqslant 0\,; & W=0 & \text{at } C. \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{2.9})$$

Since equation (**2.8**) depends on (**2.3**), a result following from (**2.1**), (**2.8**) implicitly satisfies the continuity condition

$$\nabla \cdot \mathbf{W} = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{2.10})$$

III. The Thermal Analogy

With the reservation that $\mathbf{W}$ is a vector, (**2.8**) is the equation of heat conduction. We may regard each of the three scalar components of $\mathbf{W}$ as separately analogous to temperature, the three scalar fields being linked by continuity requirement (**2.10**). Conditions (**2.9**) are then equivalent to three initial temperature distributions governing the three scalar fields, the boundary condition for each being that C is held at zero temperature ; i.e. the establishment of a steady motion from rest is analogous to the dissipation of three initial temperature distributions.

In this way established results in the mathematics of heat conduction (Carslaw and Jaeger 1947) may be employed in solving the present problem. Although the solution of (**2.8**) is facilitated by the use of heat conduction Green's functions, the fields in which analytical, or indeed even numerical, results will be readily available will be found to be limited to rather simple types of media. The main concern will often be to find the order of magnitude of the time required for the establishment of the steady motion. We shall see that the limited range of exact solutions presents no obstacle in such cases.

Szymanski (1930) treated motion in the generalized tube under an imposed potential gradient varying in time, but his treatment was limited to flow parallel to the generators of the generalized tube (two-dimensional special case). The rather different approach of this section embraces the general medium (three-dimensional problem) but is special in the sense that the applied potential gradient may change only discontinuously, i.e. the potential gradient must be a step-function of t. In Section VIII an approximate method of computing motion in the general medium is developed for the case where the variation of potential gradient with time is quite general.

IV. Two Exact Solutions

Szymanski provides the only available numerical solution of this problem—for the case of axial motion in the circular tube. It gives a most instructive picture of the manner in which the steady motion is set up more rapidly near the tube walls, the final increments of velocity being almost wholly in the region of the tube axis. His result is embodied in Figure 2.

An even simpler problem of interest (since the circle and the infinite slit give the extreme values of area : perimeter ratio for regular figures) is that of linear flow between and parallel to two parallel plates. If z is the ordinate normal to planes $z=0$, $z=h$, forming the surfaces of the two plates, we have

$$\mathbf{U}_\infty = -(\nabla\Phi)z(h-z)/2\nu. \quad \text{(4.1)}$$

The problem reduces to solving (4.2) subject to (4.3)

$$\partial\mathbf{W}/\partial t = \nu\partial^2\mathbf{W}/\partial z^2, \quad \text{(4.2)}$$

$$t=0\,;\quad \mathbf{W}=-(\nabla\Phi)z(h-z)/2\nu. \quad t\geqslant 0\,;\quad \mathbf{W}=0 \text{ at } z=0,\ z=h. \quad \text{(4.3)}$$

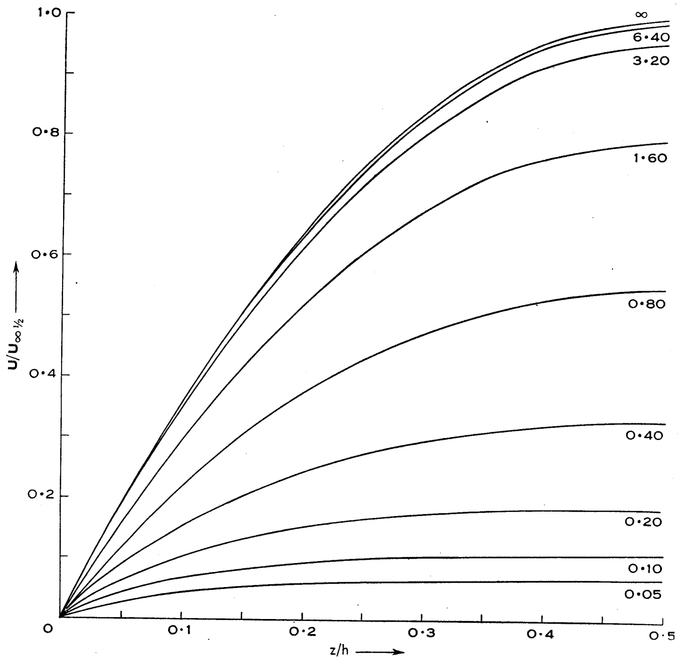

Fig. 1.—Transition from rest to steady motion for flow between parallel plates. Numbers on each curve give values of $\pi^2\nu t/h^2$. $\mathbf{U}_{\infty\frac{1}{2}}$ denotes the value of $\mathbf{U}_\infty$ for $z/h=0\cdot 5$.

From the known result in heat conduction (Carslaw and Jaeger 1947), the solution is

$$\mathbf{W} = -\frac{4}{\pi^3\nu}(\nabla\Phi)h^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \exp\left[-\nu(2n+1)^2\pi^2 t/h^2\right] \sin\left[(2n+1)\pi z/h\right]. \quad \text{(4.4)}$$

The discharge per unit width, Q, is given by

$$Q = -\frac{(\nabla\Phi)h^3}{\nu}\left[\frac{1}{12} - \frac{8}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\exp\left[-\nu(2n+1)^2\pi^2 t/h^2\right]\right]. \quad \text{(4.5)}$$

The solution is shown in dimensionless form in Figure 1. Here also the steady state tends to be established most rapidly near the fluid-solid interface and most slowly in midstream. However, as one might expect, the tendency is less marked than in Szymanski's problem.

V. An Approximate Solution

We now develop an approximate solution to the problem of the transition from rest to steady motion in the general medium. For simplicity, we take the case where the directions of **U** and $-\nabla\Phi$ coincide, i.e. the isotropic medium or a principal direction in the anisotropic medium with **K** self-conjugate. In the latter case superposition of the results for the three principal directions gives the general solution. In what follows we simplify the notation by writing the scalar S for $-\nabla\Phi$ and the scalar $\bar{U}$ for $\bar{\mathbf{U}}$.

The assumption on which this analysis depends is

$$\mathbf{U}=F\mathbf{U}_\infty, \quad\quad (5.1)$$

where F is a function of t only. It is evident that (**5.1**) cannot be exactly true, the exact behaviour being exemplified by Figure 1 and Szymanski's result. However, if (**5.1**) did hold, we could also write

$$\Psi=F^2\Psi_\infty, \text{ i.e. } \int\Psi\mathrm{d}V=F^2\int\Psi_\infty\mathrm{d}V, \quad\quad (5.2)$$

where Ψ is the point rate of viscous dissipation of energy, Ψ_∞ the steady state value of Ψ, and the integral is taken throughout the whole volume of the medium, V.

Now, *in the steady state*, the mean rate of viscous dissipation of energy per unit volume, Ψ_∞, is given by (**5.3**) in which $\bar{U}_\infty$ is written for the steady state macroscopic velocity.

$$\Psi_\infty=\rho\bar{U}_\infty S. \quad\quad (5.3)$$

(**5.3**) is simply proved by equating for any unit cube of the medium the rate at which work is done on the fluid by the potential gradient and the rate of viscous dissipation of energy.

Then (**5.2**) may be written

$$\int\Psi\mathrm{d}V=\rho F^2V\bar{U}_\infty S. \quad\quad (5.4)$$

Equating for any instant during the transient state the rate at which work is being done on the fluid passing through the medium by the potential gradient to the sum of the rate of gain of kinetic energy and the rate of viscous dissipation of energy,

$$\rho FV\bar{U}_\infty S=\alpha\rho FV\bar{U}^2_\infty\mathrm{d}F/\mathrm{d}t+\rho F^2V\bar{U}_\infty S. \quad\quad (5.5)$$

α is the ratio of the actual kinetic energy to that computed if U is assumed equal to $\bar{U}(=F\bar{U}_\infty)$.

Using the relationship

$$\bar{U}_\infty=\frac{K}{\nu}S \quad\quad (5.6)$$

and rearranging (**5.5**), we get

$$\mathrm{d}F/\mathrm{d}t=\frac{\nu}{\alpha K}(1-F). \quad\quad (5.7)$$

The integral of (**5.7**) for which F vanishes at $t=0$ is

$$F=1-e^{-\beta t}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.8)$$

that is,

$$\bar{U}=\bar{U}_\infty(1-e^{-\beta t}), \quad \ldots\ldots\ldots\ldots\ldots \quad (5.9)$$

where

$$\beta=\nu/\alpha K. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.10)$$

α is suitably replaced by

$$\gamma/p^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.11)$$

where p is the porosity, so that

$$\beta=p^2\nu/\gamma K. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.12)$$

For a circular tube $\gamma=1{\cdot}33$, and for flow between plates $\gamma=1{\cdot}20$. Probably γ will always be of about this order of magnitude.

It is perhaps worth remarking that the approximation of this subsection is exactly analogous to that made in elementary dynamics when the approach to the terminal velocity of a body falling in a viscous fluid is calculated on the assumption of a resistance proportional to velocity and independent of acceleration.

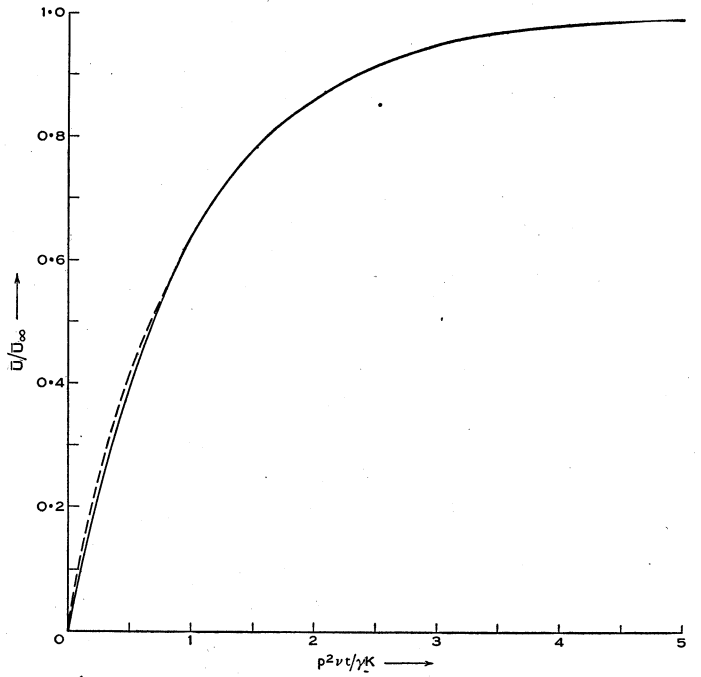

Fig. 2.—Transition from rest to steady motion. Full curve gives approximate solution (**5.9**). Broken curve shows divergence of two exact solutions from the approximate solution.

VI. The Accuracy of the Approximation

The full curve of Figure 2 represents (**5.9**) in dimensionless form. It was intended to show on the same graph the exact solutions for flow in the circular tube and between parallel plates referred to above, using the known values of K and γ for these systems. However, for these two exact solutions, values of $\bar{U}/\bar{U}_\infty$ for any particular value of βt were found to differ by less than about $0 \cdot 005$; so the two curves could only be shown as one in the figure. Further, it was only in the region $\beta t < 0 \cdot 8$ that these exact solutions differed from (**5.9**) by sufficient for them to be represented by a separate curve. The broken curve joining the full curve at about $t=0 \cdot 8$ represents the exact solutions up to that point. Beyond that the full curve represents the approximate solution and both exact solutions.

It is apparent from this result that the approximate solution promises to be surprisingly accurate—in fact more accurate than we really need for our present purpose, which is to evaluate the order of magnitude of time required for the establishment of the steady motion.

VII. The Time for the Establishment of the Steady Motion

Although this matter might have been treated in terms of the "half-life", it was felt that, since the process is not exactly exponential, it is perhaps better discussed in terms of $t_{0\cdot 99}$, the time taken for $\bar{U}$ to reach the value $0 \cdot 99\bar{U}_\infty$. Then from (**5.9**)

$$t_{0\cdot 99}=4 \cdot 6\gamma K/p^2\nu. \qquad (\mathbf{7.1})$$

Adopting typical values $p=0\cdot 5$, $\gamma=1 \cdot 3$, and for water (20 °C) $\nu=1 \cdot 005 \times 10^{-2}$ we obtain

$$t_{0\cdot 99}=2380K, \qquad (\mathbf{7.2})$$

with K in cm². For K in darcys (Muskat 1937) this becomes

$$t_{0\cdot 99}=0 \cdot 236K. \qquad (\mathbf{7.3})$$

It will be observed that $t_{0\cdot 99}$ is proportional to K and inversely proportional to ν. Since in nature and technology both values of ν less than that for water and values of K greater than 10 darcys are uncommon, it will be seen that the steady state is generally set up within a few seconds, and, in fact, usually within a fraction of a second. It will be noted that $t_{0\cdot 99}$ is quite independent of the dimensions of the medium and the magnitude of the potential gradient.

VIII. Generalization of the Approximate Analysis

(**5.5**), the basic equation upon which the preceding approximate study depends, may be recast into the form

$$(1/\beta)\mathrm{d}\bar{U}/\mathrm{d}t+\bar{U}=KS/\nu. \qquad (\mathbf{8.1})$$

Previously we restricted our attention to the case with S constant for all $t>0$, but this limitation is unnecessary as only the instantaneous values of $\bar{U}$ and S enter into (**5.5**) and (**8.1**). Hence, with S allowed to vary in time quite generally, (**8.1**) may be integrated to yield

$$\bar{U}=\bar{U}_0\mathrm{e}^{-\beta t}+(\beta K/\nu)\mathrm{e}^{-\beta t}\int_0^t S(T)\mathrm{e}^{\beta T}\mathrm{d}T. \qquad (\mathbf{8.2})$$

$\bar{U}_0$ denotes the value of $\bar{U}$ at $t=0$. (**8.2**) expresses the macroscopic fluid motion in terms of the initial conditions and the imposed $S(t)$. Because of the superposability of the motions (Strang 1948) (**8.2**) can be shown to depend only on the original assumption (**5.1**). Clearly (**5.9**) is merely a special form of (**8.2**). It is of some interest to consider the problem of S simple harmonic. For

$$S=s \sin t/\omega, \qquad (\mathbf{8.3})$$

$$\bar{U}_0=0, \qquad (\mathbf{8.4})$$

(**8.2**) becomes

$$\bar{U}=\frac{K\beta\omega s}{\nu(\beta^2\omega^2+1)}\left[\left(\beta\omega \sin\frac{t}{\omega}-\cos\frac{t}{\omega}\right)+e^{-\beta t}\right], \qquad (\mathbf{8.5})$$

which may be rearranged as

$$\bar{U}=\frac{Ks}{\nu}\cos\theta\left[\sin\left(\frac{t}{\omega}-\theta\right)+(\sin\theta)e^{-\beta t}\right], \qquad (\mathbf{8.6})$$

where

$$\sin\theta=(\beta^2\omega^2+1)^{-\frac{1}{2}}. \qquad (\mathbf{8.7})$$

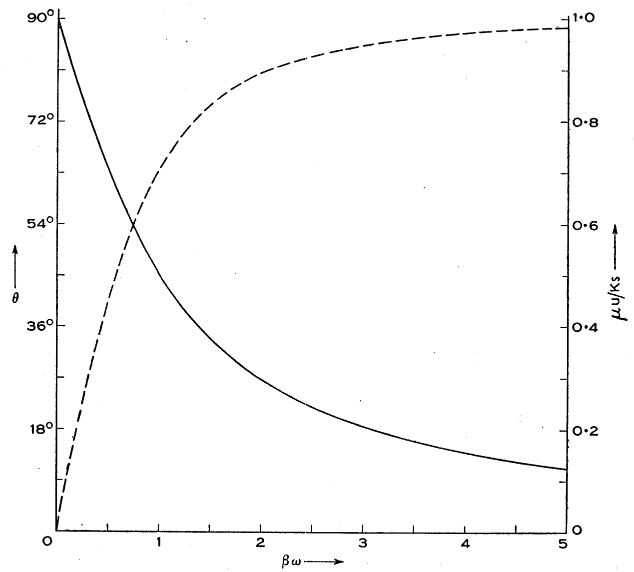

Fig. 3.—" Steady " fluid motion with simple harmonic applied potential gradient. Full curve denotes the phase lag θ. Broken curve denotes the amplitude in the dimensionless form, $\mu u/Ks$. Note that μ in the figure corresponds to ν in tne text.

In the limit as $t \to \infty$, the motion becomes simple harmonic with the same period as the impressed gradient (i.e. $2\pi\omega$) with a phase lag between S and $\bar{U}$. The amplitude of $\bar{U}$ is $(Ks/\nu)\cos\theta$. Figure 3 shows the variation of the phase lag θ and the amplitude u with the significant parameter of the system, $\beta\omega$.

Figure 4 compares the final " steady " motion with the first few cycles of the transition from rest for the case of $\beta\omega=0{\cdot}213$. This value corresponds, for example, to $2\pi\omega = 0\cdot5$ sec, $K = 2\cdot8 \times 10^{-3}$ cm^2 (equal to that of a tube of diameter $0\cdot3$ cm), $\nu = 1\cdot005 \times 10^{-2}$ cm^2 sec^{-1}.

Since the motions we consider are superposable (Strang 1948), it follows that for any S which is a periodic function of time, (**8.2**) will be of the form

$$\bar{U} = \bar{U}_1 e^{-\beta t} + \bar{U}_2 + \bar{U}_3, \qquad \text{(8.8)}$$

where (i) $\bar{U}_1 e^{-\beta t}$ represents the exponential decay of an initial motion $\bar{U}_1$, which may be interpreted as the motion corresponding to the degree that $\bar{U}_0$ is out of equilibrium with S,

(ii) $\bar{U}_2$ is periodic with the same period as S,

(iii) $\bar{U}_3$ represents the steady motion produced by the steady application of the mean value of S.

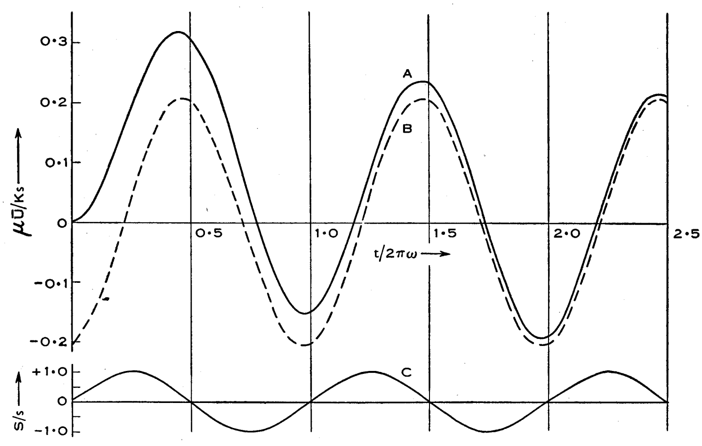

Fig. 4.—Fluid motion with simple harmonic applied potential gradient. Curve A denotes the transition from rest. Curve B denotes the final steady motion. Both curves for $\beta\omega = 0\cdot213$. Curve C denotes the applied potential gradient. Note that μ in the figure corresponds to ν in the text.

IX. Discussion

These investigations suggest that transient effects in saturated porous media will not often be large enough to invalidate the use of Darcy's law. However, for the sudden application of a potential gradient to a very permeable medium containing a fluid of low kinematic viscosity an appreciable time may elapse before the motion has effectively attained the steady state. Equation (**7.1**) provides a suitable criterion.

Moreover, where the applied potential gradient is periodic, significant deviations from Darcy's law occur even if the frequency of the system is as low as one cycle per minute. Naturally, the permeability and kinematic viscosity influence the magnitude of the deviations here also. Equation (**8.6**) enables the effect of the various factors to be evaluated for the case of a simple harmonic S function.

It is possible that in the systems of very rapidly changing potential which occur in such phenomena as the movement of liquids into initially unsaturated media, deviations from the " diffusion " description of the motion (Philip 1955) may occur due to the failure of Darcy's law. Such problems arising in unsaturated media are beyond the scope of this paper and require further study.

The approximate macroscopic analysis of transients given here has the feature in common with Darcy's law that it represents an attempt to provide a simpler and more amenable description of the motion than does the Stokes-Navier equation. Thus, if we consider the form of Darcy's law

$$\bar{U}=KS/\nu \qquad \text{(9.1)}$$

and rewrite (**5.9**) as

$$\bar{U}=(KS/\nu)(1-\mathrm{e}^{-\beta t}), \qquad \text{(9.2)}$$

the replacement of the S of (**9.1**) by the $S(1-\mathrm{e}^{-\beta t})$ of (**9.2**) may be regarded as a process of generalizing Darcy's law to include the influence of the sudden application of the potential gradient at $t=0$. In the same way, we may look on (**8.2**) as the complete generalization of Darcy's law to include any time variation of the potential gradient, the S of (**9.1**) now being replaced by

$$\beta\mathrm{e}^{-\beta t}\int_0^t S(T)\mathrm{e}^{\beta T}\mathrm{d}T+(\nu\bar{U}_0/K)\mathrm{e}^{-\beta t}. \qquad \text{(9.3)}$$

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