SHORT COMMUNICATIONS

THE RESOLUTION OF THE CLOCK PARADOX*

By H. DINGLE†

In a recent paper (Builder 1957) Dr. G. Builder discusses the well-known "clock paradox" on the basis of the restricted theory of relativity. The problem considered, to quote his own description, arises from the following hypothetical experiment: "It is supposed that two observers R and M, equipped with identical synchronized clocks, are initially at rest together, e.g. at the origin of an inertial reference system S. The observer M is sent on a journey along the x-axis of S, travelling away from R with uniform speed v for a time T, coming to rest for a time τ, and then returning with the same speed v to rejoin R after a total time $2T+\tau$ as read on R's clock. It will, in the first instance, be supposed that the times required to accelerate, or decelerate, M are so small that they can be neglected without appreciable error."

The last sentence means in effect that we may take $\tau=0$, and the claim that the restricted theory of relativity is sufficient for the solution of the problem means that we may suppose that M starts instantaneously with velocity v, reverses to velocity $-v$ also instantaneously, and (though this supposition is immaterial) finally comes to rest instantaneously on reaching R with velocity $-v$. The statement is somewhat indefinite since we are not told whether the time T, during which M recedes from R, is obtained from M's clock or is R's coordinate time for that journey, and the statement that M rejoins R after a total time $2T$ as read on R's clock contains a presupposition that R's clock at this event will in fact show twice the time T, whatever T is intended to represent. However, the context indicates that T stands for R's coordinate time for M's outward journey, and in that case it is a false assumption, as I hope to show, that R's clock will read $2T$ when M returns.

The paradox, of course, arises from the following consideration. According to the restricted theory of relativity, a moving clock runs slow as compared with a stationary one. Hence, if we regard M as moving, his clock will be behind R's when they reunite. But the principle of relativity allows us with equal justification to suppose that R is moving and M stationary, and in that case M's clock will be *ahead of* R's when they reunite. These results cannot both be true. Which, then, if either, is right?

The answer, as I think is obvious, is that neither is right: the two clocks will agree on reunion. That this must be so follows immediately from the symmetry of the situation and the principle of relativity of motion. There is

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† 104 Downs Court Road, Purley, Surrey, England.

clearly nothing whatever to distinguish R from M in this problem, so we may attach the label M to whichever we like. And since, according to the principle of relativity, the result of the experiment must be the same whichever choice we make, the only possible relation between the clock readings on reunion is equality, if the readings were the same when the motion began. Dr. Builder, however, finds that when M rejoins R his clock will be relatively retarded by an amount $2T(1-\alpha)$, where $\alpha=(1-v^2/c^2)^{\frac{1}{2}}$.* All we have to do, therefore, is to locate the point where Dr. Builder introduces an asymmetry into the problem, and to consider its significance.

He has himself indicated this point for us. He defines events, E_1, E_2, E_3, and E_4 as respectively the beginning of M's journey, the end of M's outward journey, the beginning of M's return journey, and the end of M's return journey. Since τ is being neglected, E_2 and E_3 are effectively the same event. He then points out that M is present at all these events, but R is present only at E_1 and E_4. Hence there is " an essential asymmetry in the relations of the events E_1 and E_2 to the systems S and S' and, similarly, in the relations of the events E_3 and E_4 to the systems S and S'' ". (S' and S'' are the coordinate systems in which M is successively at rest during the two parts of his journey.)

This, of course, is perfectly true, but clearly that is because Dr. Builder has selected the events with that end in view. If, instead of the events E_2 and E_3, we select events E_2' and E_3', which mark respectively the end of R's outward journey and the beginning of R's return journey, then we shall have four events, at all of which R is present while M is present at only two of them. We may then repeat Dr. Builder's argument and deduce, with precisely the same degree of validity, whatever that may be, that it is R's clock instead of M's that is retarded on reunion.

Dr. Builder's choice of E_2 and E_3 instead of E_2' and E_3' is apparently made from the consideration given in the following sentence: " Because M is the accelerated observer, i.e. the one to whom something happens, the identifiable events E_1, E_2, E_3, E_4 are all coincident with M." But in what sense is M accelerated rather than R? Dr. Builder does not use the general theory of relativity, but he gives us no reason to suppose that he regards it as essentially wrong. We must assume, therefore, that he does not believe that there are absolute accelerations, especially as he has not attempted to justify the ascription of such a thing to M rather than to R, so the acceleration of M must be relative to something. If, therefore, in spite of having decided to ignore the accelerations we reintroduce them for ulterior reasons, we must regard M as accelerated with respect to R, for there is nothing else in the problem to which to relate it. But if M is accelerated with respect to R, then R must be accelerated with respect to M; any other possibility is inconceivable. Hence the statement that " M is the accelerated observer " is meaningless. It can acquire a meaning only if we decide to take into account the " something " that has " happened " to M; for example, if M is projected by an explosion or a gravitational field or

* Dr. Builder's symbol for this quantity is γ^{-1}, but since, in previous papers on this subject, I have used the symbol α, I will keep to it here to save confusion in possible future cross-references.

something like that, then that, it is true, enables one to distinguish M from R, but it does not allow one to distinguish the *motion* of M from the *motion* of R. And if it is this physical cause of the acceleration that is supposed to produce the final difference of readings of the clocks, then Dr. Builder must show how it does so : it is a purely arbitrary statement that, because this unspecified physical process is applied to M and not to R, it will invariably produce the very precise retardation given. If M has red hair and R black, then they will again be distinguishable, and in the absence of a proved relation between the distinguishing characteristic and the behaviour of the clocks, this difference establishes an asymmetry quite as well as the explosion or whatever it may be.

When we consider the character of the effect supposed to be produced, the matter becomes still more mysterious. The amount of the relative retardation of M's clock is determined by the factor α, which is a function of velocity irrespective of how the velocity is produced. Hence, if there were no asymmetry between M and R they would both have to suffer equal retardations with respect to a " stationary " clock in relation to which their velocities were equal and opposite. In the actual case, then, the difference of readings must be produced not by the " explosion " putting the α factor into operation, for it operates without that calamity, but by *stopping* its operation in the clock which shows the greater passage of time. That is to say, the effect of doing " something " to M is not to affect M but to change the behaviour of R. This is so hard to believe that we might reasonably expect Dr. Builder to offer some elucidation of it, but he gives none.

The fact is that Dr. Builder has quite correctly identified the reason why he gets a retardation, but he has not realized the significance of that reason. There is one fundamental distinction in relativity theory, namely, that between *observed* times (or distances) and *coordinate* times (or distances). An observed time is the time of an event according to a clock which is present at the event ; a coordinate time is the time of an event according to a clock which is *not* present at the event. Observed times are absolute : coordinate times vary with the coordinate system used, i.e. they vary according to the state of motion which one ascribes to the clock in question. Consequently, M's times for the events E_1, E_2, E_3, and E_4, being all observed times, are the same for all assumed motions of M, and the total time of the journey is the difference between the readings of M's clock at the events E_1 and E_4. R's clock is present at E_1 and E_4, but not at E_2 and E_3. The difference between the readings of R's clock at E_1 and E_4 is therefore absolute and must agree with that of M's ;* but it is quite wrong to calculate that difference by taking the difference between R's *reading* at E_1 and his *coordinate* time at E_2 and doubling it, because the coordinate time at E_2 is derived by a convention which makes it vary with motion. As I have already shown in discussing a somewhat similar problem (Dingle 1956), when the motion is reversed there is a sudden change of this coordinate time ; i.e. the coordinate

* We are here dealing with the restricted theory. I am aware that the lengths of two different geodesics joining the same points are not necessarily the same in the relativity theory of gravitation, but the difference in any actual case is extremely small, and is neglected here with the accelerations.

time of E_3 is different from that of E_2, being, in fact, as much behind M's actual reading at these events as the coordinate time at E_2 is ahead of it. The result is that when M gets back to R he finds R's clock reading exactly the same as his own.

To make the matter quite clear I would like here to consider the outward half of the journey alone in somewhat more detail, and so get the same result without having to introduce this abrupt change of coordinate times. If we can understand clearly what happens in each half of the journey, we can depend less on coordinates and keep closer to actual observed times. This can be done by using to the full the requirements of special relativity that a " moving " clock runs slow, and a " moving " rod is shortened, by the factor α. These requirements are well known to be equivalent to the Lorentz transformation, but they are less dependent on the conventionality of coordinates. They do not, of course, entirely eliminate that conventionality, for the " retardation " and " contraction " are necessarily as conventional as the " motion " ascribed to the body that is said to experience them, but they eliminate reference to arbitrary zero points in space and time, and so, when they can be used without undue labour, are to be preferred to formal coordinate systems as giving greater insight into the nature of the problem. For this purpose I will give the problem a slightly more picturesque setting, which, it will easily be perceived, does not alter it in principle.

Let A and B be two railway stations connected by a straight railway line of length L when regarded as at rest. M, on an engine, travels from A to B at velocity v, and R stays at A. Let each clock, when at rest, tick n times a second. How many times will each clock tick while M goes from A to B?

Consider M first. He can regard himself as moving along the stationary line from A to B, or as remaining at rest while the line and stations move in the opposite direction. In the former case he will say that the distance he travels is L, but his moving clock ticks $n\alpha$ times a second. The time of the journey is L/v seconds, so the number of ticks is $Ln\alpha/v$. If, on the other hand, he regards himself as at rest, the distance the stations travel is $L\alpha$, and the time they take is $L\alpha/v$; but his stationary clock ticks n times a second, so the number of ticks is again $Ln\alpha/v$. Hence relativity is satisfied, and M's number of ticks for the journey is unambiguously $Ln\alpha/v$.

Now consider R. He also can regard himself as at rest or, with the line and stations, as moving. Suppose he regards himself as at rest. Then the distance M travels is L, and the time of the journey is L/v. During this time R's stationary clock ticks n times a second, so it ticks Ln/v times during the journey. This is Dr. Builder's T, and it is clearly $1/\alpha$ times M's value, as he says. From this he concludes that this is an actual physical difference, which will be preserved, and in fact repeated, during the return journey.

We can easily see, however, that there is something wrong. Suppose that R regards himself as moving. Then the distance between the moving stations is $L\alpha$, and the time of the journey $L\alpha/v$ seconds. His moving clock ticks $n\alpha$ times a second, so its total number of ticks during the journey is $Ln\alpha^2/v$. Hence R's time for the journey is either Ln/v or $Ln\alpha^2/v$, according to his caprice:

he can have which value he likes by choosing the appropriate state of motion. This is contrary to relativity. Hence there must be something wrong in the argument.

The explanation is that R's duration of L/v or $L\alpha/v$ for the journey is only a coordinate value, and therefore varies with the coordinate system (i.e. the standard of rest) adopted. R's clock is present at the beginning of the journey, but not at the end. Hence his estimate of the duration of the journey must necessarily remain a coordinate having no unique value. M, on the other hand, is present at both ends of the journey, so the difference of readings of his clock *has* a unique value. His estimates of the unrecorded space-interval and time-interval, which combine to give the absolute number of ticks which his clock dial records, will vary according to his choice of a standard of rest, but they will necessarily yield the same value, $Ln\alpha/v$, for the number of ticks.

When the engine returns to A, both M's and R's clocks are present, so the total time of the double journey can be determined uniquely from either of them, no matter what assumption is made about which one is at rest. That unique time can be found immediately from M's clock, by simply doubling its time for the outward journey, for that time is independent of coordinates. The time for the total journey is thus seen to be $2Ln\alpha/v$. But how can we get this from R's readings? There are two ways. The first I have already indicated as having been done in my Physical Society paper (Dingle 1956). Choose formal coordinate systems in which R and M respectively are at rest and separate at the common time $t=0$. Then R's coordinate time for the end of M's outward journey is Ln/v. On reversal this coordinate time undergoes an abrupt change, and instead of being $Ln(1-\alpha)/v$ ahead of M's reading, it becomes this same amount behind it. (This, of course, does not mean that an abrupt change occurs in either clock. Coordinate readings are purely ideal, and change when one changes one's mind about the state of motion of the clock.) The result is that during the return journey this difference is gradually diminished until, when M arrives at R, it is exactly cancelled out and R's clock is seen to read $2Ln\alpha/v$ in agreement with M's.

The second way is to give R the same opportunity as M of being present at the end of the outward journey. To do this, let M's engine be provided with a train which, when it is standing at rest on the line, is exactly equal in length to L, the distance between the stations A and B. We may then regard the outward journey as ending when the guard's van, G, at the rear of this train, arrives at A. Let us now consider this half of the journey from R's point of view, on the same two suppositions as before, namely, first that he is at rest, and second that he is moving while the train remains at rest.

In the first case, the length of the moving train is $L\alpha$, and the time of the journey therefore $L\alpha/v$. R's stationary clock ticks n times a second, so it ticks $Ln\alpha/v$ times during this journey. In the second case, the stationary train has length L, but R's moving clock ticks $n\alpha$ times a second. Hence again the time of the journey is $Ln\alpha/v$. Hence relativity is again satisfied, and the time of the journey by R's clock is unambiguously $Ln\alpha/v$. The time of the double journey by R's clock is therefore $2Ln\alpha/v$, exactly the same as the time shown by M's clock.

It is almost superfluous to show that if we consider the arrival of G at A from M's point of view we shall get the same ambiguity as that which arose when we considered the arrival of M at B from R's point of view. For the sake of completeness, however, let it be done. If M regards himself and the train as moving, and the line and stations as at rest, he will consider the length of the train to be $L\alpha$ and the number of ticks of his moving clock to be $n\alpha$ a second. The total number of ticks for the journey will therefore be $Ln\alpha^2/v$. On the other hand, if he regards himself and the train as at rest, the length of the train will be L and his clock will tick n times a second. The total number of ticks for the journey will therefore then be Ln/v. This is precisely the same difference as in the former case.

Now the problem itself merely concerns the separation and reunion at A of M and R. Whether we introduce a distant station B, or a distant guard's van G, as an intermediary in solving it is quite immaterial: we have exactly the same title to use one as the other. If we choose the former and make the mistake of regarding coordinate times as observed times, we shall get Dr. Builder's result. If we choose the latter, and make the same mistake, we shall get the opposite result, namely, that M's clock is *ahead of* R's when they reunite. That is an inescapable proof that Dr. Builder's result must be wrong.

It should be clear now that the asymmetry that Dr. Builder has introduced into the problem is not the fact that " M is the accelerated observer, i.e. the one to whom something happens ". M is the one to whom something happens, no matter whether we use the station B or the guard's van G as an auxiliary (i.e. whether we consider the intermediate events E_2, E_3 or E_2', E_3'), but the relative readings of the clocks, resulting from Dr. Builder's line of argument, are opposite in the two cases. The asymmetry is simply that Dr. Builder has forced R, and not M, to use coordinate times, and has then mistakenly treated those times as though they were observed. I hope that he will now be able to perceive that all this analysis is really redundant. The principle of relativity—i.e. the principle that nature allows of no criterion for deciding which of two relatively moving bodies is the " moving " one—necessarily requires that the clocks in question shall agree on reunion: otherwise the prohibited criterion would have been found.

References

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