DISPERSAL OF DUST PARTICLES FROM ELEVATED SOURCES

II. LIMITATIONS OF THE APPROXIMATE THEORY

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Summary

This paper forms a continuation of a previous one (Csanady 1955) wherein Sutton's continuous point source equations were extended to the case of heavy particles of uniform free falling speed. The solution obtained there was an approximate one; limits of validity are discussed in detail in the present paper. This discussion is preceded by a generalization of the earlier solution and two charts are presented to aid calculation of the " mirror image multiplier ", the basic function of the approximate theory.

I. Introduction

In a previous communication (Csanady 1955) an approximate expression was derived for concentration and deposition rate of particles of uniform free falling speed f, discharged from an elevated, continuous point source into a horizontal wind. The method of derivation was, however, not sufficiently rigorous to establish the validity of the solution and no experimental evidence seems to be available for comparison. Substitution of the derived expression into the differential equations of turbulent diffusion (Batchelor 1949) shows in fact that the former is not an exact solution. It will be shown in the present paper that the solution is approximate but that it is likely to be useful in a certain class of practical problems, notably in estimating dust fall from a *tall* chimney, when the dust plume is *flat* (i.e. nearly horizontal).

Another improvement on the earlier result would be to consider more general forms of the standard deviation function and to allow the " chimney " height to vary with distance in order to take into account the possibility of thermal rise. These tasks present no difficulty and will be carried out below. To render the practical application of the theory still easier, it is also possible to draw up charts for the computation of the " mirror image multiplier " as a function of suitably defined non-dimensional variables. Two such charts are presented below.

II. Generalization of the Earlier Solution

Given that the effective chimney height $h(x)$ may increase with distance, the mean height above ground level of a dust cloud of uniform free falling velocity f, in a horizontal wind of speed u, will be, at a distance x from the source,

$$z^*=h(x)-fx/u. \qquad (1)$$

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Where a given "streamline" (or rather mean-path-line) reaches ground there is a vertical component of mean velocity, in addition to the bodily drift velocity f of the cloud,

$$\begin{aligned}\Delta w &= \frac{z^*}{\sigma_z} \cdot \frac{\mathrm{d}\sigma_z}{\mathrm{d}t} - \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= (hu - fx) \cdot \sigma_z'/\sigma_z - uh' \end{aligned} \qquad \ldots\ldots\ldots\ldots \quad (2)$$

(dashes denote differentiation with respect to x).

By the reasoning given in the earlier paper we find for the ground level value of the mirror image multiplier

$$\alpha_0 = \frac{\Delta w}{2f + \Delta w}. \qquad \ldots\ldots\ldots\ldots \quad (3)$$

Combining the last two equations we find the result

$$\alpha_0 = \frac{(hu - fx) \cdot \sigma_z'/\sigma_z - uh'}{2f + (hu - fx) \cdot \sigma_z'/\sigma_z - uh'}. \qquad \ldots\ldots\ldots\ldots \quad (4)$$

Equation (4) now allows for thermal rise and arbitrary standard deviations. In order to calculate α_0 it is necessary to assume some particular function for the standard deviations. On theoretical grounds it seems objectionable to extrapolate Sutton's formulae to large distances. It is possible, however, to fit a standard-deviation formula based on an exponentially decaying Lagrangian correlation coefficient to Sutton's numerical data and use the resulting expression for extrapolation. This approach is not new; it has been pointed out by the referee of this paper that Inoue (1952) stated the standard deviation equation in the following form:

$$\sigma_z^2 = 2g_z^2 x_0 [x - x_0(1 - \mathrm{e}^{-x/x_0})], \qquad \ldots\ldots\ldots\ldots \quad (5)$$

where g_z = gustiness in the vertical,
x_0 = scale of turbulence.

Sutton derives his numerical values from diffusion parameters measured at 100 m from the source. On the same basis the numerical values of equation (5) become

$$x_0 = 80 \text{ m},$$
$$g_z = 0 \cdot 085.$$

Equation (5) with these numerical values has been used in drawing up the charts presented below.

Taking first the case of constant "chimney" height, i.e. of a horizontal gas plume, $h' = 0$, and introducing the nondimensional variables,

$$X = x/x_0, \quad Y = fx/hu,$$

equation (4) may be transformed into

$$\frac{1}{Y} = 1 + \frac{4\alpha_0}{(1 - \alpha_0)} \left(\frac{1}{1 - \mathrm{e}^{-X}} - \frac{1}{X} \right), \qquad \ldots\ldots\ldots\ldots \quad (6)$$

which is represented in Figure 1. To find values of α_0 with an *inclined* gas plume it is convenient to use the results in Figure 1 and draw up a subsidiary chart to find a "modified" multiplier α_{0m}, valid for a given inclination of plume. It is thus possible to recast equation (4) into the following form :

$$\beta=\frac{\alpha_0-\alpha_{0m}}{(1-\alpha_0)(1-\alpha_{0m})}, \qquad (7)$$

where $\beta=uh'/2f$ and α_0 is the value obtained from Figure 1, i.e. with a horizontal gas plume, but for identical X and Y. Equation (7) is represented in Figure 2.

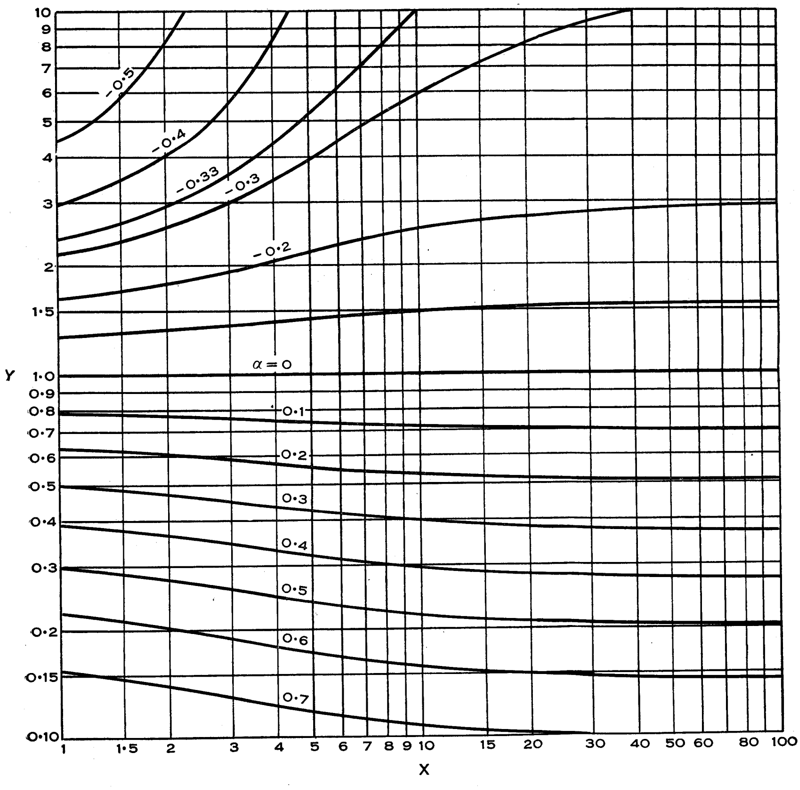

Fig. 1.—Chart to facilitate computation of the mirror image multiplier α_0 for horizontal plume. X and Y are non-dimensional distance variables.

In industrial dust clouds conditions are of most interest at the point where the centre of the plume reaches ground level. Here we have $\alpha_0=0$, so that the mirror image term may be ignored in a first approximation if the gaseous plume is horizontal. However, if there is a thermal rise,

$$\alpha_{0m,c}=-\beta/(1-\beta), \qquad (8)$$

where $\alpha_{0m,c}$ is the value of α_{0m} at $x=hu/f$, so that, taking into account the definition of β,

$$1+\alpha_{0m,c}=1-(xh'/2h)/(1-xh'/2h). \qquad (9)$$

Thus one may take the reduction of deposition due to the inclined gas plume into account in a first approximation by simply applying the factor $(1+\alpha_{0m,c})$. Physically, the reduction is due to "stretching" of distance elements; the inclination of the gas plume causes the deposition to be distributed over a larger area and the amplitudes must be diminished to preserve continuity. The expression for dust fall rates is best written as

$$D=\frac{Wf\,.\,(1+\alpha_0)}{2\pi\sigma_y\sigma_z u}\exp\left\{-\frac{y^2}{2\sigma_y^2}-\frac{(h-fx/u)^2}{2\sigma_z^2}\right\}, \quad \ldots\ldots \quad (\mathbf{10})$$

where W is the total rate of dust emission and α_0 is to be found from the charts presented here, once the three non-dimensional variables X, Y, and β are known.

The introduction of the factor α_0, which is a function of x, into equation (**10**) conflicts with the differential equations of turbulent diffusion. The reason for the discrepancy is that in the derivation of equation (**3**) (see Csanady 1955)

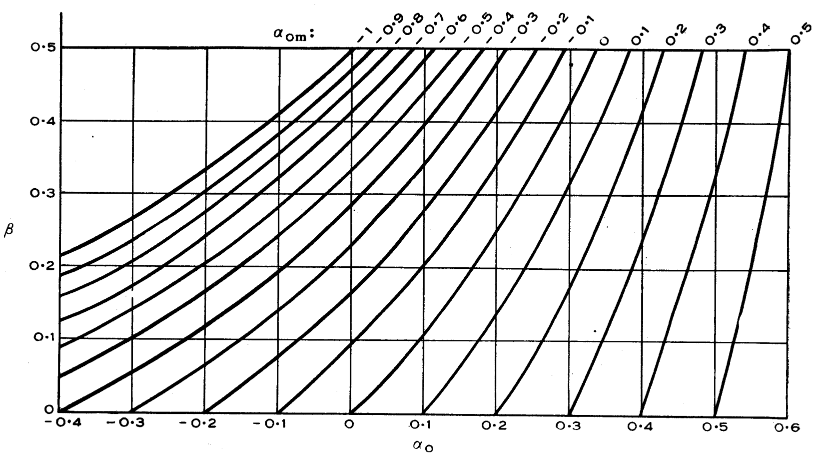

Fig. 2.—Correction chart for the mirror image multiplier when plume axis is inclined.

the transport due to a gradient in α has not been considered. It is in principle possible to rectify this but the resulting expression for α_0 becomes impracticably complex. A more fundamental mathematical investigation starting from the differential equations may some day yield an exact solution. As subsequent arguments will show, the present approximate theory should meanwhile be useful in certain practical applications.

III. Validity of Solution

(1) *Group-effect in Dust Clouds.*—Mutual drag-interference at the concentrations encountered in chimneys is negligible. The average distance between particles is of the order of 100 particle diameters and the probability that two particles travel within a distance of 10 diameters is usually less than 1 per cent.

(2) Even assuming h to be constant, the substitution of $h-fx/u$ in Sutton's (1953) equations in place of h is not a rigorous step. The conditions under which the approximation may be expected to hold can be investigated following the

method indicated by Sutton (1953, p. 136). For simplicity we discuss the isotropic case and assume that the distance from the source is large enough for the Lagrangian correlation to be approximately zero:

$$\sigma_x^2=\sigma_y^2=\sigma_z^2=2K(t-t_c), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(11)}$$

where $K=\overline{v'^2}\,.\,t_0$, the limiting value of turbulent diffusivity, with $\overline{v'^2}$ the mean square turbulent velocity fluctuation, t_0 the area under the Lagrangian correlation coefficient, and t_c is the centre of gravity of the same area. Equation (**11**) is best deduced from Kampè de Feriet's form of Taylor's well-known theorem (see e.g. Batchelor 1949).

If there is a steady horizontal wind and a uniform downward drift, the cloud emitted at t' at $x=y=z=0$ will at time t be centred at

$$x=u(t-t'), \quad y=0, \quad z=-f(t-t'). \quad \ldots\ldots\ldots\ldots \quad \textbf{(12)}$$

By the method indicated in Sutton's book one finds, after integrating over all the smoke-puffs,

$$c=\frac{W}{4\pi Kr_0}\exp\left[\frac{u}{2K}\left\{x-x_c-r_0\left(1+\frac{f^2}{u^2}\right)^{\frac{1}{2}}-\frac{f}{u}\left(z+\frac{f}{u}x_c\right)\right\}\right], \quad \ldots \quad \textbf{(13)}$$

where

$$x=ut_c,$$
$$r_0^2=x^2+y^2+z^2+(u^2+f^2)t_c^2-2(ux-fz)t_c.$$

It is now easily verified that, provided

$$\frac{f}{u}\ll 1 \text{ and } \frac{y^2+(z+fx_c/u)^2}{(x-x_c)^2}\ll 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(14)}$$

the above expression reduces approximately to

$$c=\frac{W}{4\pi K(x-x_c)}\exp\left\{-\frac{y^2+(z+fx/u)^2}{(4K/u)(x-x_c)}\right\}, \quad \ldots\ldots\ldots \quad \textbf{(15)}$$

which is of the type used in the author's previous paper and equation (**10**) above. Substitution into the differential equation of turbulent diffusion (Batchelor 1949) shows that the physical interpretation of this approximation is neglect of diffusion along wind, which is valid at large distances from the source and for a flat dust plume, according to the conditions just found. One may thus expect that when there is a thermal rise, $\mathrm{d}h/\mathrm{d}x$ must also be much smaller than unity for the same kind of expression to be valid.

(3) The introduction of a multiplier $\alpha(x, z)$ disturbs the flux between what have previously been streamlines and may also conflict with continuity in the sense that the total dust deposition might appear higher than the emission.

Investigating continuity first, we have for the total deposition after integrating equation (**10**)

$$\frac{Wf}{\sqrt{(2\pi)}\,.\,u}\int_0^\infty \frac{1+\alpha_0}{\sigma_z}\,exp\left\{-\frac{(fx/u-h)^2}{2\sigma_z^2}\right\}dx=W, \quad \ldots \quad \textbf{(16)}$$

which has here been set equal to the emission W. Equation (**16**) may be written as

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} M \exp(-p^2)\mathrm{d}p=1, \qquad (\mathbf{17})$$

where

$$M=(1+\alpha_0)\frac{f/u}{f/u-\sqrt{2}\sigma_z p},$$
$$p=\frac{fx/u-h}{\sqrt{2}\sigma_z}.$$

Equation (**17**) will be satisfied, for example, if $M=1+$odd powers of p.

If we assume $h'=0$ for simplicity, with equation (**4**) M may be reduced to

$$M=1/(1-Bp)=1+Bp+B^2p^2+\ldots, \qquad (\mathbf{18})$$

with

$$B=(1/\sqrt{2})\sigma_z' u/f.$$

Close to the source $\sigma_z=g_z$, while at large distances, using an asymptotic formula such as equation (**11**), one obtains

$$B=\frac{g_z}{2}\left[\frac{x_0 u/f}{h-x_c f/u}\right]^{\frac{1}{2}}, \qquad (\mathbf{19})$$

The integral in equation (**17**) will now contain a term, $\frac{1}{2}B^2$, and in order to satisfy continuity it is necessary that $B^2 \ll 1$. Substituting the value of B from equation (**19**) and ignoring x_c against $x=hu/f$ (i.e. against that distance where the centre of the dust plume reaches ground level, which owing to the previous condition on the " flatness " of the dust plume should indeed be large) we have the required condition

$$f/u \gg g_z^2 x_0/4h. \qquad (\mathbf{20})$$

Closer to the source we should have $f/u \gg g_z$, which would often be difficult to satisfy, hence the need for a " tall " stack.

Disturbances in flux are likely to be most important near the centre of the dust cloud as the latter reaches ground level. Here the approximate theory shows a concentration of $c_s\,.\,(1+\alpha_0)$, with c_s the concentration due to the simple term alone (not including the mirror image term in equation (**10**)). In deriving an expression for α_0 the balance in vertical transport has been considered, ignoring, however, transport due to a gradient in α. It is possible to take this gradient into account in writing down the transport balance equation (see the author's previous paper, p. 548 on) but the resulting equations are excessively complicated.

Near the point of maximum deposition streamlines are nearly straight and have the inclination f/u. One may thus write

$$\frac{\partial\alpha}{\partial z} \simeq -\frac{\mathrm{d}\alpha_0}{\mathrm{d}x}\,.\,\frac{u}{f}, \qquad (\mathbf{21})$$

since one may assume that α is constant along a streamline, see the author's previous paper. From equation (4), assuming $h'=0$ and using the asymptotic behaviour at large x of σ_z, one finds, ignoring x_0 and x_c against x,

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}x} \simeq -\frac{f}{4hu} \qquad (22)$$

at $x=hu/f$, the centre of the dust cloud.

The transport due to a gradient in α is

$$F=-K_z c_s \partial\alpha/\partial z, \qquad (23)$$

where $K_z=\frac{1}{2}\mathrm{d}\sigma_z^2/\mathrm{d}x$, turbulent diffusivity. This must be small compared to the transport due to the free fall velocity f (at the centre of the dust cloud, $\alpha_0=0$) :

$$c_s f \gg K_z c_s/4h, \qquad (24)$$

using the results in equations (21) and (22). Substituting the asymptotic value of the diffusivity, $K_z=g_z^2 u x_0$, we find again

$$f/u \gg g_z^2 x_0/4h, \qquad (25)$$

which is identical with equation (20). Using the numerical values quoted in connexion with equation (5) we may write (h in metres)

$$f/u \gg 1\cdot 16/8h.$$

For a 50 m chimney, for example, the limits of validity of the approximate theory are

$$1 \gg f/u \gg 0\cdot 003.$$

In practice one could take perhaps

$$0\cdot 25 \geqslant f/u \geqslant 0\cdot 01,$$

which often covers an important range.

IV. References

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