ELECTROMAGNETIC PROPAGATION IN AN ALMOST HOMOGENEOUS MEDIUM

By V. W. BOLIE*

[*Manuscript received September* 30, 1957]

Summary

This paper concerns the development from Maxwell's electromagnetic equations of an equation of propagation in an almost homogeneous medium. The equation is applied to the problem of determining the secondary wave produced by an isolated Gaussian-shaped perturbation in the refractive index. An exact solution is obtained for points located on the axis of symmetry parallel to the direction of propagation of the incident primary wave. An approximate solution for points remote from the anomaly is obtained and its validity is compared with the more restricted exact solution. An interesting limit process is encountered in the derivation of the formula for the scattering cross section of the refractive index perturbation.

I. INTRODUCTION

In the field of radio communication there is currently a great deal of interest in the scattering of high-frequency waves by a turbulent atmosphere and the trans-horizon propagation of measurable signal strengths. Experiments with propagation of microwaves beyond the radio horizon show signal characteristics which cannot be explained in terms of free-space propagation, horizon diffraction, or mode theory of tropospheric ducts. In 1950 Booker and Gordon† proposed a scattering theory based on the random space-dependence of the dielectric constant due to atmospheric turbulence. Although this theory was promising in some respects, the detail of the scattering mechanism was restricted to a random array of dipole scatterers. A number of other theoretical papers and a large amount of experimental data have recently appeared in the literature, none covering extensions of the simple dipole theory.

The purpose of this paper is to develop fundamental equations for the propagation and scatter of electromagnetic energy in a nearly transparent medium, and to apply these equations to the problem of scattering by an isolated, Gaussian-shaped perturbation in the refractive index. The resultant solution for the scattered field provides a means of determining the scattering caused by a refractive anomaly of arbitrary size.

* Collins Radio Company, Cedar Rapids, Iowa; present address: Iowa State College, Ames, Iowa, U.S.A.

† BOOKER, H. F., and GORDON, W. E. (1950).—A theory of radio scattering in the troposphere. *Proc. Inst. Radio Engrs., N.Y.* **38**: 401–12.

II. Propagation in a Nearly Homogeneous Medium

The propagation of electromagnetic energy is, in general, described by the partial differential equations of Maxwell. For a linear, isotropic, charge-free medium of zero conductivity, Maxwell's equations can be used to show that the electric intensity vector $E(x, y, z, t)$ is governed by the equation

$$\nabla^2 E - \nabla(\nabla \cdot E) = \mu\varepsilon(\partial^2 E/\partial t^2), \qquad \text{(1)}$$

where μ and ε represent the permeability and permittivity of the medium. The free-space values for μ and ε are $\mu_0 = 4\pi/10^7$ H/m and $\varepsilon_0 = 10^{-9}/36\pi$ F/m respectively.

For a stationary, almost homogeneous medium, the refractive index $n = \sqrt{(\mu\varepsilon/\mu_0\varepsilon_0)}$ may be expressed as

$$n = n_1[1 + p(x, y, z)], \qquad \text{(2)}$$

where n_1 is a dimensionless constant slightly greater than unity, and where $|p(x, y, z)| \ll 1$. For a medium like the atmosphere, the permeability $\mu = \mu_1$ may be assumed constant, so that the permittivity is approximated by the equation

$$\varepsilon = \varepsilon_1[1 + 2p(x, y, z)], \qquad \text{(3)}$$

where $\varepsilon_1 = n_1^2 \mu_0 \varepsilon_0/\mu_1$.

A convenient substitution for the divergence of E in equation (**1**) can be obtained by substituting equation (**3**) into the Maxwell relation $\nabla \cdot (\varepsilon E) = 0$. After suitable manipulation of vector identities, followed by a logarithmic expansion, the approximation

$$\nabla \cdot E = -2E \cdot \nabla p \qquad \text{(4)}$$

is obtained. The substitution of equation (**4**) into equation (**1**) gives,

$$\nabla^2 E + 2\nabla(E \cdot \nabla p) = \mu_1 \varepsilon_1 (1 + 2p)\partial^2 E/\partial t^2. \qquad \text{(5)}$$

Assuming the sinusoidal time-dependence, $E = E' \sin \omega t$, gives,

$$\nabla^2 E' + \nabla(E' \cdot \nabla p) + (4\pi^2/\lambda_1^2)(1 + 2p)E' = 0, \qquad \text{(6)}$$

in which the conventional notation $\lambda_1 = 2\pi/\omega\sqrt{(\mu_1\varepsilon_1)}$ has been used.

It is convenient at this point to assume the total field E' as being composed of a weak scattered field $\check{\mathsf{E}}$, and a dominant homogeneous field $\bar{E}$ which satisfies the simple propagation law $\nabla^2 \bar{E} + (4\pi^2/\lambda_1^2)\bar{E} = 0$. Making this assumption, and neglecting the effects of secondary scattering of the scattered field, gives

$$\nabla^2 \mathsf{E} + (4\pi^2/\lambda_1^2)\,\mathsf{E} = -(8\pi^2/\lambda_1^2)p\bar{E} - 2\nabla(\bar{E} \cdot \nabla p), \qquad \text{(7)}$$

which shows how the direct homogeneous field $\bar{E}$ produces sources for the scattered field $\check{\mathsf{E}}$.

In the foregoing propagation equation for the scattered field $\check{\mathsf{E}}$ it is seen that the scattering excitation is represented by two separate terms on the right side of the equation. The scattering associated with the term $(8\pi^2/\lambda_1^2)p\bar{E}$ has been examined on intuitive grounds by other investigators. The existence of the term $2\nabla(\bar{E} \cdot \nabla p)$ has been neglected or ignored in most of the literature on scattering theory, even though its effect may be appreciable under certain conditions. If $p(x, y, z)$ is an arbitrary function, then ∇p is a vector of arbitrary magnitude and direction, and $\nabla(\bar{E}\cdot\nabla p)$ will also be arbitrary in magnitude and direction. This is in contrast with the vector $(8\pi^2/\lambda_1^2)p\bar{E}$, which is always parallel to the incident direct-wave field vector $\bar{E}$.

III. Scattering from a Nearly Transparent Anomaly

In order to examine in detail the mechanism of radiation scattering in a nearly homogeneous medium, it is convenient to consider the effect of an isolated perturbation in the refractive index. While the incident-direct field may be taken as a uniform plane wave without appreciable loss of generality, the refractive-index perturbation should be three-dimensional and continuous. Such a model is well approximated by assuming the fractional variation $p(x, y, z)$ of the refractive index has its maximum value at the origin and decreases with radial distance from the origin in a Gaussian manner.

This is expressed mathematically as $p=p_0 \exp(-r^2/s^2)$, where p_0 is the maximum value of p, s represents the " anomaly radius ", and $r=\sqrt{(x^2+y^2+z^2)}$. The incident-direct field may be taken as a plane wave having its electric intensity parallel to the x-axis and travelling in the z-direction. Thus $\bar{E}=\hat{u}_1 E_0 \exp(-\mathrm{j}2\pi z/\lambda_1)+\hat{u}_2\cdot 0+\hat{u}_3\cdot 0$, where E_0 is the necessary amplitude constant, where j is the unit imaginary number, and where $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ represent unit vectors in the x, y, z directions.

Substituting the above assumptions into equation (**7**), and making use of the well-known retarded potential theory, shows that the components of the scattered-field vector $\mathsf{E}=u_1\mathsf{E}_1+u_2\mathsf{E}_2+u_3\mathsf{E}_3$ are given by the integrals

$$\mathsf{E}_1=\frac{p_0E_0}{4\pi}\underbrace{\overbrace{\iiint}^{\infty}}_{-\infty}\frac{4}{\rho}\left[\frac{2\pi^2}{\lambda_1^2}+\frac{2\xi^2-s^2}{s^4}\right]h(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \quad \ldots\ldots \quad (\mathbf{8})$$

$$\mathsf{E}_2=\frac{p_0E_0}{4\pi}\underbrace{\overbrace{\iiint}^{\infty}}_{-\infty}\frac{8}{\rho}\;\frac{\xi\eta}{s^4}h(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{9})$$

$$\mathsf{E}_3=\frac{p_0E_0}{4\pi}\underbrace{\overbrace{\iiint}^{\infty}}_{-\infty}\frac{8}{\rho}\left[\frac{\xi\zeta}{s^4}+\mathrm{j}\frac{\pi}{\lambda_1}\;\frac{\xi}{s^2}\right]h(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \quad \ldots\ldots\ldots \quad (\mathbf{10})$$

where

$$h(\xi,\eta,\zeta)=\exp\left[-\xi^2+\eta^2+\zeta^2)/s^2-\mathrm{j}2\pi(\zeta+\rho)/\lambda_1\right]$$

and

$$\rho^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

These expressions are too complicated to be readily evaluated without the aid of automatic computing equipment. Fortunately, however, there are two important cases of practical interest for which the integrals can be simplified in terms of known functions. One case is when the field point (x, y, z) is on the z-axis, and the other case is when $r \gg s$ so that the integrands are negligibly small except where the value of ρ in the denominators may be assumed equal to r and the value of ρ in the exponents is well approximated by its projection in the direction of the field point.

IV. Exact Solution in the Incident-wave Direction

At points far removed from the scattering anomaly the scattered field integrals given in equations (**8**), (**9**), and (**10**) can be simplified and evaluated in terms of elementary functions. The resultant solutions, however, can only be considered as approximations which are asymptotic to the exact solutions. The approximations will degenerate as the field point approaches the scattering anomaly. In order to estimate the validity of the approximations it is convenient to determine the exact solutions under conditions sufficiently restricted to permit the evaluation of the complicated integrals. Such conditions result from assuming the field point is on the z-axis.

If the field point (x, y, z) is on the z-axis, then $x=y=0$, and the integrands in equations (**9**) and (**10**) are odd in ξ, with the result that E_2 and E_3 are both zero. This result can also be reasoned from the symmetry of the physical problem. In order to simplify the integral for E_1, on the z-axis, it is convenient to introduce the translated polar coordinate notation $\xi=\rho \sin\alpha \cos\beta$, $\eta=\rho \sin\alpha \sin\beta$, $\zeta=z+\rho\cos\alpha$, and to replace the volume element $d\xi d\eta d\zeta$ by $\rho^2 \sin\alpha d\alpha d\beta d\rho$. Under these conditions the integrand in equation (**8**) is independent of β, permitting immediate integration with respect to that variable. Integration with respect to α is easily done by replacing $\cos\alpha$ by a dummy variable. The final integration with respect to ρ gives

$$\mathsf{E}_1=[C_1I_1-C_2I_2+C_3]p_0E_0\frac{\sqrt{\pi}}{2}\frac{s}{z}\left(\frac{2\pi s}{\lambda_1}\right)^2\exp(-j2\pi z/\lambda_1), \quad \ldots (\mathbf{11})$$

where

$$C_1=\frac{1}{(1+j\pi s^2/\lambda_1 z)}-\frac{j\lambda_1/2\pi z}{(1+j\pi s^2/\lambda_1 z)^2}-\frac{\lambda_1^2/4\pi^2 z^2}{(1+j\pi s^2/\lambda_1 z)^3}, \quad \ldots\ldots (\mathbf{12})$$

$$C_2=\frac{1}{(1+j\pi s^2/\lambda_1 z)}-\frac{j\lambda_1/2\pi z}{(1+j\pi s^2/\lambda_1 z)^2}-\frac{\lambda_1^2/4\pi^2 z^2}{(1+j\pi s^2/\lambda_1 z)^3}, \quad \ldots\ldots (\mathbf{13})$$

$$C_3=\frac{\lambda_1^2/2\pi^2 z}{(1+j\pi s^2/\lambda_1 z)^2}\,\frac{\exp(-z^2/s^2)}{s\sqrt{\pi}}, \quad \ldots\ldots\ldots\ldots (\mathbf{14})$$

and where

$$I_1=\frac{1}{2}+\frac{1}{s\sqrt{\pi}}\int_0^z \exp(-u^2/s^2)du, \quad \ldots\ldots\ldots\ldots (\mathbf{15})$$

$$I_2=\frac{\exp(j4\pi z/\lambda_1)}{s\sqrt{\pi}}\int_z^\infty \exp(-u^2/s^2-j4\pi u/\lambda_1)du, \quad \ldots\ldots (\mathbf{16})$$

The quantities C_1, C_2, and C_3 are complex numbers which are readily evaluated in terms of s, λ_1, and z. The integral I_1 is easily interpreted in terms of the error function of well-known statistical theory. The integral I_2 can be represented in terms of the error function of a complex variable.

Equation (**11**) represents the exact solution for the scattered field on the z-axis. The formula is valid for all values of z, including points at or near the origin, where the centre of the scattering anomaly is located. When the anomaly radius is large compared to one wavelength, the above solution can be further simplified for points near the origin. Thus, if $s \gg \lambda_1$, and $-s \leqslant z \leqslant 3s$, it can be shown that the quantity $C_1I_1 - C_2I_2 + C_3$ is well approximated by the value $C_1I_1 - C_2I_2 + C_3 \cong (-j\lambda_1 z/2\pi s^2)[1+\mathrm{erf}\,(z)]$. Substituting this expression into equation (**11**) gives

$$\mathsf{E}_1 = -jp_0E_0[1+\mathrm{erf}\,(z)]\frac{\sqrt{\pi}}{2}\,\frac{2\pi s}{\lambda_1}\exp\,(-j2\pi z/\lambda_1), \qquad \ldots\ldots \quad (\mathbf{17})$$

which is valid for $-s \leqslant z \leqslant 3s$ when $s \gg \lambda_1$. As might be anticipated on intuitive grounds, the magnitude of the scattered field intensity increases with z in the vicinity of the origin. This effect is illustrated in Figure 1, where the space factor

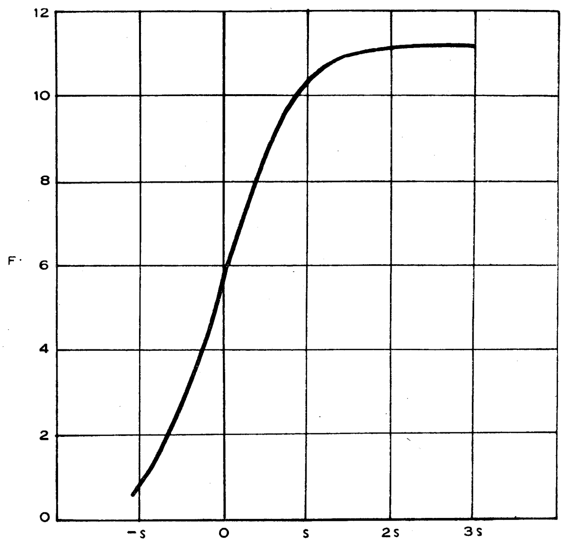

Fig. 1.—Space factor for the scattered field. On the z-axis the scattered field is given by $|\,\mathsf{E}/E_0\,| = p_0sF/\lambda_1$. When $s \gg \lambda_1$ and $-s \leqslant z \leqslant 3s$, $F = \pi\sqrt{\pi}\{1+(2/s\sqrt{\pi})\int_0^z \exp\,(-u^2/s^2)\mathrm{d}u\}$.

$F = \pi\sqrt{\pi}[1+\mathrm{erf}\,(z)]$ is plotted as a function of z. It is apparent from the graph that the scattered field intensity does not attain its greatest magnitude until the incident wave has passed through the greater part of the refractive region.

Equation (**11**) is easily simplified for large values of z. It is readily shown that when $s > \lambda_1$ and $2s < z < \infty$, the value of I_1 is nearly unity, the magnitudes of C_3 and C_2I_2 are negligible, while the expression for C_1 simplifies so that $C_1I_1 - C_2I_2 + C_3 \cong [1+j\pi s^2/\lambda_1 z]^{-1}$. Equation (**11**) then reduces to

$$\mathsf{E}_1 = p_0E_0\frac{\sqrt{\pi}}{2}\left(\frac{2\pi s}{\lambda_1}\right)^2\frac{\exp\,(-j2\pi z/\lambda_1)}{z/s + j\pi s/\lambda_1}, \qquad \ldots\ldots\ldots \quad (\mathbf{18})$$

which is valid when $s > \lambda_1$ and $z > 2s$. It is seen from this expression that as z increases in the positive direction the amplitude of the scattered field eventually varies inversely with distance. Thus, when $z \gg \pi s^2/\lambda_1$, the above formula becomes

$$\mathsf{E}_1 = p_0 E_0 \frac{\sqrt{\pi}}{2}\left(\frac{s}{z}\right)\left(\frac{2\pi s}{\lambda_1}\right)^2 \exp\left(-j2\pi z/\lambda_1\right), \quad \ldots\ldots \quad (\mathbf{19})$$

which confirms the expected inverse-distance behaviour at remote points. This result is also useful in checking the validity of the far field solution to be considered in the next section.

The actual behaviour of the scattered field in the direction of the incident wave is illustrated in Figure 2, where equations (**17**), (**18**), and (**19**) are plotted as functions of z for the case in which $s = 10\lambda_1$. The graph shows a rather rapid

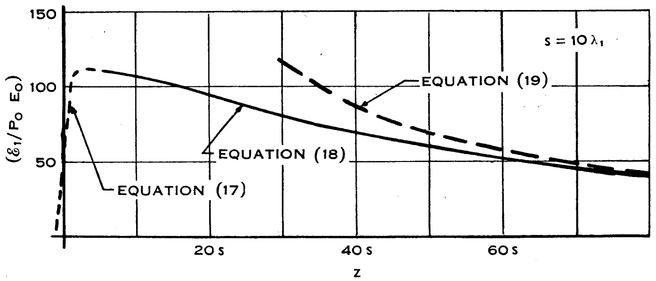

Fig. 2.—Scattered field intensity in the incident-wave direction for a 10-wavelength anomaly radius.

increase of the scattered field in the neighbourhood of the origin, to a maximum value of $|\mathsf{E}_1| \simeq 112 \,|p_0 E_0|$ when $z \simeq 3s$, and then a gradual decrease, to become nearly identical to the inverse-distance behaviour at $z \simeq 100s$. The corresponding plot for the case in which $s = 100\lambda_1$ would be similar except that the scale markings on the vertical axis would be increased by a factor of 10, and that the transition to the inverse-distance asymptote would not be adequately completed until $z \simeq 1000s$. It is clear that the strongest interference between the direct and scattered fields occurs when $z \simeq 3s$. The early assumption that the incident field is essentially undisturbed by the scattered field is seen to be valid, provided that $|p_0| < (\lambda_1/100s)$. This important condition is easily satisfied in all cases of tropospheric scattering of v.h.f. and microwaves.

V. Solution for the Far Field

When the field point (x, y, z) is sufficiently far removed from the origin, the expression for ρ can be approximated as $\rho \simeq r - x\xi/r - y\eta/r - z\zeta/r$ for use in the $h(\xi, \eta, \zeta)$ function in equations (**8**), (**9**), and (**10**). The approximation $\rho \simeq r$ is adequate for use in the denominators, since each integrand is of negligibly small magnitude except near the origin. These assumptions reduce the formulas for E_1, E_2, E_3 to expressions which yield to integration. The results show that, when $r \gg s$, equations (**8**), (**9**), and (**10**) reduce to

$$\mathsf{E}_1 = p_0 E_0 \frac{\sqrt{\pi}}{2} \frac{s}{r} \left(\frac{2\pi s}{\lambda_1}\right)^2 \left(1 - \frac{x^2}{r^2}\right) \exp\left\{-\frac{2\pi^2 s^2}{\lambda_1^2}\left(1-\frac{z}{r}\right) - \frac{j2\pi}{\lambda_1} r\right\}, \quad \text{.. (20)}$$

$$\mathsf{E}_2 = -p_0 E_0 \frac{\sqrt{\pi}}{2} \frac{s}{r} \left(\frac{2\pi s}{\lambda_1}\right)^2 \frac{xy}{r^2} \exp\left\{-\frac{2\pi^2 s^2}{\lambda_1^2}\left(1-\frac{z}{r}\right) - \frac{j2\pi}{\lambda_1} r\right\}, \quad \text{.... (21)}$$

$$\mathsf{E}_3 = -p_0 E_0 \frac{\sqrt{\pi}}{2} \frac{s}{r} \left(\frac{2\pi s}{\lambda_1}\right)^2 \frac{xz}{r^2} \exp\left\{-\frac{2\pi^2 s^2}{\lambda_1^2}\left(1-\frac{z}{r}\right) - \frac{j2\pi}{\lambda_1} r\right\}. \quad \text{.... (22)}$$

On the z-axis, these equations agree with the far field previously obtained from the exact solution. It is easy to confirm that the scattered field represented by these formulas propagates radially outward from the origin, with the electric vector perpendicular to the direction of propagation.

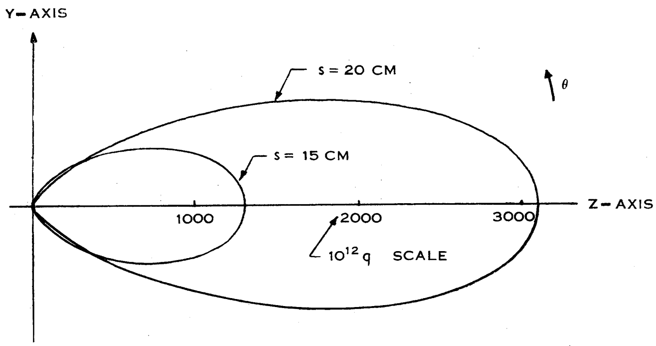

Fig. 3.—Scattered field patterns.
$10^{12}q = 10^{12}p_0 \cdot \frac{1}{2}\sqrt{\pi} \cdot (s/r)(2\pi s/\lambda_1)^2 \exp\{-(2\pi s/\lambda_1)^2 \sin^2 \frac{1}{2}\theta\}$,
$p_0 = 10^{-6}$, $\lambda = 30$ cm, $r = 1000$ m.

In order to investigate the magnitude $|\mathsf{E}| = \sqrt{(\mathsf{E}_1^2 + \mathsf{E}_2^2 + \mathsf{E}_3^2)}$ of the scattered field at points far from the origin, it is convenient to introduce the polar coordinate transformation $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r\cos\theta$. The ratio $q = |\mathsf{E}/E_0|$ of the scattered-field strength to the incident-field strength may then be expressed as

$$q = p_0 \frac{\sqrt{\pi}}{2} \frac{s}{r} \left(\frac{2\pi s}{\lambda_1}\right)^2 \sqrt{(1 - \sin^2\theta \cos^2\varphi)} \exp\left\{-\left(\frac{2\pi s}{\lambda_1}\right)^2 \sin^2 \frac{\theta}{2}\right\}, \quad \text{................ (23)}$$

which exhibits the expected inverse-distance behaviour. The angular dependence of the scattered field in the yz-plane is shown in Figure 3, where equation (**23**) is plotted for two different anomaly sizes.

The scattering cross section σ, being the total scattered power radiated out of a surface enclosing the scattering anomaly divided by the incident power density, is defined as

$$\sigma = \int_0^{2\pi} \int_0^{\pi} q^2 r^2 \sin\theta \, d\theta \, d\varphi.$$

The evaluation of this expression with the aid of equation (**23**) shows that the ratio $g=\sigma/\pi s^2$ of the scattering cross section σ to the " geometrical cross section " πs^2 is

$$g=\frac{\pi p_0^2}{2k^2}\{[k^4-k^2+1]-[k^4+k^2+1]\exp(-2k^2)\}, \quad \ldots\ldots\ldots \quad (\mathbf{24})$$

where $k=2\pi s/\lambda_1$. The behaviour of this equation is shown in Figure 4, where the scaled cross-section ratio g/p_0^2 is plotted as a function of the normalized anomaly radius s/λ_1. From equation (**24**) it can be shown that

$$\lim_{s\to\infty} g=(32\pi^5 p_0^2/3)(s/\lambda_1)^4,$$

which is in good agreement with the well-known Rayleigh theory of small dielectric spheres.

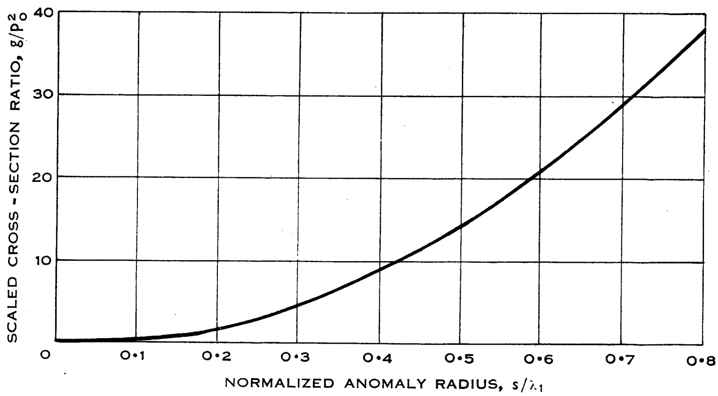

Fig. 4.—Cross-section ratio *v.* normalized anomaly radius.
$g=(\pi p_0^2/2k^2)\{(k^4-k^2+1)-(k^4+k^2+1)\exp(-2k^2)\}$,
where $k=2\pi s/\lambda_1$.

VI. Conclusions

The foregoing theory describes in detail the scattering of electromagnetic energy by a single Gaussian perturbation in the refractive index. A turbulent atmosphere may be considered as being composed of many such perturbations, distributed randomly in location, size, and intensity. A statistical treatment of a large number of independent scattering elements, based on the detailed theory discussed here, should yield a worth-while contribution to the understanding of trans-horizon propagation of microwaves.