COULOMB WAVE FUNCTIONS*

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Introduction

Coulomb wave functions are required for the solution of many physical problems involving charged particles. Recently, Froberg's (1955) review article has shown the need for a skeleton table of the values corresponding to the higher incident energies now so readily available. The region of interest is that formed by the parallelogram with corners in $(\rho, \eta)=(10, 0)$, $(20, 5)$, $(20, 15)$, and $(10, 10)$. In the present paper, such a table of values for f_L and f'_L (defined in equations (**3c**) and (**4a**)) is given for $\eta \leqslant 10$, $L=5$, 6, 10, and 11, $\Delta\rho=1$ and $\Delta\eta=1$. These integrals were calculated on the CSIRAC (Physics Department, University of Melbourne). Work is in progress to obtain the corresponding values for g_L and g'_L (defined in equations (**3d**) and (**4b**)).

Notation

The differential equation is

$$d^2y/d\rho^2+\{1-2\eta/\rho-L(L+1)/\rho^2\}y=0, \quad \ldots\ldots\ldots \quad (1)$$

which has a regular solution $F_L(\eta, \rho)$ and a logarithmic solution $G_L(\eta, \rho)$, which are normalized to act as sine-cosine functions as $\rho\to\infty$,

$$F_L(\eta, \rho)\sim\sin\theta_L, \quad \ldots\ldots\ldots\ldots\ldots \quad (2a)$$

$$G_L(\eta, \rho)\sim\cos\theta_L, \quad \ldots\ldots\ldots\ldots\ldots \quad (2b)$$

where

$$\theta_L=\{\rho-\eta\ln 2\rho-L\pi/2+\sigma_L\}, \quad \ldots\ldots\ldots\ldots \quad (2c)$$

$$\sigma_L=\arg\Gamma(i\eta+L+1). \quad \ldots\ldots\ldots\ldots\ldots \quad (2d)$$

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These functions are given by the integral representation (Bloch *et al.* 1950)

$$F_L=A_L\rho^{L+1}f_L, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3a)$$

$$G_L=A_L\rho^{L+1}g_L, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3b)$$

where

$$f_L=\int_0^1 (1-z^2)^L \cos(2\eta \operatorname{artanh} z-\rho z)\mathrm{d}z, \quad \ldots\ldots\ldots\ldots \quad (3c)$$

$$g_L=\int_0^1 (1-z^2)^L \sin(2\eta \operatorname{artanh} z-\rho z)\mathrm{d}z + e^{\pi\eta}\int_0^\infty (1+u^2)^L \exp(-u\rho-2\eta \operatorname{artan} 1/u)\mathrm{d}u, \quad \ldots\ldots \quad (3d)$$

and

$$A_L=2(L^2+\eta^2)^{-\frac{1}{2}}A_{L-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3e)$$

$$A_0=(1-e^{-2\pi\eta})^{\frac{1}{2}}(2\pi\eta)^{-\frac{1}{2}}=\{C_0(\eta)\}^{-1}. \quad \ldots\ldots\ldots\ldots\ldots \quad (3f)$$

The derivatives f'_L $(\partial F_L/\partial\rho)$ and g'_L are thus

$$f'_L=\int_0^1 z(1-z^2)^L \sin(2\eta \operatorname{artanh} z-\rho z)\mathrm{d}z, \quad \ldots\ldots\ldots\ldots \quad (4a)$$

$$g'_L=-\int_0^1 z(1-z^2)^L \cos(2\eta \operatorname{artanh} z-\rho z)\mathrm{d}z - e^{\pi\eta}\int_0^\infty u(1+u^2)^L \exp(-u\rho-2\eta \operatorname{artan} 1/u)\mathrm{d}u. \quad \ldots\ldots \quad (4b)$$

Use and Accuracy of Tables

The required values of F_L and F'_L are obtained by using (**3a**), (**3e**), and (**3f**) with the tabulated f_L, f'_L and noting that

$$F'_L=A_L\rho^{L+1}f'_L+(L+1)F_L/\rho. \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

$C_0(\eta)$ may be obtained from the National Bureau of Standards tables (1952). For convenience, the tabulated values of L are adjacent 5, 6 and 10, 11 so that intermediate and other L values may be obtained by the recurrence formulae

$$\rho^2 L f_{L+1}=2(2L+1)\{\eta\rho+L(L+1)\}f_L-4(L+1)(L^2+\eta^2)f_{L-1}, \quad \ldots \quad (6a)$$

and

$$\rho L f'_L=2(L^2+\eta^2)f_{L-1}-\{\eta\rho+L(2L+1)\}f_L. \quad \ldots\ldots\ldots\ldots \quad (6b)$$

These formulae permit a check on the accuracy of the tables, which was generally found to be at least 0·01 per cent. For the transition line $\rho=2\eta$, $2\eta \operatorname{artanh} Z-\rho Z\simeq 0$ and one cannot expect a sensible answer for the method of computation used. To the left of the transition line F_L oscillates rapidly and this causes round-off errors to be important. Any error in F_L means an admixture of G_L, and since G_L becomes large as ρ decreases, the corresponding error in F_L increases rapidly. For $\eta=10$, $\rho=10$, $L=5$ and 6, the possible error is $\sim$0·1 per cent. This represents the maximum error in the table except near a zero of f_L. Similar comments apply to f'_L.

TABLE 1

VALUES OF f_L AND f_L'

ρ	$L=5$		$L=6$		$L=10$		$L=11$	
	f_5	f_5'	f_6	f_6'	f_{10}	f_{10}'	f_{11}	f_{11}'
				$\eta=10$				
10	0·01157	0·006781	0·01492	0·008046	0·02973	0·011845	0·03334	0·012424
11	0·02024	0·010808	0·02497	0·012242	0·04347	0·015715	0·04756	0·016104
12	0·03365	0·016251	0·03981	0·017629	0·06127	0·019930	0·06555	0·019918
13	0·05320	0·023042	0·06056	0·024009	0·08332	0·024122	0·08734	0·023595
14	0·08005	0·030763	0·08799	0·030866	0·10934	0·027779	0·11253	0·026636
15	0·11475	0·038547	0·12216	0·037323	0·13850	0·030290	0·14022	0·028510
16	0·15673	0·045078	0·16210	0·042185	0·16933	0·031030	0·16899	0·028698
17	0·20394	0·048710	0·20554	0·044082	0·19978	0·029454	0·19691	0·026781
18	0·25260	0·047743	0·24884	0·041734	0·22734	0·025232	0·22176	0·022538
19	0·29743	0·040836	0·28729	0·034272	0·24935	0·018349	0·24122	0·016026
				$\eta=9$				
10	0·03254	0·016329	0·03895	0·017819	0·06121	0·020245	0·06561	0·020220
11	0·05231	0·023429	0·06003	0·024474	0·08363	0·024541	0·08774	0·023972
12	0·07974	0·031532	0·08807	0·031631	0·11010	0·028261	0·11332	0·027046
13	0·11540	0·039691	0·12314	0·038335	0·13974	0·030763	0·14142	0·028889
14	0·15866	0·046462	0·16416	0·043295	0·17100	0·031400	0·17051	0·028969
15	0·20726	0·050064	0·20866	0·045050	0·20173	0·029626	0·19862	0·026872
16	0·25711	0·048688	0·25274	0·042266	0·22934	0·025127	0·22346	0·022395
17	0·30251	0·040972	0·29138	0·034100	0·25108	0·017931	0·24264	0·015631
18	·····	·····	·····	·····	·····	·····	·····	·····
19	0·35354	0·006207	0·33102	0·002682	0·26752	—0·002434	0·25626	—0·002753
20	0·34804	—0·017509	0·32375	—0·017371	0·25946	—0·013644	0·24853	—0·012672
				$\eta=8$				
10	0·07940	0·032379	0·08817	0·032468	0·11092	0·028766	0·11417	0·027471
11	0·11613	0·040952	0·12422	0·039436	0·14106	0·031250	0·14267	0·029274
12	0·16079	0·047974	0·16641	0·044483	0·17276	0·031768	0·17210	0·029235
13	0·21091	0·051505	0·21203	0·046050	0·20375	0·029775	0·20038	0·026942
14	0·26199	0·049627	0·25688	0·042755	0·23137	0·024980	0·22516	0·022218
15	0·30788	0·040972	0·29563	0·033798	0·25280	0·017459	0·24405	0·015194
16	·····	·····	·····	·····	·····	·····	·····	·····
17	0·35646	0·003756	0·33284	0·000623	0·26776	—0·003380	0·25638	—0·003541
18	0·34804	—0·020791	0·32323	—0·019904	0·25871	—0·014619	0·24784	—0·013466
19	0·31517	—0·044420	0·29346	—0·039144	0·23891	—0·024714	0·22980	—0·022361
20	0·26095	—0·062834	0·24646	—0·053894	0·21007	—0·032485	0·20378	—0·029264
				$\eta=7$				
10	0·16318	0·049630	0·16888	0·045753	0·17460	0·032129	0·17375	0·029490
11	0·21492	0·053032	0·21567	0·047072	0·20584	0·029896	0·20219	0·026987
12	0·26726	0·050532	0·26127	0·043175	0·23343	0·024786	0·22688	0·022002
13	0·31355	0·040784	0·30002	0·033330	0·25450	0·016930	0·24542	0·014716
14	·····	·····	·····	·····	·····	·····	·····	·····
15	0·35907	0·000945	0·33438	—0·001662	0·26786	—0·004360	0·25641	—0·004349
16	0·34739	—0·024348	0·32224	—0·022577	0·25782	—0·015596	0·24705	—0·014257
17	0·31095	—0·047857	0·28986	—0·041574	0·23712	—0·025507	0·22828	—0·022997
18	0·25377	—0·065174	0·24082	—0·055436	0·20765	—0·032939	0·20175	—0·029629
19	0·18385	—0·072927	0·18152	—0·061821	0·17234	—0·037105	0·16995	—0·033514
20	0·11159	—0·069793	0·11999	—0·059890	0·13465	—0·037727	0·13575	—0·034411

TABLE 1 (*Continued*)

ρ	$L=5$		$L=6$		$L=10$		$L=11$	
	f_5	f'_5	f_6	f'_6	f_{10}	f'_{10}	f_{11}	f'_{11}
				$\eta=6$				
10	0·27298	0·051364	0·26594	0·043495	0·23550	0·024539	0·22861	0·021746
11	0·31949	0·040349	0·30453	0·032659	0·25615	0·016341	0·24675	0·014193
12	·····	·····	·····	·····	·····	·····	·····	·····
13	0·36122	—0·002259	0·33552	—0·004177	0·26783	—0·005367	0·25631	—0·005174
14	0·34598	—0·028153	0·32069	—0·025357	0·25679	—0·016567	0·24614	—0·015040
15	0·30585	—0·051265	0·28569	—0·043929	0·23521	—0·026256	0·22667	—0·023599
16	0·24589	—0·067162	0·23478	—0·056716	0·20516	—0·033323	0·19968	—0·029944
17	0·17499	—0·072814	0·17492	—0·061635	0·16969	—0·037055	0·16774	—0·033491
18	0·10396	—0·067513	0·11432	—0·058261	0·13225	—0·037266	0·13372	—0·034070
19	0·043023	—0·053141	0·060734	—0·047946	0·096220	—0·034323	0·10051	—0·031937
20	—0·000601	—0·033656	0·019830	—0·033446	0·064397	—0·029015	0·070548	—0·027705
				$\eta=5$				
11	0·36278	—0·005870	0·33621	—0·006914	0·26765	—0·006395	0·25610	—0·006011
12	0·34366	—0·032146	0·31852	—0·028199	0·25560	—0·017520	0·24513	—0·015808
13	0·29980	—0·054519	0·28093	—0·046134	0·23318	—0·026953	0·22498	—0·024161
14	0·23740	—0·068662	0·22842	—0·057663	0·20263	—0·033632	0·19758	—0·030204
15	0·16613	—0·072029	0·16837	—0·061040	0·16707	—0·036929	0·16554	—0·033412
16	0·09697	—0·064631	0·10908	—0·056292	0·12996	—0·036754	0·13176	—0·033686
17	0·039623	—0·049051	0·057932	—0·045186	0·094584	—0·033549	0·099049	—0·031344
18	0·000162	—0·029644	0·019850	—0·030699	0·063594	—0·028156	0·069735	—0·027026
19	—0·019993	—0·011204	—0·003454	—0·016166	0·038653	—0·021631	0·045380	—0·021568
20	—0·023861	0·002448	—0·013409	—0·004381	0·020357	—0·015026	0·026696	—0·015803
				$\eta=4$				
10	0·34029	—0·036223	0·31570	—0·031033	0·25427	—0·018447	0·24400	—0·016554
11	0·29283	—0·057463	0·27564	—0·048105	0·23106	—0·027590	0·22322	—0·024679
12	0·22847	—0·069532	0·22184	—0·058215	0·20007	—0·033861	0·19546	—0·030407
13	0·15752	—0·070520	0·16203	—0·060026	0·16450	—0·036725	0·16338	—0·033279
14	0·09090	—0·061225	0·10442	—0·054037	0·12778	—0·036187	0·12988	—0·033265
15	0·037448	—0·044864	0·055864	—0·042386	0·093094	—0·032761	0·097694	—0·030736
16	0·020031	—0·026016	0·020510	—0·028174	0·062930	—0·027323	0·069028	—0·026363
17	—0·015275	—0·009205	—0·000697	—0·014585	0·038772	—0·020920	0·045309	—0·020974
18	—0·018159	—0·002410	—0·009687	—0·004043	0·021077	—0·014552	0·027143	—0·015393
				$\eta=3$				
10	0·21935	—0·069653	0·21519	—0·058326	0·19750	—0·034010	0·19332	—0·030553
11	0·14948	—0·068304	0·15605	—0·058616	0·16199	—0·036452	0·16127	—0·033096
12	0·085936	—0·057454	0·10043	—0·051586	0·12572	—0·035579	0·12809	—0·032814
13	0·036512	—0·040813	0·054523	—0·039660	0·091749	—0·031971	0·096446	—0·030124
14	0·004712	—0·022966	0·021702	—0·025958	0·062393	—0·026524	0·068416	—0·025719
15	—0·010384	—0·011118	0·002215	—0·015514	0·038972	—0·020679	0·045305	—0·020693
16	—0·013000	0·001761	—0·006202	—0·004053	0·021816	—0·014140	0·027616	—0·015006

TABLE 1 (*Continued*)

ρ	$L=5$		$L=6$		$L=10$		$L=11$	
	f_5	f'_5	f_6	f'_6	f_{10}	f'_{10}	f_{11}	f'_{11}
				$\eta=2$				
10	0·082219	—0·053548	0·097149	—0·049049	0·12379	—0·034943	0·12639	—0·032339
11	0·036698	—0·037133	0·053852	—0·037117	0·090545	—0·031188	0·095298	—0·029515
12	0·007982	—0·020616	0·023274	—0·024107	0·061972	—0·025766	0·067890	—0·025102
13	—0·005664	—0·007437	0·005119	—0·012637	0·039236	—0·019672	0·045357	—0·019902
14	—0·008589	0·000703	—0·003060	—0·004320	0·022560	—0·013785	0·028103	—0·014659
				$\eta=1$				
10	0·011458	—0·018979	0·025065	—0·022631	0·061649	—0·025055	0·067448	—0·024514
11	—0·001389	—0·007453	0·007878	—0·012181	0·039549	—0·019133	0·045455	—0·019424
12	—0·004991	—0·000522	—0·000309	—0·004743	0·023295	—0·013481	0·028597	—0·014350

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