THE GENERAL RELATIONSHIPS BETWEEN THE ELASTIC CONSTANTS OF ISOTROPIC MATERIALS IN n DIMENSIONS

By N. W. TSCHOEGL*

[*Manuscript received December 30, 1957*]

Summary

The relationships between the elastic constants of homogeneous isotropic materials in n dimension are derived and are shown to depend on n. The maximal value of the generalized Poisson's ratio is $1/(n-1)$. The n-dimensional formulae reproduce the well-known three-dimensional relations for $n=3$, while $n=2$ produces the relations appropriate for monomolecular films. The correct degeneration is shown for $n=1$.

I. INTRODUCTION

In the physical chemistry of monomolecular films it is customary to regard the film (*or monolayer*) as a two-dimensional system since there is no bonding between molecules of the same kind in the third dimension. One accordingly defines a *surface bulk* (*compressional*) *modulus* and a *surface shear modulus* having the dimensions of surface traction (force per unit length) instead of traction (force per unit area) (Langmuir and Schaefer 1937).† This procedure prompts the development of a general n-dimensional elastic theory comprising both two- and three-dimensional theory as special cases.

In this paper it is proposed to derive the general relationships connecting the four elastic material constants of homogeneous isotropic bodies in n dimensions. It will be shown that these relations are generally dependent on the number of dimensions considered.

II. THE FUNDAMENTAL ELASTIC MODULI IN n DIMENSIONS

An infinitesimal elastic deformation of a homogeneous isotropic body can be resolved into a change in size (dilatation or contraction) and an independent change in shape (distortion). Homogeneous isotropic bodies therefore possess two fundamental elastic moduli, the *bulk modulus* relating to changes in size and the *shear modulus* relating to changes in shape.

It follows from the independence of the two fundamental moduli that the general n-dimensional stress tensor s_{ij} is not proportional to the strain tensor e_{ij}. These tensors may, however, be resolved into separately proportional component tensors such that

$$s''_{ij}=s_{ij}-s'_{ij}, \qquad \text{(1a)}$$

$$e''_{ij}=e_{ij}-e'_{ij}, \qquad \text{(1b)}$$

* Bread Research Institute of Australia, North Sydney, N.S.W.

† LANGMUIR, I., and SCHAEFER, N. J. (1937).—*J. Amer. Chem. Soc.* **59**: 2400.

where s'_{ij} and e'_{ij} may be called *mean normal tensors* while s''_{ij} and e''_{ij} are termed *deviators*.

We can now state

$$s'_{ij}=nK_ne'_{ij}, \qquad (2a)$$
$$s''_{ij}=2G_ne''_{ij}, \qquad (2b)$$

where K_n and G_n are the n-dimensional bulk modulus and shear modulus respectively.

The dimensional factor in (**2a**) appears because K_n is traditionally defined as the proportionality constant between the mean normal stress s_{ii}/n and the (hyper)volumetric strain e_{ii}

$$s_{ii}/n=K_ne_{ii}. \qquad (2a')$$

That (**2a**) and (**2a'**) are equivalent may be seen from the following development.

Since the deviatoric tensors relate to changes in shape only, the sums of their principal components vanish. Using the summation convention

$$s''_{ii}\equiv 0, \qquad (3a)$$
$$e''_{ii}\equiv 0, \qquad (3b)$$

and, from (**1a**) and (**1b**)

$$s'_{ii}=s_{ii}, \qquad (4a)$$
$$e'_{ii}=e_{ii}. \qquad (4b)$$

The mean normal tensors relate to changes in size only. Clearly their non-principal components must vanish while their principal components must be equal. It then follows from (**4a**) and (**4b**) that

$$s'_{ij}=s_{ii}/n\,.\,d_{ij}, \qquad (5a)$$
$$e'_{ij}=e_{ii}/n\,.\,d_{ij}, \qquad (5b)$$

where d_{ij} is the unit tensor. Substituting (**5a**) and (**5b**) into (**2a**) then leads to (**2a'**).

The numerical factor 2 in (**2b**) arises because of the traditional definition of shear which is twice the corresponding shear component of the strain tensor.

III. The Relationships between the Elastic Constants

To derive the general relationships between K_n, G_n, the n-dimensional *extensional (Young's) modulus* Y_n, and *Poisson's ratio* μ_n we consider a pure tensile stress in the nth dimension. For such a stress all components of the stress tensor vanish except s_{nn}. All non-principal components of the strain tensor also vanish since a pure tensile stress produces no distortion. The extension e_{nn} in the nth dimension is accompanied by $(n-1)$ contractions $-\mu_ne_{nn}$ in the remaining dimensions. Consequently

$$s_{ii}=s_{nn}, \qquad (6a)$$
$$e_{ii}=e_{nn}-(n-1)\mu_ne_{nn}. \qquad (6b)$$

Substituting into (**5a**) and (**5b**) and introducing Y_n from Hooke's law, the mean normal tensors become

$$s'_{ij}=s_{nn}/n\,.\,d_{ij}, \quad \text{(7a)}$$
$$e'_{ij}=[1-(n-1)\mu_n]s_{nn}/nY_n\,.\,d_{ij}. \quad \text{(7b)}$$

The non-principal components of the deviators also vanish since there is no distortion. The principal components can be expressed by the following set of equations

$$s''_{11}=s''_{22}=\ldots=s''_{(n-1)(n-1)}=0-s_{ii}/n, \quad \text{(8a)}$$
$$s''_{nn}=s_{nn}-s_{ii}/n, \quad \text{(8a')}$$
$$e''_{11}=e''_{22}=\ldots=e''_{(n-1)(n-1)}=-\mu_n e_{nn}-e_{ii}/n, \quad \text{(8b)}$$
$$e''_{nn}=e_{nn}-e_{ii}/n, \quad \text{(8b')}$$

obtained by subtracting the principal components of the mean normal tensors from those of the total tensors according to (**1a**) and (**1b**).

Substituting for s_{ii} from (**6a**) and for e_{ii} from (**6b**) and using Hooke's law

$$s''_{11}=s''_{22}=\ldots=s''_{(n-1)(n-1)}=-s_{nn}/n, \quad \text{(9a)}$$
$$s''_{nn}=(n-1)s_{nn}/n, \quad \text{(9a')}$$
$$e''_{11}=e''_{22}=\ldots=e''_{(n-1)(n-1)}=-(1+\mu_n)s_{nn}/nY_n, \quad \text{(9b)}$$
$$e''_{nn}=(n-1)(1+\mu_n)s_{nn}/nY_n. \quad \text{(9b')}$$

Defining a tensor g_{ij} such that

$$g_{11}=g_{22}=\ldots=g_{(n-1)(n-1)}=-1, \quad \text{(10)}$$
$$g_{nn}=n-1, \quad \text{(10')}$$

while all non-principal components vanish, the deviatoric tensors may be written

$$s''_{ij}=s_{nn}/n\,.\,g_{ij}, \quad \text{(11a)}$$
$$e''_{ij}=(1+\mu_n)s_{nn}/nY_n\,.\,g_{ij}. \quad \text{(11b)}$$

Now from (**2a**), (**7a**), and (**7b**)

$$K_n=\frac{Y_n}{n[1-(n-1)\mu_n]}, \quad \text{(12)}$$

and from (**2b**), (**11a**), and (**11b**)

$$G_n=\frac{Y_n}{2(1+\mu_n)}. \quad \text{(13)}$$

From (**12**) and (**13**)

$$Y_n=\frac{2n^2K_nG_n}{n(n-1)K_n+2G_n}, \quad \text{(14)}$$
$$\mu_n=\frac{nK_n-2G_n}{n(n-1)K_n+2G_n}. \quad \text{(15)}$$

IV. The Maximal Value of Poisson's Ratio

Writing (**15**) in the form

$$nK_n[1-(n-1)\mu_n]=2G_n(1+\mu_n), \qquad (15')$$

it is apparent that μ_n cannot be less than -1 and cannot exceed $1/(n-1)$. The maximal value of Poisson's ratio, $\mu_{n\,(\mathrm{max.})}$, is thus seen to depend on n.

The identical result can also be obtained directly by the following consideration: let an n-dimensional hypercube of unit length be extended by a small amount λ in one dimension. The change ΔH in the hypervolume then is

$$\Delta H=(1+\lambda)(1-\mu_n\lambda)^{(n-1)}-1. \qquad (16)$$

Expanding $(1-\mu_n\lambda)^{(n-1)}$ by the binomial theorem and neglecting all higher powers of λ

$$\Delta H=(1+\lambda)[1-(n-1)\mu_n\lambda]-1. \qquad (17)$$

If the hypercube is incompressible ($K_n=\infty$), $\Delta H=0$, and again neglecting λ^2,

$$\mu_{n\,(max.)}=1/(n-1). \qquad (18)$$

Either from (**13**) by substituting $\mu_n=1/(n-1)$, or from (**14**) by substituting $K_n=\infty$, we obtain the general relationships between the extensional and shear moduli of incompressible bodies (for $n>1$) as

$$Y_n=\frac{2n}{(n-1)}G_n. \qquad (19)$$

V. Special Cases

For $n=3$, (**12**)–(**15**) and (**18**) and (**19**) reproduce the well-known three-dimensional relationships. The two-dimensional relations can be obtained simply by substituting $n=2$ and are seen to be different from the relations in three dimensions. For $n=1$, (**12**) and (**14**) correctly show that the bulk modulus of a " one-dimensional body " degenerates into the extensional modulus

$$K_1\equiv Y_1. \qquad (20)$$

This modulus is the only elastic constant a " one-dimensional body " can have. Neither G_n nor μ_n have any meaning for $n=1$. In fact, expressing G_1 and μ_1 in terms of K_1 and Y_1 from (**14**) and (**12**) and using identity (**20**) we obtain, for $n=1$,

$$G_1=\frac{n(n-1)K_1}{2(n-1)(n+1)}=0/0, \qquad (21)$$

$$\mu_1=\frac{n-1}{n-1}=0/0. \qquad (22)$$