

ON THE GALVANOMAGNETIC, THERMOMAGNETIC, AND THERMOELECTRIC EFFECTS IN ISOTROPIC METALS AND SEMICONDUCTORS

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Summary

The calculation of the galvanomagnetic, thermomagnetic, and thermoelectric effects in those isotropic metals and semiconductors which can be represented by the two-band model is simplified and extended to cases not previously treated. General expressions are obtained for effects involving time-independent electric fields, magnetic fields, and thermal gradients. Analytic formulae are given for the isothermal galvanomagnetic effects when a high frequency electric field is applied together with a time-independent magnetic field.

Throughout, emphasis is placed on coefficients which are experimentally or theoretically important. The paper concludes with an appendix on the estimation of parameters occurring in the two-band model of metals and semiconductors.

I. INTRODUCTION

The exact calculation of the galvanomagnetic, thermomagnetic, and thermoelectric effects in isotropic metals or semiconductors is extremely difficult and has not yet been attempted. Approximate treatments have been developed by many authors (e.g. Wilson 1953, Ch. 8; Donovan 1954; Madelung 1954), but even these are complicated. Our object in the present paper is to show how these approximate calculations may be simplified and extended.

The two main stages in all these approximate treatments are:

(1) the derivation from the Boltzmann transport equation of expressions for the electric and thermal current densities in terms of the applied electric fields, magnetic fields, and thermal gradients;

(2) the calculation from these equations of formulae for the above effects.

Due to the complex structure of real metals and semiconductors it is necessary to base stage (1) of the calculations on some simplified theoretical model. In past work, the free-electron model and the two-band model have been the two most frequently employed. Since the two-band model (Wilson 1953, pp. 43, 198-9) is the most general one for which the calculations can be carried to completion, we adopt it in the present discussion.

For the most commonly studied effects which are either one-dimensional (e.g. conductivities) or two-dimensional (e.g. Hall effect), two components of each current density must be specified. In the past, one equation has been given for each component (cf. Wilson 1953, p. 219; Donovan 1954; Madelung 1954). Recently, however, Dingle (1956) has shown that, if two imaginaries are used

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instead of one, it is possible to describe the two electric current components with only one equation. This work, which suggested the present investigation, indicated that a simplification could probably be effected in the calculation of the isothermal galvanomagnetic effects.

Dingle based his treatment on the one-band (free-electron) model and assumed that harmonically varying electric and time-independent magnetic fields were applied. In order to simplify the calculation of the galvanomagnetic, thermomagnetic, and thermoelectric effects for the isotropic two-band model, both the electric and thermal current densities are required for the case when time-independent thermal gradients also exist. To satisfy these requirements it is necessary to extend Dingle's work. This is done in Section II, where two equations (2.9, 2.10) are obtained for the current densities to replace the four customarily introduced in earlier work.

Previously, stage (2) of the calculations has been attempted separately for :

(a) steady electric field effects in (i) metals (e.g. Wilson 1953, Ch. 8) and (ii) semiconductors (e.g. Madelung, loc. cit.);

(b) high frequency electric field effects in (i) metals (Donovan, loc. cit.; Donovan and Sondheimer 1953), and (ii) semiconductors (left untreated).

In the present discussion, we still consider (a) and (b) separately (Sections III, IV) as the effects in each case are of basically different type (cf. Section IV (b)). As most of the important effects in (a) have been calculated previously, our main object here is to simplify these calculations. However, in (b) past work has been confined to metals, and, moreover, the results obtained for the surface resistance and magnetoresistance have only been extracted numerically (Donovan, loc. cit.). Here, our main aims are to extend the calculations to include semiconductors, and to obtain analytic expressions for the principal galvanomagnetic effects.

For both (a) and (b), most previous authors appear to have been primarily interested in either metals or semiconductors, not both, and consequently their results apply to either the one or the other. In the present paper, we derive (Sections III (c), IV (b)) general formulae which may easily be specialized to either case.

II. GENERAL EQUATIONS FOR THE ELECTRIC AND THERMAL CURRENT DENSITIES

As indicated in the introduction we base this first stage of the calculations on the metal or semiconductor model in which there are two conduction bands of standard form (Wilson 1953, pp. 43, 198-9). The band of greater energy is assumed to be normal (i.e. the charge carriers are negative) and that of lesser energy inverted (positive charge carriers). We make the usual assumption that these bands are independent, and therefore we may consider each one separately.

(a) Normal Band

The following derivation of the formulae for the current densities broadly follows that given by Wilson (1953, p. 210), but with modifications suggested by Dingle's paper (loc. cit.). We assume that a "universal" time of relaxation

can be defined, and therefore in the presence of harmonically varying electric fields, and time-independent magnetic fields and thermal gradients, the Boltzmann transport equation may be written (in Gaussian units) :

$$(-2\pi\varepsilon/h)(\mathbf{E} + c^{-1}\mathbf{v} \wedge \mathbf{H}) \cdot \text{grad}_{\mathbf{k}} f_n + \mathbf{v} \cdot \text{grad}_{\mathbf{r}} f_n = -(f_n - f_{0n})[(1 + i\omega\tau_n)/\tau_n], \dots\dots\dots (2.1)$$

where $i^2 = -1$, ε is the numerical electronic charge, \mathbf{v} , \mathbf{r} , and \mathbf{k} are respectively the velocity, position vector, and wave vector of the electrons, τ_n is the relaxation time, f_n and f_{0n} are the equilibrium distribution functions of the electrons in the presence and absence respectively of applied fields, \mathbf{E} is the electric field which is proportional to $e^{i\omega t}$, and \mathbf{H} is the steady magnetic field. We also make the usual assumption that f_{0n} is the Fermi-Dirac function $\{\exp [(E_n - \zeta_n)/kT] + 1\}^{-1}$.

To terms linear in the electric fields and thermal gradients, this equation has the solution

$$f_n - f_{0n} = -\mathbf{k} \cdot \mathbf{c}_n(E_n)(\partial f_{0n}/\partial E_n),$$

where $E_n = \frac{1}{2}\hbar^2 |\mathbf{k}|^2/m_n$, m_n being the effective mass of the electrons, and $\mathbf{c}_n = (c_{1n}, c_{2n}, c_{3n})$.

The analysis then proceeds as in Wilson (loc. cit.), and for the special case of fields given by :

$$\begin{aligned} \text{Electric field} \quad \mathbf{E} &= (E_x, E_y, 0), \\ \text{Magnetic field} \quad \mathbf{H} &= (0, 0, H), \\ \text{Thermal gradient} \quad \mathbf{K} &= (K_x, K_y, 0) = (\partial T/\partial x, \partial T/\partial y, 0), \end{aligned}$$

we obtain (dropping the subscript n)*

$$\left. \begin{aligned} \frac{2\pi m}{\hbar\tau}(1 + i\omega\tau)c_1 - \frac{2\pi m\Omega}{\hbar}c_2 &= -\varepsilon E_x - \left[T \frac{\partial}{\partial T} \left(\frac{\zeta}{T} \right) + \frac{E}{T} \right] K_x, \\ \frac{2\pi m\Omega}{\hbar}c_1 + \frac{2\pi m}{\hbar\tau}(1 + i\omega\tau)c_2 &= -\varepsilon E_y - \left[T \frac{\partial}{\partial T} \left(\frac{\zeta}{T} \right) + \frac{E}{T} \right] K_y, \end{aligned} \right\} \dots (2.2)$$

where $\Omega = (-\varepsilon)H/me$ is the circular frequency of precessional motion of an electron.

Now, if we define $\mathbf{C} = c_1 + j c_2$, $\mathbf{E} = E_x + j E_y$, and $\mathbf{K} = K_x + j K_y$, with $j^2 = -1$ (but $ij \neq -1$), equations (2.2) reduce to

$$\mathbf{C} = -\frac{\hbar}{2\pi m} \left[\frac{\tau(\varepsilon\mathbf{E} + \beta\mathbf{K})}{1 + (i\omega + j\Omega)\tau} \right], \dots\dots\dots (2.3)$$

where

$$\beta = T \frac{\partial}{\partial T} \left(\frac{\zeta}{T} \right) + \frac{E}{T}.$$

* It is implicitly assumed that the Fermi energy ζ is a function of temperature only, i.e. that the substance is homogeneous.

(i) *Electric Current Density.*—The electric current density $\mathbf{J}=(J_x, J_y, 0)$ has components :

$$J_x = \frac{32\sqrt{2}\pi^2\epsilon m^{3/2}}{3h^4} \int_0^\infty E^{3/2} c_1 \frac{\partial f_0}{\partial E} dE, \quad J_y = \frac{32\sqrt{2}\pi^2\epsilon m^{3/2}}{3h^4} \int_0^\infty E^{3/2} c_2 \frac{\partial f_0}{\partial E} dE.$$

We introduce $\mathbf{J}=J_x+jJ_y$, and obtain

$$\mathbf{J} = \frac{32\sqrt{2}\pi^2\epsilon m^{3/2}}{3h^4} \int_0^\infty E^{3/2} \mathbf{c} \frac{\partial f_0}{\partial E} dE,$$

that is,

$$J_n = -Mm_n^{\frac{1}{2}} \int_0^\infty \frac{\tau_n(\epsilon\mathbf{E} + \beta_n\mathbf{K})}{1 + (i\omega + j\Omega_n)\tau_n} E_n^{3/2} \frac{\partial f_{0n}}{\partial E_n} dE_n, \quad \dots\dots (2.4)$$

where $M=16\sqrt{2}\pi\epsilon/3h^3$.

(ii) *Thermal Current Density.*—The thermal current density $\mathbf{W}=(W_x, W_y, 0)$ has components :

$$\left. \begin{aligned} W_x &= -\frac{32\sqrt{2}\pi^2 m^{3/2}}{3h^4} \int_0^\infty E' E^{3/2} c_1 \frac{\partial f_0}{\partial E} dE, \\ W_y &= -\frac{32\sqrt{2}\pi^2 m^{3/2}}{3h^4} \int_0^\infty E' E^{3/2} c_2 \frac{\partial f_0}{\partial E} dE, \end{aligned} \right\} \dots\dots (2.5)$$

where $E'=A_n+E$ and A_n is the energy of the base of the normal band when the energy at the metal or semiconductor surface is defined to be zero. In the past, the energy zero has generally been taken either at the bottom of the normal band (e.g. Wilson 1953, p. 219) or as the mean of the energies of the bottom of the normal and the top of the inverted bands (e.g. Madelung 1954). Here, we have defined the energy zero as above, since all experimental measurements are made at the metal or semiconductor surface. However, the choice of energy zero only affects the results when both the electric current density and the thermal gradient are simultaneously non-zero.

From (2.3) and (2.5) we find that

$$W_n = W_x + jW_y = -\frac{A_n J_n}{\epsilon} + \frac{Mm_n^{\frac{1}{2}}}{\epsilon} \int_0^\infty \frac{\tau_n(\epsilon\mathbf{E} + \beta_n\mathbf{K})}{1 + (i\omega + j\Omega_n)\tau_n} E_n^{5/2} \frac{\partial f_{0n}}{\partial E_n} dE_n. \quad \dots (2.6)$$

(b) *Inverted Band*

The current densities for the inverted band are derived by a method similar to that in Section II (a) (see Wilson 1953, p. 211). We find

$$J_p = -Mm_p^{\frac{1}{2}} \int_0^\infty \frac{\tau_p(\epsilon\mathbf{E} - \beta_p\mathbf{K})}{1 + (i\omega + j\Omega_p)\tau_p} E_p^{3/2} \frac{\partial f_{0p}}{\partial E_p} dE_p, \quad \dots\dots\dots (2.7)$$

$$W_p = -\frac{A_p J_p}{\epsilon} - \frac{Mm_p^{\frac{1}{2}}}{\epsilon} \int_0^\infty \frac{\tau_p(\epsilon\mathbf{E} - \beta_p\mathbf{K})}{1 + (i\omega + j\Omega_p)\tau_p} E_p^{5/2} \frac{\partial f_{0p}}{\partial E_p} dE_p, \quad \dots\dots (2.8)$$

where $\Omega_p = \epsilon H/m_p c$ and A_p is the energy of the top of the inverted band.

(c) *Total Current Densities*

As we have assumed that the two bands are independent, the total current densities are the sums of the current densities for each band. We therefore obtain from (2.4) and (2.7)

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_p = -M \int_0^\infty \left[\frac{m_n^{\frac{1}{2}} \tau_n (\epsilon \mathbf{E} + \beta_n \mathbf{K})}{1 + (i\omega + j\Omega_n) \tau_n} E_n^{3/2} \frac{\partial f_{0n}}{\partial E_n} dE_n + \frac{m_p^{\frac{1}{2}} \tau_p (\epsilon \mathbf{E} - \beta_p \mathbf{K})}{1 + (i\omega + j\Omega_p) \tau_p} E_p^{3/2} \frac{\partial f_{0p}}{\partial E_p} dE_p \right], \dots\dots\dots (2.9)$$

and from (2.6) and (2.8)

$$\mathbf{W} = -\kappa_L \mathbf{K} + \mathbf{W}_n + \mathbf{W}_p = -\kappa_L \mathbf{K} - \epsilon^{-1} (A_n \mathbf{J}_n + A_p \mathbf{J}_p) + \frac{M}{\epsilon} \int_0^\infty \left[\frac{m_n^{\frac{1}{2}} \tau_n (\epsilon \mathbf{E} + \beta_n \mathbf{K})}{1 + (i\omega + j\Omega_n) \tau_n} E_n^{5/2} \frac{\partial f_{0n}}{\partial E_n} dE_n - \frac{m_p^{\frac{1}{2}} \tau_p (\epsilon \mathbf{E} - \beta_p \mathbf{K})}{1 + (i\omega + j\Omega_p) \tau_p} E_p^{5/2} \frac{\partial f_{0p}}{\partial E_p} dE_p \right], \dots\dots\dots (2.10)$$

where κ_L is the thermal conductivity of the lattice.

We are now in a position to commence the second stage of the calculations, which, as indicated in the introduction, must be carried out separately for the two special cases of steady, and high frequency, electric fields. As the time-independent case is the more important it is considered first.

III. STEADY ELECTRIC FIELD EFFECTS

(a) *Introduction*

All the steady electric field effects are defined in terms of the components of the fields and the current densities. Since the x and y components of the current densities are the real and imaginary parts respectively of (2.9) and (2.10), it is now advisable to rationalize these equations (with $\omega = 0$). This is a simple process which yields

$$\mathbf{J} = [\Phi_{EL}(\Omega) + jH\Phi_{ET}(\Omega)] \mathbf{E} + [\Phi_{KL}(\Omega) + jH\Phi_{KT}(\Omega)] \mathbf{K}, \dots\dots\dots (3.1)$$

$$\mathbf{W} = [\Psi_{EL}(\Omega) + jH\Psi_{ET}(\Omega)] \mathbf{E} + [\Psi_{KL}(\Omega) + jH\Psi_{KT}(\Omega)] \mathbf{K}, \dots\dots\dots (3.2)$$

where

$$\Phi_{EL}(\Omega) = \epsilon [I_{11n}(\Omega_n) + I_{11p}(\Omega_p)], \dots\dots\dots (3.3)$$

$$\Phi_{ET}(\Omega) = \epsilon^2 c^{-1} [m_n^{-1} I_{21n}(\Omega_n) - m_p^{-1} I_{21p}(\Omega_p)], \dots\dots\dots (3.4)$$

$$\Phi_{KL}(\Omega) = T \left[I_{11n}(\Omega_n) \frac{\partial}{\partial T} \left(\frac{\zeta_n}{T} \right) - I_{11p}(\Omega_p) \frac{\partial}{\partial T} \left(\frac{\zeta_p}{T} \right) \right] + \frac{1}{T} [I_{12n}(\Omega_n) - I_{12p}(\Omega_p)], \dots\dots\dots (3.5)$$

$$\Phi_{KT}(\Omega) = \epsilon c^{-1} \left\{ T \left[m_n^{-1} I_{21n}(\Omega_n) \frac{\partial}{\partial T} \left(\frac{\zeta_n}{T} \right) + m_p^{-1} I_{21p}(\Omega_p) \frac{\partial}{\partial T} \left(\frac{\zeta_p}{T} \right) \right] + T^{-1} [m_n^{-1} I_{22n}(\Omega_n) + m_p^{-1} I_{22p}(\Omega_p)] \right\}, \dots\dots\dots (3.6)$$

$$\Psi_{EL}(\Omega) = -\epsilon^{-1} [A_n \Phi_{ELn}(\Omega_n) + A_p \Phi_{ELp}(\Omega_p)] - [I_{12n}(\Omega_n) - I_{12p}(\Omega_p)], \dots\dots (3.7)$$

$$\Psi_{ET}(\Omega) = -\epsilon^{-1} [A_n \Phi_{ETn}(\Omega_n) + A_p \Phi_{ETp}(\Omega_p)] - \{ \epsilon c^{-1} [m_n^{-1} I_{22n}(\Omega_n) + m_p^{-1} I_{22p}(\Omega_p)] \}, \dots\dots\dots (3.8)$$

$$\Psi_{KL}(\Omega) = -\kappa_L - \varepsilon^{-1} [A_n \Phi_{KLn}(\Omega_n) + A_p \Phi_{KLp}(\Omega_p)] - \varepsilon^{-1} \left\{ T \left[I_{12n}(\Omega_n) \frac{\partial}{\partial T} \left(\frac{\zeta_n}{T} \right) + I_{12p}(\Omega_p) \frac{\partial}{\partial T} \left(\frac{\zeta_p}{T} \right) \right] + \frac{1}{T} [I_{13n}(\Omega_n) + I_{13p}(\Omega_p)] \right\}, \dots \dots \dots (3.9)$$

$$\Psi_{KT}(\Omega) = -\varepsilon^{-1} [A_n \Phi_{KTn}(\Omega_n) + A_p \Phi_{KTp}(\Omega_p)] - c^{-1} \left\{ T \left[m_n^{-1} I_{22n}(\Omega_n) \frac{\partial}{\partial T} \left(\frac{\zeta_n}{T} \right) - m_p^{-1} I_{22p}(\Omega_p) \frac{\partial}{\partial T} \left(\frac{\zeta_p}{T} \right) \right] + T^{-1} [m_n^{-1} I_{23n}(\Omega_n) - m_p^{-1} I_{23p}(\Omega_p)] \right\}, \dots (3.10)$$

and, for instance,

$$I_{stn}(x_n) = M m_n^{\frac{1}{2}} L_{stn}(x_n),$$

$$L_{stn}(x_n) = - \int_0^\infty \frac{\tau_n^s E_n^{t+\frac{1}{2}}}{1 + x_n^2 \tau_n^2} \frac{\partial f_{0n}}{\partial E_n} dE_n.$$

In order to express the Φ 's and Ψ 's entirely in terms of elementary or tabulated functions of the parameters, it is only necessary to evaluate the integral $L_{st}(x)$. This can be readily achieved in the following special cases (see Appendix I) :

- (1) The electrons (or holes) form a strongly degenerate system—at normal temperatures this applies to all metals and semi-metals, and to some semi-conductors.
- (2) The electrons (or holes) form a weakly degenerate system—most semi-conductors are in this group. For the more important scattering mechanisms (thermal, neutral-impurity, and ionic-impurity) the above integral can be expressed in terms of functions tabulated by Johnson and Whitesell (1953), Dingle, Arndt, and Roy (1957*a*, 1957*b*, 1957*c*), and Beer, Armstrong, and Greenberg (1957).

We are now in a position to derive formulae for the coefficients which describe the various steady electric field effects. In general, previous authors (e.g. Wilson 1953, Ch. 8 ; Bass and Tsidil'kovskii 1956 ; Beer, Armstrong, and Greenberg 1957) have expressed these coefficients directly in terms of the metal or semiconductor parameters, i.e. as functions of the expressions equivalent to equations (3.3)–(3.10). In the present paper we give all our formulae in terms of the Φ 's and Ψ 's (cf. Madelung 1954). The advantages of introducing these eight functions are :

- (a) The calculations are simplified and condensed.
- (b) The final expressions obtained for the coefficients are simpler, and hence any relationships between the various effects are more easily deduced.
- (c) Once the values of these functions are determined, either theoretically or experimentally, any essentially one- or two-dimensional coefficient may be readily evaluated. Possible methods of measuring these functions experimentally are indicated in the subsequent work (Section III (c)).
- (d) From Φ_{EL} , Φ_{ET} , Ψ_{EL} , and Ψ_{ET} , the values of the principal two-band model metal and semiconductor parameters may be estimated (see Appendix I).

For metals, the parameters A_n , A_p , τ_n , τ_p , m_n , m_p , ζ_n , ζ_p , and hence the number of free electrons (n) and holes (p) per unit volume may be determined. As far as the author can ascertain, no method has previously been given for the evaluation of all the above metal parameters from measurements of the galvanomagnetic, thermomagnetic, and thermoelectric effects alone. Moreover, earlier authors have made simplifying assumptions (e.g. $n=p$, $\tau_n=\tau_p$; cf. Wilson (1953, p. 217)), thus reducing the value of the two-band model.

It is interesting to note that, as all the essentially one- or two-dimensional effects can be expressed in terms of these eight functions, only a maximum of eight of these effects may be considered to be independent. It can easily be shown that this applies to anisotropic as well as to isotropic materials.

(b) Notation

As the experimental conditions under which each effect is measured may vary considerably, it is an advantage to adopt some standard notation to indicate the combination under consideration. In the present paper we indicate the conditions by the superscripts in Table 1.

TABLE 1
NOTATION EMPLOYED FOR INDICATING EXPERIMENTAL
CONDITIONS

| Superscript | Condition |
|-------------|-----------|
| E | $E_x=0$ |
| e | $E_y=0$ |
| J | $J_x=0$ |
| j | $J_y=0$ |
| K | $K_x=0$ |
| k | $K_y=0$ |
| W | $W_x=0$ |
| w | $W_y=0$ |
| 0 | $H=0$ |

The x direction is taken as longitudinal and the y direction as transverse.

For instance, the isothermal electrical conductivity for non-zero magnetic fields and zero transverse electric current is denoted by σ^{jKk} .

(c) General Formulae* for the Steady Electric Field Effects

Since there are many coefficients describing the various effects, it is impossible to include them all in a paper of reasonable length. We therefore consider only those which satisfy one or other of the following criteria:

(a) the coefficient is experimentally important, or

(b) it is an extremely simple function of the Φ 's and Ψ 's, and is experimentally measurable.

* Unless stated otherwise, the formulae in this subsection are derived from equations (3.1) and/or (3.2).

The reason for employing (a) is obvious. Condition (b) is adopted since the coefficients satisfying it provide the simplest method of determining the Φ 's and Ψ 's, and hence of evaluating the parameters of a metal or semiconductor [Section III (a) (d)].

(i) *Conductivities*.—(1) *Electrical conductivity*.—The electrical conductivity is defined by $\sigma = J_x/E_x$. Only the isothermal conductivities are treated here since they are the more important experimentally and also the simpler theoretically.

The principal cases are :

Zero Magnetic Field

$$\sigma^{K0} = \Phi_{EL}(0), \dots\dots\dots (3.11)$$

Non-zero Magnetic Field

(a) $E_y = 0$

$$\sigma^{eKk} = \Phi_{EL}(\Omega). \dots\dots\dots (3.12)$$

The measurement of this quantity immediately yields experimental values for the function Φ_{EL} , the exact determination of which is essential for reliable parameter estimation (see Appendix I). This conductivity is also required in the derivation of the Corbino magnetoresistance (Section III (c) (ii) (1)).

(b) $J_y = 0$

Although it is not particularly important, this conductivity is included here since it is required for the calculation of the normal magnetoresistance.

We find that

$$\sigma^{jKk} = \Phi_{EL}(\Omega) + [H^2 \Phi_{ET}^2(\Omega) / \Phi_{EL}(\Omega)]. \dots\dots\dots (3.13)$$

(2) *Thermal conductivity*. — The thermal conductivity is defined by $\kappa = -W_x/K_x$. The usual experimental conditions are $J=0$ and $W_y=0$, but in this case the formulae obtained for κ contain products of the Φ 's and Ψ 's. For $E=0$ and $K_y=0$, the expressions derived for κ are much simpler, and therefore they would be expected to yield more precise values for the Φ 's and Ψ 's. However, these latter conditions are more difficult to apply in practice, and little, if any, work has been carried out for them.

We can readily deduce the following equations for these two cases :

Zero Magnetic Field

$$(a) \quad \kappa^{E0} = -\Psi_{KL}(0), \dots\dots\dots (3.14)$$

$$(b) \quad \kappa^{J0} = -\Psi_{KL}(0) + [\Psi_{EL}(0)\Phi_{KL}(0)/\Phi_{EL}(0)]. \dots\dots\dots (3.15)$$

Non-zero Magnetic Field

$$(a) \quad \kappa^{Eek} = -\Psi_{KL}(\Omega). \dots\dots\dots (3.16)$$

The measurement of κ^{Eek} would therefore yield values of $\Psi_{KL}(\Omega)$.

(b)

$$\chi^{jw} = - \frac{\begin{vmatrix} \Phi_{EL} & -H\Phi_{ET} & \Phi_{KL} & -H\Phi_{KT} \\ H\Phi_{ET} & \Phi_{EL} & H\Phi_{KT} & \Phi_{KL} \\ \Psi_{EL} & -H\Psi_{ET} & \Psi_{KL} & -H\Psi_{KT} \\ H\Psi_{ET} & \Psi_{EL} & H\Psi_{KT} & \Psi_{KL} \end{vmatrix}}{\begin{vmatrix} \Phi_{EL} & -H\Phi_{ET} & -H\Phi_{KT} \\ H\Phi_{ET} & \Phi_{EL} & \Phi_{KL} \\ H\Psi_{ET} & \Psi_{EL} & \Psi_{KL} \end{vmatrix}} \quad (3.17)$$

(ii) *Galvanomagnetic Effects*.—(1) *Corbino magnetoresistance*.—The Corbino magnetoresistive effect is described by the coefficient $M_{\rho e} = (\rho^e - \rho^0)/\rho^0$, where ρ is the resistivity. For the more important isothermal conditions we obtain from (3.11) and (3.12)

$$M_{\rho e}^{Kk} = [\Phi_{EL}(0)/\Phi_{EL}(\Omega)] - 1. \quad (3.18)$$

(2) *Normal magnetoresistance*.—The normal magnetoresistance $M_{\rho j}^{Kk}$ is defined by $M_{\rho j}^{Kk} = (\rho^{jKk} - \rho^{K0})/\rho^{K0}$. Equations (3.11) and (3.13) yield

$$M_{\rho j}^{Kk} = - \frac{H^2\Phi_{ET}^2(\Omega) + \Phi_{EL}(\Omega)[\Phi_{EL}(\Omega) - \Phi_{EL}(0)]}{\Phi_{EL}^2(\Omega) + H^2\Phi_{ET}^2(\Omega)}. \quad (3.19)$$

(3) *Hall coefficient*.—The Hall coefficient is $R = E_y/HJ_x$; $J_y = 0$. For the more common isothermal conditions we easily obtain

$$R^{Kk} = - \frac{\Phi_{ET}(\Omega)}{\Phi_{EL}^2(\Omega) + H^2\Phi_{ET}^2(\Omega)}. \quad (3.20)$$

A simpler closely related effect is defined by

$$S = E_y/HE_x; \quad (J_y = 0) = (\tan \theta)/H,$$

where θ is the Hall angle. Under isothermal conditions we then find that

$$S^{Kk} = -\Phi_{ET}(\Omega)/\Phi_{EL}(\Omega). \quad (3.21)$$

As $\Phi_{EL}(\Omega) = \sigma^{eKk}$, this coefficient gives experimental values for $\Phi_{ET}(\Omega)$, a function which is extremely important for the estimation of parameters (see Appendix I).

(4) *Ettingshausen coefficient*.—The Ettingshausen coefficient A_E is defined by $A_E = -K_y/HJ_x$; $W_y = 0$. The usual experimental conditions are $J_y = W_x = 0$, in which case

$$A_E^{jW} = - \frac{\begin{vmatrix} H\Phi_{ET} & \Phi_{EL} & H\Phi_{KT} \\ \Psi_{EL} & -H\Psi_{ET} & \Psi_{KL} \\ H\Psi_{ET} & \Psi_{EL} & H\Psi_{KT} \end{vmatrix}}{H \begin{vmatrix} \Phi_{EL} & -H\Phi_{ET} & \Phi_{KL} & -H\Phi_{KT} \\ H\Phi_{ET} & \Phi_{EL} & H\Phi_{KT} & \Phi_{KL} \\ \Psi_{EL} & -H\Psi_{ET} & \Psi_{KL} & -H\Psi_{KT} \\ H\Psi_{ET} & \Psi_{EL} & H\Psi_{KT} & \Psi_{KL} \end{vmatrix}} \quad (3.22)$$

A more useful coefficient, closely related to the above, is $B_E = -K_y/HE_x$; $W_y = 0$. For the conditions $E_y = K_x = 0$, it reduces to the extremely simple form

$$B_E^{eK} = \Psi_{ET}(\Omega)/\Psi_{KL}(\Omega). \quad \dots\dots\dots (3.23)$$

Unfortunately, however, it possesses the great practical disadvantage of requiring mixed thermal conditions. If instead of $K_x = 0$, we apply the restriction $W_x = 0$, the coefficient so defined is more easily obtained experimentally. In this case

$$B_E^{eW} = \frac{\Psi_{KL}\Psi_{ET} - \Psi_{EL}\Psi_{KT}}{\Psi_{KL}^2 + H^2\Psi_{KT}^2}. \quad \dots\dots\dots (3.24)$$

Equation (3.24) provides an experimental method for determining the function Ψ_{ET} , the exact evaluation of which is important for estimating both metal and semiconductor parameters.

(iii) *Thermomagnetic Effects.*—(1) *Magneto thermal conductivity.*—This effect is described by the coefficient $M_x = (\alpha - \alpha^0)/\alpha^0$, which is usually measured subject to the restriction $\mathbf{J} = 0$. However, in this case, M_x involves fourth order determinants (see Section III (c) (i) (2)), and therefore it has few theoretical applications. For zero electric field conditions, M_x is much simpler. We find

$$M_x^{Eek} = [\Psi_{KL}(\Omega)/\Psi_{KL}(0)] - 1, \quad \dots\dots\dots (3.25)$$

$$M_x^{Eew} = \frac{\Psi_{KL}(\Omega) - \Psi_{KL}(0)}{\Psi_{KL}(0)} + \frac{H^2\Psi_{KT}^2(\Omega)}{\Psi_{KL}(\Omega)\Psi_{KL}(0)}. \quad \dots\dots (3.26)$$

(2) *Righi-Leduc coefficient.*—The Righi-Leduc coefficient is $B_{RL} = K_y/HK_x$; $W_y = 0$, and is therefore the thermomagnetic analogue of S (Section III (c) (ii) (3)). It is generally measured under conditions of zero electric current, but theoretically, zero electric field conditions possess a distinct advantage. However, experimentally, the reverse is true, and it is doubtful if B_{RL} has yet been determined for these latter conditions. We obtain

$$B_{RL}^{Ee} = -\Psi_{KT}(\Omega)/\Psi_{KL}(\Omega), \quad \dots\dots\dots (3.27)$$

Measurements of this quantity would therefore yield values of $\Psi_{KT}(\Omega)$.

$$B_{RL}^{Jj} = - \frac{H \begin{vmatrix} \Phi_{EL} & -H\Phi_{ET} & \Phi_{KL} \\ H\Phi_{ET} & \Phi_{EL} & H\Phi_{KT} \\ H\Psi_{ET} & \Psi_{EL} & H\Psi_{KT} \end{vmatrix}}{\begin{vmatrix} \Phi_{EL} & -H\Phi_{ET} & -H\Phi_{KT} \\ H\Phi_{ET} & \Phi_{EL} & \Phi_{KL} \\ H\Psi_{ET} & \Psi_{EL} & \Psi_{KL} \end{vmatrix}}. \quad \dots (3.28)$$

(3) *Ettingshausen-Nernst coefficient.*—This coefficient is defined by $B_{EN} = -E_y/HK_x$; $J_y = 0$, and is thus the thermomagnetic analogue of B_E (Section III (c) (ii) (4)). In practice, B_{EN} is usually determined subject to the

restrictions $J_x=0$ and either $K_y=0$ or $W_y=0$. However, theoretically, the simplest and therefore the most useful conditions are $E_x=0$ and $K_y=0$. We find

$$B_{EN}^{Ek} = \Phi_{KT}(\Omega)/\Phi_{EL}(\Omega), \dots\dots\dots (3.29)$$

A measurement of this quantity would thus determine $\Phi_{KT}(\Omega)$.

$$B_{EN}^{Jk} = \frac{\Phi_{EL}\Phi_{KT} - \Phi_{ET}\Phi_{KL}}{\Phi_{EL}^2 + H^2\Phi_{ET}^2} \dots\dots\dots (3.30)$$

As values of Φ_{EL} , Φ_{ET} , and Φ_{KL} are obtainable, equation (3.30) provides an alternative method of evaluating $\Phi_{KT}(\Omega)$.

(iv) *Thermoelectric Effects*.—In the present discussion we consider only those effects which apply to a single metal or semiconductor. Thus, for example, the Peltier effect is not treated.

(1) *Thomson coefficient*.—The Thomson coefficient μ is defined as the coefficient of $-K_x J_x$ in the equation $Q = \mathbf{E} \cdot \mathbf{J} - \text{div } \mathbf{W}$, which gives the rate of energy production per unit volume of a substance in which an electric field, electric current, and thermal current are present. For zero magnetic fields we readily obtain

$$\mu^0 = [\Phi_{KL}(0)/\Phi_{EL}(0)] + (\partial/\partial T)[\Psi'_{EL}(0)/\Phi_{EL}(0)]. \dots (3.31)$$

It is also easy to calculate μ for a non-zero magnetic field, but as this case does not appear to be of any experimental or theoretical importance it is not treated here.

(2) *Other coefficients*.—Other thermoelectric effects which may become important due to their theoretical simplicity are as follows:

(a)* $E_x^k = E_x/K_x; J_x=0, (E_y=K_y=0) = -\Phi_{KL}/\Phi_{EL} \dots (3.32)$

A determination of this quantity therefore enables Φ_{KL} to be obtained.

(b) $D_W^{Ek} = W_x/E_x; K_x=0, (E_y=K_y=0) = \Psi'_{EL}(\Omega) \dots\dots (3.33)$

(c) $D_K^{Ek} = K_x/E_x; W_x=0, (E_y=K_y=0) = -\Psi'_{EL}/\Psi'_{KL} \dots (3.34)$

A measurement of either (b) or (c) would thus yield the value of $\Psi'_{EL}(\Omega)$. The evaluation of this function is important for estimating both metal and semiconductor parameters (see Appendix I). Unfortunately, neither of the last two effects appears to have been investigated experimentally up to the present time.

IV. HIGH FREQUENCY ELECTRIC FIELD EFFECTS

(a) Introduction

For alternating electric fields only the isothermal galvanomagnetic effects are experimentally important. Under isothermal conditions the equation (2.9) for the electric current density becomes

$$\mathbf{J} = -M\epsilon \int_0^\infty \left[\frac{m_n^{\frac{1}{2}} \tau_n E_n^{3/2}}{1 + (i\omega + j\Omega_n)\tau_n} \frac{\partial f_{0n}}{\partial E_n} dE_n + \frac{m_p^{\frac{1}{2}} \tau_p E_p^{3/2}}{1 + (i\omega + j\Omega_p)\tau_p} \frac{\partial f_{0p}}{\partial E_p} dE_p \right] \mathbf{E} \dots (4.1)$$

* The name "Beer-Willardson" effect has been suggested for this by Armstrong and Greenberg (Beer, Armstrong, and Greenberg 1957).

As in Section III, it is now desirable to separate the real and imaginary parts of this expression. By substituting

$$\frac{2}{1+(i\omega+j\Omega)\tau} = \left[\frac{1}{1+(\omega+\Omega)^2\tau^2} + \frac{1}{1+(\omega-\Omega)^2\tau^2} \right] - i \left[\frac{(\omega+\Omega)\tau}{1+(\omega+\Omega)^2\tau^2} + \frac{(\omega-\Omega)\tau}{1+(\omega-\Omega)^2\tau^2} \right] - j \left[\frac{(\omega+\Omega)\tau}{1+(\omega+\Omega)^2\tau^2} + \frac{(\Omega-\omega)\tau}{1+(\omega-\Omega)^2\tau^2} \right] - ij \left[\frac{1}{1+(\omega+\Omega)^2\tau^2} - \frac{1}{1+(\omega-\Omega)^2\tau^2} \right]$$

in (4.1) we can readily derive the following rationalized equation

$$J = \{[\Pi_{ELR}(\Omega, \omega) + i\Pi_{ELI}(\Omega, \omega)] + jH[\Pi_{ETR}(\Omega, \omega) + i\Pi_{ETI}(\Omega, \omega)]\}E, \dots (4.2)$$

where

$$\Pi_{ELR}(\Omega, \omega) = \frac{1}{2}\epsilon [I_{11n}(\omega + \Omega_n) + I_{11n}(\omega - \Omega_n) + I_{11p}(\omega + \Omega_p) + I_{11p}(\omega - \Omega_p)], \dots (4.3)$$

$$\Pi_{ELI}(\Omega, \omega) = -\frac{1}{2}\epsilon \{ \omega [I_{21n}(\omega + \Omega_n) + I_{21n}(\omega - \Omega_n) + I_{21p}(\omega + \Omega_p) + I_{21p}(\omega - \Omega_p)] + \Omega_n [I_{21n}(\omega + \Omega_n) - I_{21n}(\omega - \Omega_n)] + \Omega_p [I_{21p}(\omega + \Omega_p) - I_{21p}(\omega - \Omega_p)] \}, \dots (4.4)$$

$$\Pi_{ETR}(\Omega, \omega) = \frac{1}{2}\epsilon^2 c^{-1} \{ m_n^{-1} [I_{21n}(\omega + \Omega_n) + I_{21n}(\omega - \Omega_n)] - m_p^{-1} [I_{21p}(\omega + \Omega_p) + I_{21p}(\omega - \Omega_p)] - (c\omega/\epsilon H) [I_{21n}(\omega + \Omega_n) - I_{21n}(\omega - \Omega_n) + I_{21p}(\omega + \Omega_p) - I_{21p}(\omega - \Omega_p)] \}, \dots (4.5)$$

$$\Pi_{ETI}(\Omega, \omega) = -\frac{1}{2}\epsilon H^{-1} [I_{11n}(\omega + \Omega_n) - I_{11n}(\omega - \Omega_n) + I_{11p}(\omega + \Omega_p) - I_{11p}(\omega - \Omega_p)], \dots (4.6)$$

and, as in Section III,

$$I_{stin}(x_n) = -Mm_n^{\frac{1}{2}} \int_0^\infty \frac{\tau_n^2 E_n^{t+\frac{1}{2}}}{1+x_n^2\tau_n^2} \frac{\partial f_{0n}}{\partial E_n} dE_n.$$

(b) General Formulae for the High Frequency Electric Field Effects

Before attempting to derive any formulae we must first distinguish between two basically different types of effects.

Point Effects.—These are defined in terms of the currents and fields at a particular point in the metal or semiconductor, and are therefore independent of the geometry of the specimen. In general, they are not directly measurable, i.e. the fields and currents appearing in the definition cannot be directly determined. The conductivity is an example of a point effect.

Surface Effects.—In these it is necessary to take into account the variation of the electric field with distance from the surface of the sample, and therefore the geometry of the metal or semiconductor is important. In general, they are directly measurable. Examples of surface effects are the surface impedance and the surface magnetoresistance.

(i) *Point Effects*.—These effects are defined in a similar manner to those in the steady electric field case, and the expressions obtained for them are formally the same, but with $\Pi_{ELR} + i\Pi_{ELI}$ replacing Φ_{EL} , and $\Pi_{ETR} + i\Pi_{ETI}$ replacing Φ_{ET} .

(1) *Electrical conductivity*.—As in Section III (c) (i) (1), the electrical conductivity is defined by $\sigma = J_x/E_x$. The only experimental conditions of any importance are :

Zero Magnetic Field

From (4.2) we obtain

$$\sigma^0 = [\Pi_{ELR}^2(0, \omega) + \Pi_{ELI}^2(0, \omega)]^{\frac{1}{2}} \exp \{i \tan^{-1}[\Pi_{ELI}(0, \omega)/\Pi_{ELR}(0, \omega)]\}. \dots\dots\dots (4.7)$$

Non-zero Magnetic Field

(a) $E_y = 0$

$$\sigma^e = [\Pi_{ELR}^2(\Omega, \omega) + \Pi_{ELI}^2(\Omega, \omega)]^{\frac{1}{2}} \exp \{i \tan^{-1}[\Pi_{ELI}(\Omega, \omega)/\Pi_{ELR}(\Omega, \omega)]\}, \dots\dots\dots (4.8)$$

(b) $J_y = 0$

$$\sigma^j = \sigma^e + \frac{H^2[\Pi_{ETR}^2 + \Pi_{ETI}^2]}{[\Pi_{ELR}^2 + \Pi_{ELI}^2]^{\frac{1}{2}}} \exp \left\{ i \left[2 \tan^{-1} \left(\frac{\Pi_{ETI}}{\Pi_{ETR}} \right) - \tan^{-1} \left(\frac{\Pi_{ELI}}{\Pi_{ELR}} \right) \right] \right\}. \dots\dots\dots (4.9)$$

(ii) *Surface Effects*.—The following calculations do not take account of the anomalous skin effect (Reuter and Sondheimer 1948). The displacement current is included, although for metals it is only significant when the anomalous terms in the skin effect should also be taken into account. It is considered here since it may be required for semiconductors, due to their lower conductivity, and also because it is as easy to include as to exclude.

It is assumed that the metal or semiconductor specimen occupies the volume defined by $z \geq 0$. The magnetic field is then in the direction of the outward normal from the surface, and the electric fields are parallel to the surface (see Section II).

As indicated in Section IV (b), the electric field is a function of position. With the above assumptions this function satisfies Maxwell's equations, which yield

$$\frac{d^2 \mathbf{E}}{dz^2} = \mathbf{E}'' = \frac{\mu}{c^2} \left(\kappa \frac{\partial^2 \mathbf{E}}{\partial t^2} + 4\pi \frac{\partial \mathbf{J}}{\partial t} \right), \dots\dots\dots (4.10)$$

where μ is the magnetic permeability and κ is the dielectric constant. As it has been implicitly assumed in Section II that $\mu = 1$, we must also make this assumption here, thus excluding ferromagnetic materials from consideration.

On combining (4.2) and (4.10), and remembering that $\partial \mathbf{E} / \partial t = i\omega \mathbf{E}$ (see Section II (a)) we obtain

$$\mathbf{E}'' = (A + jHB)\mathbf{E}, \dots\dots\dots (4.11)$$

where

$$A = \omega c^{-2} \{-\omega \kappa + 4\pi i [\Pi_{ELR}(\Omega, \omega) + i \Pi_{ELI}(\Omega, \omega)]\}, \dots (4.12)$$

$$B = 4\pi \omega c^{-2} i [\Pi_{ETR}(\Omega, \omega) + i \Pi_{ETI}(\Omega, \omega)]. \dots (4.13)$$

To simplify later work we define here

$$\theta = \arg A = \tan^{-1} \left[-\frac{4\pi \Pi_{ELR}(\Omega, \omega)}{\omega \kappa + 4\pi \Pi_{ELI}(\Omega, \omega)} \right], \dots (4.14)$$

$$\varphi = \arg B = \tan^{-1} \{ -[\Pi_{ETR}(\Omega, \omega) / \Pi_{ETI}(\Omega, \omega)] \}. \dots (4.15)$$

Under the boundary condition that $E(z)$ is always finite, (4.11) possesses the solution

$$\left. \begin{aligned} E_x(z) &= a \exp(-\lambda_1 z) + b \exp(-\lambda_2 z), \\ E_y(z) &= -ja \exp(-\lambda_1 z) + jb \exp(-\lambda_2 z), \end{aligned} \right\} \dots (4.16)$$

where $a = \frac{1}{2}[E_x(0) + jE_y(0)]$, $b = \frac{1}{2}[E_x(0) - jE_y(0)]$, $\lambda_1 = (A + jHB)^{\frac{1}{2}}$, $\lambda_2 = (A - jHB)^{\frac{1}{2}}$, and $-\frac{1}{2}\pi < \arg \lambda_1, \lambda_2 \leq \frac{1}{2}\pi$, i.e. the real parts of both λ_1 and λ_2 are positive. Since we have taken $E \propto e^{i\omega t}$ (Section II) and we require the solution to represent waves travelling in the positive z direction, the imaginary parts of λ_1 and λ_2 must also be positive, i.e. $0 \leq \arg \lambda_1, \lambda_2 \leq \pi$. The final restriction on $\arg \lambda_1, \lambda_2$ is therefore $0 \leq \arg \lambda_1, \lambda_2 \leq \frac{1}{2}\pi$.

We are now in a position to calculate the various surface effects.

(1) *Surface impedance.*—The surface impedance is defined by

$$Z(\omega, H) = E_x(0) \bigg/ \int_0^\infty J_x(z) dz = -(4\pi \omega c^{-2}) i E_x(0) / E'_x(0). \dots (4.17)$$

For non-zero magnetic fields, one more condition is necessary. The most convenient restrictions are either, (i) $E_y(0) = 0$, or (ii) $\int_0^\infty J_y(z) dz = 0$, i.e. $E'_y(0) = 0$. As (ii) appears to be the more common condition (Donovan 1954) it is assumed here. It is also required in the calculation of the normal magnetoresistance. Condition (i) would define the high frequency analogue of the Corbino magnetoresistance (Section III (e) (ii) (1)), but no attempt is made to calculate this effect here as it is neither experimentally nor theoretically important.

With restriction (ii), we find from (4.16) and (4.17) that

$$\begin{aligned} Z(\omega, H) &= (4\pi \omega c^{-2}) i (\lambda_1 + \lambda_2) / 2\lambda_1 \lambda_2 \dots (4.18) \\ &= (2\pi \omega c^{-2}) i [(A + jHB)^{\frac{1}{2}} + (A - jHB)^{\frac{1}{2}}] [A^2 + H^2 B^2]^{-\frac{1}{2}}. \dots (4.19) \end{aligned}$$

Since A and B are complex numbers, any attempt to separate the real and imaginary parts of (4.19) exactly will result in extremely complicated expressions. However, for weak magnetic fields ($H \perp B \parallel A \mid < 1$), it is comparatively simple

to expand (4.19) in ascending powers of HB/A and then rationalize term by term. We find after expanding and rationalizing (4.19) that

$$Z(\omega, H) = (4\pi\omega c^{-2})i |A|^{-\frac{1}{2}} \exp(-\frac{1}{2}i\theta) \left\{ 1 - \frac{3}{8} \frac{H^2|B|^2}{|A|^2} \exp[2i(\varphi - \theta)] + \frac{35}{128} \frac{H^4|B|^4}{|A|^4} \exp[4i(\varphi - \theta)] \dots \right\}, \dots \dots \dots (4.20)$$

and

$$Z(\omega, 0) = (4\pi\omega c^{-2})i |A_0|^{-\frac{1}{2}} \exp(-\frac{1}{2}i\theta_0), \dots \dots \dots (4.21)$$

where A_0 and θ_0 are the values of A and θ for $H=0$.

Equation (4.19) may also be expanded in descending powers of HB/A to obtain formal expressions for the strong magnetic field case ($H|B|/|A| > 1$). But for metals at least, this condition implies, in general, that $|\Omega_n \tau_n|$ and/or $|\Omega_p \tau_p| > 1$ and the derivation of the Boltzmann equation then breaks down (Wilson 1953, p. 210). We therefore do not consider this case here.

(2) *Surface resistance and reactance.*—The surface resistance $R(\omega, H)$ and surface reactance $X(\omega, H)$ are defined as the real and imaginary parts respectively of $Z(\omega, H)$. From (4.20) and (4.21) we obtain

$$R(\omega, H) = (4\pi\omega c^{-2}) |A|^{-\frac{1}{2}} \left\{ \sin \frac{1}{2}\theta - \frac{3}{8} \frac{H^2|B|^2}{|A|^2} \sin [\frac{1}{2}(5\theta - 4\varphi)] + \frac{35}{128} \frac{H^4|B|^4}{|A|^4} \sin [\frac{1}{2}(9\theta - 8\varphi)] + \dots \right\}, \dots \dots \dots (4.22)$$

$$R(\omega, 0) = (4\pi\omega c^{-2}) |A_0|^{-\frac{1}{2}} \sin \frac{1}{2}\theta_0, \dots \dots \dots (4.23)$$

$$X(\omega, H) = (4\pi\omega c^{-2}) |A|^{-\frac{1}{2}} \left\{ \cos \frac{1}{2}\theta - \frac{3}{8} \frac{H^2|B|^2}{|A|^2} \cos [\frac{1}{2}(5\theta - 4\varphi)] + \frac{35}{128} \frac{H^4|B|^4}{|A|^4} \cos [\frac{1}{2}(9\theta - 8\varphi)] + \dots \right\}, \dots \dots \dots (4.24)$$

$$X(\omega, 0) = (4\pi\omega c^{-2}) |A_0|^{-\frac{1}{2}} \cos \frac{1}{2}\theta_0, \dots \dots \dots (4.25)$$

(3) *Surface magnetoresistance.*—The magnetoresistance $M_R(\omega, H)$ is defined by

$$M_R(\omega, H) = [R(\omega, H) - R(\omega, 0)]/R(\omega, 0).$$

The substitution of (4.22) and (4.23) into the above leads to

$$M_R(\omega, H) = \left[\frac{\sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta_0} \frac{|A_0|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} - 1 \right] - \frac{3}{8} \frac{H^2|B|^2}{|A|^{5/2}} \frac{|A_0|^{\frac{1}{2}} \sin [\frac{1}{2}(5\theta - 4\varphi)]}{\sin \frac{1}{2}\theta_0} \times \left\{ 1 - \frac{35}{48} \frac{H^2|B|^2}{|A|^2} \frac{\sin [\frac{1}{2}(9\theta - 8\varphi)]}{\sin [\frac{1}{2}(5\theta - 4\varphi)]} + \dots \right\}. \dots \dots (4.26)$$

(4) *Hall coefficient*.—Following Donovan and Sondheimer (1953) we define the Hall coefficient $Y(\omega, H)$ by

$$Y(\omega, H) = +E_y(0) \left/ \left[H \int_0^\infty J_x(z) dz \right] \right. ; \quad \int_0^\infty J_y(z) dz = 0$$

$$= -(4\pi\omega c^{-2}) i E_y(0) / H E'_x(0) ; \quad E'_y(0) = 0.$$

From (4.16) we obtain

$$Y(\omega, H) = ij(4\pi\omega c^{-2})(\lambda_1 - \lambda_2) / (H.2\lambda_1\lambda_2)$$

$$= ij(2\pi\omega c^{-2}) H^{-1} [(A + jHB)^\frac{1}{2} - (A - jHB)^\frac{1}{2}] [A^2 + H^2B^2]^{-\frac{1}{2}}.$$

..... (4.27)

As in Section IV (b) (ii) (1), equation (4.27) can be expanded in ascending powers of HB/A , yielding

$$Y(\omega, H) = - \frac{2\pi\omega |B|}{c^2 |A|^{\frac{3}{2}}} i \exp [\frac{1}{2}i(2\phi - 3\theta)] \left\{ 1 - \frac{5}{8} \frac{H^2 |B|^2}{|A|^2} \exp [2i(\phi - \theta)] \right.$$

$$\left. + \frac{63}{128} \frac{H^4 |B|^4}{|A|^4} \exp [4i(\phi - \theta)] - \dots \right\} \dots \dots \dots (4.28)$$

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APPENDIX I

Estimation of Two-band Model Parameters

The parameters of a metal or semiconductor may be determined from the experimentally measurable functions Φ_{EL} , Φ_{ET} , Ψ_{EL} , and Ψ_{ET} defined in Section III (a). It is convenient to consider this problem separately for metals and semiconductors since, in general, the above functions assume different forms for each case.

(i) *Metals.*—In metals the electrons and holes form strongly degenerate systems, i.e. Fermi energy $\zeta \gg kT$. Under these conditions, convenient experimental parameters are A_n , A_p , m_n , m_p , τ_n , τ_p , ζ_n , and ζ_p (see Section II). We now proceed to show how these quantities may be estimated from the four functions Φ_{EL} , Φ_{ET} , Ψ_{EL} , and Ψ_{ET} .

All the above functions have been expressed in Section III (a) in terms of the integral

$$L_{st}(x) = - \int_0^\infty \frac{\tau^s E^{t+\frac{1}{2}}}{1+x^2\tau^2} \frac{\partial f_0}{\partial E} dE,$$

which, for $\zeta \gg kT$, may be evaluated by applying the asymptotic expansion first given by Sommerfeld (1928)

$$- \int_0^\infty \varphi(E) \frac{\partial f_0}{\partial E} dE = \varphi(\zeta) + \frac{1}{6}(\pi kT)^2 \frac{\partial^2 \varphi(\zeta)}{\partial \zeta^2} + \dots \quad (1)$$

From (1) we obtain, to first order terms in kT/ζ ,

$$L_{st}(x) = \frac{\tau^s \zeta^{t+\frac{1}{2}}}{1+x^2\tau^2} \dots \quad (2)$$

Substitution of (2) in (3.3), (3.4), (3.7), and (3.9) yields

$$\Phi_{EL}(0) = M\varepsilon(\tau_n m_n^{\frac{1}{2}} \zeta_n^{3/2} + \tau_p m_p^{\frac{1}{2}} \zeta_p^{3/2}) = \sigma_n + \sigma_p, \quad (3)$$

$$\Phi_{EL}(\Omega) = \frac{\sigma_n}{1+\mu_n^2 H^2} + \frac{\sigma_p}{1+\mu_p^2 H^2}, \quad (4)$$

$$\Phi_{ET}(0) = \mu_n \sigma_n - \mu_p \sigma_p, \quad (5)$$

$$\Phi_{ET}(\Omega) = \frac{\mu_n \sigma_n}{1+\mu_n^2 H^2} - \frac{\mu_p \sigma_p}{1+\mu_p^2 H^2}, \quad (6)$$

$$\varepsilon \Psi_{EL}(0) = -(A_n \sigma_n + A_p \sigma_p) - \zeta_n \sigma_n + \zeta_p \sigma_p, \quad (7)$$

$$\varepsilon \Psi_{EL}(\Omega) = - \left(\frac{A_n \sigma_n}{1+\mu_n^2 H^2} + \frac{A_p \sigma_p}{1+\mu_p^2 H^2} \right) - \frac{\zeta_n \sigma_n}{1+\mu_n^2 H^2} + \frac{\zeta_p \sigma_p}{1+\mu_p^2 H^2}, \quad (8)$$

$$\varepsilon \Psi_{ET}(0) = -(A_n \mu_n \sigma_n - A_p \mu_p \sigma_p) - \mu_n \zeta_n \sigma_n - \mu_p \zeta_p \sigma_p, \quad (9)$$

$$\varepsilon \Psi_{ET}(\Omega) = - \left(\frac{A_n \mu_n \sigma_n}{1+\mu_n^2 H^2} - \frac{A_p \mu_p \sigma_p}{1+\mu_p^2 H^2} \right) - \frac{\mu_n \zeta_n \sigma_n}{1+\mu_n^2 H^2} - \frac{\mu_p \zeta_p \sigma_p}{1+\mu_p^2 H^2}, \quad (10)$$

where the electron conductivity mobility $e\mu_n = \varepsilon\tau_n/m_n$, and the hole mobility $e\mu_p = \varepsilon\tau_p/m_p$.

From the equations (3), (4), (5), and (6), which are four independent relations involving the four unknowns σ_n , σ_p , μ_n , and μ_p , we find that

$$\mu_n = \frac{\Phi_{ET}(0) - \Phi_{ET}(\Omega)(1 + \mu_p^2 H^2)}{\Phi_{EL}(0) - \Phi_{EL}(\Omega)(1 + \mu_p^2 H^2)}, \dots\dots\dots (11)$$

$$\mu_p = -\frac{\Phi_{ET}(0) - \Phi_{ET}(\Omega)(1 + \mu_n^2 H^2)}{\Phi_{EL}(0) - \Phi_{EL}(\Omega)(1 + \mu_n^2 H^2)}. \dots\dots\dots (12)$$

Equations (11) and (12) may be solved by successive approximations.

The partial conductivities σ_n , σ_p are then obtainable from (3) and (5). The number (n) of conduction electrons and the number (p) of conduction holes per unit volume may also be deduced, since (see Wilson 1953, p. 198)

$$n = \sigma_n / \varepsilon c \mu_n, \quad p = \sigma_p / \varepsilon c \mu_p. \dots\dots\dots (13)$$

The substitution of μ_n , μ_p , σ_n , and σ_p in (7), (8), (9), and (10) yields four linear equations which may be solved for the four unknowns A_n , A_p , ζ_n , and ζ_p .

The parameters τ_n , τ_p , m_n , and m_p are then obtainable, since from (3), (4), and (5) we have

$$\tau_n = \frac{(c \mu_n \sigma_n^2)^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}} \zeta_n} \left(\frac{3 h^3}{16 \sqrt{2} \pi \varepsilon^2} \right)^{\frac{2}{3}},$$

$$m_n = \varepsilon \tau_n / \mu_n c,$$

with similar expressions for τ_p and m_p .

Finally, the difference in energy (ΔE) between the base of the normal band and the top of the inverted band may be deduced from the relation $\Delta E = A_n - A_p$.

(ii) *Semiconductors*.—In many semiconductors the electrons and holes form non-degenerate systems ($\zeta \ll 0$). In this case the most convenient parameters are :

- (a) The electron and hole densities n and p .
- (b) The conductivity mobilities μ_n and μ_p .

As we have assumed (Section II) that both electrons and holes have Fermi-Dirac distributions, n and p are given by (Wilson 1953, p. 15)

$$\left. \begin{aligned} n &= -\frac{16 \sqrt{2} \pi m_n^{3/2}}{3 h^3} \int_0^\infty E_n^{3/2} \frac{\partial f_{0n}}{\partial E_n} dE_n, \\ p &= -\frac{16 \sqrt{2} \pi m_p^{3/2}}{3 h^3} \int_0^\infty E_p^{3/2} \frac{\partial f_{0p}}{\partial E_p} dE_p. \end{aligned} \right\} \dots\dots\dots (14)$$

In terms of the integral $I_{st}(x)$ (Section III (a)), equations (14) may be written

$$\left. \begin{aligned} n &= m_n \varepsilon^{-1} I_{01n}(0), \\ p &= m_p \varepsilon^{-1} I_{01p}(0). \end{aligned} \right\} \dots\dots\dots (15)$$

The mobilities are defined by (Shockley 1950, p. 16)

$$\sigma^{K0} = \varepsilon (n \mu_n + p \mu_p).$$

From (3.3) and (3.11) we have

$$\sigma^{K0} = \Phi_{EL}(0) = \varepsilon [I_{11n}(0) + I_{11p}(0)], \dots\dots\dots (16)$$

and hence

$$\left. \begin{aligned} \mu_n &= I_{11n}(0)/n = \varepsilon I_{11n}(0)/m_n I_{01n}(0), \\ \mu_p &= I_{11p}(0)/p = \varepsilon I_{11p}(0)/m_p I_{01p}(0). \end{aligned} \right\} \dots\dots\dots (17)$$

The above four parameters may be determined from Φ_{EL} and Φ_{ET} as we now proceed to show. On combining (3.3), (3.4), and (16) we obtain for these functions

$$\Phi_{EL}(\Omega) = \varepsilon \{ n \mu_n [L_{11n}(\Omega_n)/L_{11n}(0)] + p \mu_p [L_{11p}(\Omega_p)/L_{11p}(0)] \}, \dots (18)$$

$$\Phi_{ET}(\Omega) = \frac{\varepsilon}{c} \left[n \mu_n^2 \frac{L_{01n}(0)L_{21n}(\Omega_n)}{L_{11n}^2(0)} - p \mu_p^2 \frac{L_{01p}(0)L_{21p}(\Omega_p)}{L_{11p}^2(0)} \right]. \dots\dots\dots (19)$$

It is then necessary to evaluate the integral

$$L_{st}(x) = - \int_0^\infty \frac{\tau^s E^{t+\frac{1}{2}}}{1+x^2\tau^2} \frac{\partial f_0}{\partial E} dE.$$

For non-degenerate systems, $f_0 = \exp [(\zeta - E)/kT]$, and therefore (unlike case (i)) the energy dependence of τ must be taken into account. This varies with the scattering mechanism. Here, we consider in detail only the special case of scattering by the thermal vibrations of the lattice ($\tau \propto E^{-\frac{1}{2}}$). For ionic-impurity scattering [$\tau = g(E)E^{3/2}$] an exactly similar method is applicable, provided the slowly varying function $g(E)$ is replaced by a constant $g(E_m)$ —for a further discussion of this point see Dingle (1955), Beer, Armstrong, and Greenberg (1957)—and only the numerical constants differ in the final results. The third important case of mixed scattering is considerably more complicated since one further parameter (the ratio of thermal to ionic scattering) must also be evaluated.

For thermal scattering we have $\tau \propto E^{-\frac{1}{2}}$ (Wilson 1953, p. 265) and therefore L_{st} may be written

$$L_{st}(x) = (kT)^{t+\frac{1}{2}} a^s e^{\zeta/kT} \int_0^\infty \frac{y^{t+3/2-s/2}}{y+x^2a^2} e^{-y} dy \dots\dots\dots (20)$$

$$= (kT)^{t+\frac{1}{2}} a^s e^{\zeta/kT} \left(t + \frac{3}{2} - \frac{1}{2}s \right)! A_{t+3/2-s/2}(x^2a^2), \dots\dots\dots (21)$$

where $y = E/kT$ and $\tau = ay^{-\frac{1}{2}}$. The integral

$$A_r(u) = \frac{1}{r!} \int_0^\infty \frac{y^r}{y+u} e^{-y} dy$$

has been tabulated by Dingle, Arndt, and Roy (1957a).

For the two special cases of $x=0$ and $x\tau \gg 1$, equation (21) reduces to

$$L_{st}(0) = (kT)^{t+\frac{1}{2}} a^s e^{\zeta/kT} \left(t + \frac{1}{2} - \frac{1}{2}s \right)! \dots\dots\dots (22)$$

$$L_{st}(x) = (kT)^{t+\frac{1}{2}} a^{s-2} e^{\zeta/kT} x^{-2} \left(t + \frac{3}{2} - \frac{1}{2}s \right)! \dots\dots\dots (23)$$

On combining (17), (18), (19), (22), and (23) we obtain

$$\Phi_{EL}(0) = \varepsilon(n\mu_n + p\mu_p), \dots\dots\dots (24)$$

$$H^2\Phi_{EL}(\Omega) = \frac{32c^2\varepsilon}{9\pi} \left(\frac{n}{\mu_n} + \frac{p}{\mu_p} \right), \text{ for } \Omega^2\tau^2 \gg 1, \dots\dots (25)$$

$$\Phi_{ET}(0) = (3\pi\varepsilon/8c)(n\mu_n^2 - p\mu_p^2), \dots\dots\dots (26)$$

$$H^2\Phi_{ET}(\Omega) = \varepsilon c(n - p) \text{ for } \Omega^2\tau^2 \gg 1. \dots\dots\dots (27)$$

These four algebraic equations can be solved for n , p , μ_n , and μ_p .

The values of the parameters A_n , A_p may then be deduced from Ψ_{EL} and Ψ_{ET} since equations (3.7) and (3.8) reduce to

$$\begin{aligned} \Psi_{EL}(0) &= -A_n n \mu_n - A_p p \mu_p - I_{12n}(0) + I_{12p}(0) \\ &= -(A_n + 2kT)n\mu_n - (A_p - 2kT)p\mu_p, \dots\dots\dots (28) \end{aligned}$$

$$\begin{aligned} \Psi_{ET}(0) &= -(3\pi/8c)(A_n n \mu_n^2 - A_p p \mu_p^2) - \varepsilon c^{-1} [m_n^{-1} I_{22n}(0) + m_p^{-1} I_{22p}(0)] \\ &= -(3\pi/8c) \left[\left(A_n + \frac{3}{2}kT \right) n \mu_n^2 - \left(A_p - \frac{3}{2}kT \right) p \mu_p^2 \right]. \dots\dots\dots (29) \end{aligned}$$

Finally, the energy gap (ΔE) between the bands may be deduced from the relation $\Delta E = A_n - A_p$.