

# ON THE CYLINDRICAL PROBE METHOD OF MEASURING THERMAL CONDUCTIVITY WITH SPECIAL REFERENCE TO SOILS

## I. EXTENSION OF THEORY AND DISCUSSION OF PROBE CHARACTERISTICS

By D. A. DE VRIES\* and A. J. PECK\*

[Manuscript received December 3, 1957]

### *Summary*

The theory of cylindrical probes for measuring thermal conductivity is extended to the case of a probe of finite conductivity containing a line source at its centre. This provides a more realistic approximation to most actual probes than the theory for a probe of infinite conductivity developed by other authors. New experimental results are presented which are in complete agreement with theory. It is shown how an estimate can be obtained of the magnitude of a possible thermal contact resistance between the probe and the medium and how its influence on the measured conductivity can be assessed.

Conditions under which the theory of the infinite line source can be applied with a sufficient degree of accuracy are treated. The properties of various probes described in the literature are reviewed in this respect. The importance of measuring both the heating and the cooling branch of the temperature against time curve is emphasized. Difficulties in measuring thermal diffusivity with the probe are briefly discussed.

## I. INTRODUCTION

Measurement of the thermal conductivity of soils and thermal insulating materials has received much attention over the past 10 years. One can distinguish between two groups of investigators interested in the subject who apparently have not always been aware of each other's activities in this field (see for instance a recent discussion in *Nature*: de Vries (1956), Webb (1956, 1957), Makowski and Mochlinski (1957)). The first group is that of engineers, who are interested in thermal properties of the soil in connexion with heat transfer from buried cables and coils of heat pumps, road construction, etc. The second group consists of soil scientists, hydrologists, meteorologists, oceanographers, and agronomists, who are concerned with such problems as the energy balance of the Earth's surface, the temperature regime of the upper soil and lower air layers, and the measurement of soil moisture content by thermal methods.

Steady state methods are not very suitable for use with soils. They cannot be applied *in situ* and in addition the soil water will redistribute itself under the influence of a temperature gradient. Various non-stationary methods have been proposed in which the temperature rise is measured of heated test bodies of various shapes which are inserted in the soil (Chudnowskii 1946, 1954; Skeib 1950; Misener 1952).

\* Division of Plant Industry, C.S.I.R.O., Deniliquin, N.S.W.

Most actual measurements have been made with needle-shaped cylindrical test bodies (which we shall call cylindrical probes or, briefly, probes). These probes contain as a heat source a thin metal wire which is heated electrically; the temperature rise is measured by means of a thermocouple with its "warm" junction inside the probe near its centre. They can be installed *in situ* fairly easily without appreciable disturbance of the soil near the measuring thermojunction. When properly constructed and dimensioned the theoretical interpretation of the results is simple. In its simplest form the probe consists of a single heating wire, the temperature of which is measured by means of a thermocouple or by a resistance method.

The cylindrical probe method was first suggested by Schleiermacher (1888) and independently by Stålhane and Pyk (1931). The method was developed and used for measuring the thermal conductivity of liquids by Weishaupt (1940) and by van der Held and van Drunen (1949). The work of the latter authors served as the basis for many further developments. More recent work on the thermal conductivity of liquids was published by van der Held, Hardebol, and Kalshoven (1953), Gillam and Lamm (1955), Gillam *et al.* (1955), Hill (1957).

The method was first applied to soils by Hooper and Lepper (1950) and independently by Skeib (1950). Later work on soils was published by Hooper (1952), Mason and Kurtz (1952), de Vries (1952*a*, 1952*b*, 1953), van Duin and de Vries (1954), de Vries and de Wit (1954), Buettner (1955*a*), Makowski and Mochlinski (1956). Closely related to the work on soils is that of Bullard, Maxwell, and Revelle (1956) on sediments deposited on the ocean floor.

Cylindrical probes were used to measure the conductivity of thermal insulating materials by Hooper and Lepper (1950), d'Eustachio and Schreiner (1952), Mann and Forsyth (1956).

Measurements on rocks were published by Beck, Jaeger, and Newstead (1956).

The interpretation of the measurements is in most cases based on the theory of the infinite line source, supplemented by theoretical estimates or calculations of the influence of finite dimensions. A first attempt by van der Held and van Drunen (1949) to calculate the influence of the finite (i.e. non-zero) probe radius was unsatisfactory. This was recognized subsequently by van der Held, Hardebol, and Kalshoven (1953) and independently by Blackwell (1954). These authors treated the theory for a probe of finite radius and infinite thermal conductivity. The theory for this case was elaborated by Jaeger (1956). Blackwell (1954) also discussed the case of a hollow probe of finite conductivity with heat supplied at its surface. In a later paper Blackwell (1956) treated the influence of the finite probe length.

In the present paper the theory is extended to the case of a homogeneous probe of finite conductivity with a line source of heat at its centre. This gives a more realistic approximation to most actual probes than that of preceding theories. A detailed interpretation of experimental results obtained on dry sand with probes of various construction is given on the basis of this theory.

Following the discussion of probe theory and characteristics in this paper we shall treat the complications due to moisture effects in experiments on moist soils in a second paper.

II. NOTATION AND UNITS

- $a$ , Thermal diffusivity of medium ( $\text{cm}^2 \text{sec}^{-1}$ ),
- $a_i$ , thermal diffusivity of probe ( $\text{cm}^2 \text{sec}^{-1}$ ),
- $c$ , specific heat ( $\text{cal g}^{-1} \text{ }^\circ\text{C}^{-1}$ ),
- $C = \rho c$ , volumetric heat capacity of medium ( $\text{cal cm}^{-3} \text{ }^\circ\text{C}^{-1}$ ),
- $C_i$ , volumetric heat capacity of probe ( $\text{cal cm}^{-3} \text{ }^\circ\text{C}^{-1}$ ),
- $H$ , heat transfer coefficient ( $\text{cal cm}^{-2} \text{sec}^{-1} \text{ }^\circ\text{C}^{-1}$ ),
- $I_n$ , modified Bessel function of first kind and order  $n(=0,1)$ ,
- $K_n$ , modified Bessel function of second kind and order  $n(=0,1)$ ,
- $l$ , half length of probe (cm),
- $p$ , variable in Laplace transformation ( $\text{sec}^{-1}$ ),
- $q = (p/a)^{\frac{1}{2}}$  ( $\text{cm}^{-1}$ ),
- $q_i = (p/a_i)^{\frac{1}{2}}$  ( $\text{cm}^{-1}$ ),
- $Q$ , heat production per unit length of probe ( $\text{cal cm}^{-1} \text{sec}^{-1}$ ),
- $r$ , radial distance from axis of probe (cm),
- $R$ , radius of probe (cm),
- $R_1$ , outer radius of soil sample (cm),
- $t$ , time (sec),
- $t_1$ , time at end of heating (sec),
- $T$ , temperature ( $^\circ\text{C}$ ),
- $T_0$ , initial temperature ( $^\circ\text{C}$ ),
- $T_i$ , probe temperature ( $^\circ\text{C}$ ),
- $\alpha = \lambda/\lambda_i$ ,
- $\beta = C_i/C$ ,
- $\gamma = 0.5772$ , Euler's constant,
- $\delta$ , thickness of air gap (cm),
- $\eta = \lambda/RH$ , dimensionless contact resistance,
- $\lambda$ , thermal conductivity of medium ( $\text{cal cm}^{-1} \text{sec}^{-1} \text{ }^\circ\text{C}^{-1}$ ),
- $\lambda_{\text{air}}$ , thermal conductivity of air ( $\text{cal cm}^{-1} \text{sec}^{-1} \text{ }^\circ\text{C}^{-1}$ ),
- $\lambda_i$ , thermal conductivity of probe ( $\text{cal cm}^{-1} \text{sec}^{-1} \text{ }^\circ\text{C}^{-1}$ ),
- $\rho$ , density ( $\text{g cm}^{-3}$ ),
- $\tau = at/R^2$ ,
- $\tau_1 = at_1/R^2$ ,
- $\tau_i = at/l^2$ .

III. PROBE THEORY

(a) *The Infinite Line Source*

In its simplest form the theory of the cylindrical probe is based on that of the infinite line source embedded in an infinite, homogeneous, isotropic medium. The Fourier equation of heat conduction can then be written

$$\frac{\partial T}{\partial t} = a \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right), \dots\dots\dots (1)$$

The initial and boundary conditions are :

$$T = T_0, \text{ for } t = 0, \text{ and for } r = \infty \text{ (} t \text{ finite),} \quad \dots\dots (2)$$

$$Q = -\lim_{r \rightarrow 0} 2\pi\lambda r \partial T / \partial r, \text{ for } t \geq 0. \quad \dots\dots (3)$$

The solution to this problem is (Carslaw and Jaeger 1948)

$$T - T_0 = (Q/4\pi\lambda)[-Ei(-r^2/4at)]. \quad \dots\dots (4)$$

The exponential integral can be expanded as

$$-Ei(-x) = -\gamma - \ln x + x - \frac{1}{2}x^2 + 0(x^3). \quad \dots\dots (5)$$

Hence for  $4at/r^2 \gg 1$  we have to a good degree of approximation

$$T - T_0 = (Q/4\pi\lambda)(-\gamma + \ln t + \ln 4a/r^2). \quad \dots\dots (6)$$

When the source discontinues to operate at time  $t_1$  we must replace condition (3) by

$$Q = -\lim_{r \rightarrow 0} 2\pi\lambda r \partial T / \partial r, \text{ for } 0 \leq t \leq t_1; \quad Q = 0, \text{ for } t > t_1. \quad \dots (3a)$$

The solution now becomes

$$T - T_0 = (Q/4\pi\lambda)[-Ei(-r^2/4at) + Ei\{-r^2/4a(t-t_1)\}], \text{ for } t \geq t_1, \quad \dots (4a)$$

or, for  $4a(t-t_1)/r^2 \gg 1$ ,

$$T - T_0 = (Q/4\pi\lambda) \ln t / (t-t_1). \quad \dots\dots (6a)$$

As we shall see below, a line source can be realized to a good degree of approximation by a thin metal wire (e.g. diameter 0.01 cm) which is heated by an electric current during the time interval 0 to  $t_1$ . In that case  $\lambda$  can be found from an experiment by plotting  $T - T_0$  against  $\ln t$  for  $t \leq t_1$ , and also by plotting  $(Q/4\pi\lambda)(-\gamma + \ln 4at/r^2) - (T - T_0)$  against  $\ln(t-t_1)$ . Values of the first term in this expression for  $t > t_1$  are found by extrapolation of the line observed for  $t < t_1$ . The derivation of  $\lambda$  from both the heating and the cooling branches of the temperature-time curve provides a useful check on the procedure. This is of special importance in the case of measurements on moist porous media as will be discussed in Part II (de Vries and Peck 1958, in press).

(b) *The Probe of Finite Thickness containing a Line Source*

We shall now extend the theory to that of a homogeneous, isotropic, cylindrical probe of infinite length and radius  $R$ , which has an infinite line source at its centre.

The heat conduction problem must then be solved for a composite medium. The basic differential equation is again equation (1), this time with the thermal diffusivity  $a$  for  $r > R$  and  $a_i$  for  $r < R$ . Instead of the boundary condition (3) or (3a) we now have a similar condition with  $\lambda_i$  substituted for  $\lambda$ . Finally, there are additional boundary conditions at the probe-medium interface. Assuming a contact resistance at this boundary with heat transfer coefficient  $H$ , we have

$$H[T_i(R,t) - T(R,t)] = -\lambda_i \partial T_i / \partial r = -\lambda \partial T / \partial r, \text{ for } r = R. \quad \dots (7)$$

Here  $T_i(R,t)$  and  $T(R,t)$  are respectively the temperatures of the probe and the medium at the interface.

The solution to this problem can be found by the method of the Laplace transformation. Denoting transformed temperatures by a bar the transformed equations are in the usual notation :

$$\frac{d^2\bar{T}_i}{dr^2} + \frac{1}{r} \frac{d\bar{T}_i}{dr} - \frac{p\bar{T}_i}{a_i} = 0, \quad \text{for } 0 \leq r < R, \quad \dots\dots\dots (8)$$

$$\frac{d^2\bar{T}}{dr^2} + \frac{1}{r} \frac{d\bar{T}}{dr} - \frac{p\bar{T}}{a} = 0, \quad \text{for } r > R, \quad \dots\dots\dots (9)$$

$$Q/p = - \lim_{r \rightarrow 0} 2\pi\lambda_i r d\bar{T}_i/dr, \quad \dots\dots\dots (10)$$

$$H(\bar{T}_i - \bar{T}) = -\lambda_i d\bar{T}_i/dr = -\lambda d\bar{T}/dr, \quad \text{for } r = R. \quad \dots (11)$$

The solution is :

$$\bar{T}_i - T_0/p = AI_0(q_i r) + QK_0(q_i r)/2\pi\lambda_i p, \quad \dots\dots\dots (12)$$

$$\bar{T} - T_0/p = Qa^{\frac{1}{2}}a^{\frac{1}{2}}K_0(qr)/2\pi\Delta p^{3/2}R, \quad \dots\dots\dots (13)$$

with

$$A = (Q/2\pi\Delta p)[a^{\frac{1}{2}}K_0(qR)K_1(q_i R) + a^{\frac{1}{2}}\eta q R K_1(qR)K_1(q_i R) - \lambda a^{\frac{1}{2}}\lambda_i^{-1}K_0(q_i R)K_1(qR)], \quad \dots\dots\dots (14)$$

$$\Delta = \lambda a^{\frac{1}{2}}I_0(q_i R)K_1(qR) + \lambda_i a^{\frac{1}{2}}I_1(q_i R)[K_0(qR) + \eta q R K_1(qR)], \quad \dots\dots\dots (15)$$

where  $\eta = \lambda/RH$ .

The temperatures  $T_i$  and  $T$  can now be found by applying an inverse Laplace transformation.\* However, numerical calculation of  $T_i$  and  $T$  from the resulting analytical expressions is very laborious due to the oscillatory character of the Bessel functions occurring in the solutions.

Here we are principally interested in the behaviour of the solution for large values of time ( $4at/R^2 \gg 1$ ). This can be found along the lines set out by Blackwell (1954). The expressions for  $\bar{T}_i$  and  $\bar{T}$  are expanded in ascending powers of  $p$  and the resultant series is integrated term by term along a contour in the  $p$ -plane which is cut along the negative real axis. This contour follows the lower negative axis ( $xe^{-\pi i}$ ) from  $-\infty$  to the origin, circles the origin counter-clockwise, and returns to  $-\infty$  along the upper negative axis ( $xe^{\pi i}$ ). Using the contour integrals listed by Blackwell and

$$\frac{1}{2\pi i} \int p \ln(cp)e^{tp} dp = \frac{1}{t^2}, \quad \dots\dots\dots (16)$$

$$\frac{1}{2\pi i} \int p \ln^2(cp)e^{tp} dp = -\frac{2}{t^2} \left( \ln \frac{t}{c} + \gamma - 1 \right), \quad \dots\dots\dots (17)$$

$$\frac{1}{2\pi i} \int p \ln^3(cp)e^{tp} dp = \frac{1}{t^2} \left[ 3 \left( \ln \frac{t}{c} + \gamma \right)^2 - 6 \left( \ln \frac{t}{c} + \gamma \right) - \frac{1}{2}\pi^2 \right] \quad \dots (18)$$

\* D. A. de Vries, unpublished result (for  $\eta=0$  only).

(where the integration is along the contour and  $c$  is a real constant), we obtain :

$$T_i(r,t) - T_0 = (Q/4\pi\lambda)G_i(\tau,\eta,\alpha,\beta,r/R), \quad \text{for } r \leq R, \quad \dots \quad (19)$$

$$T(R,t) - T_0 = (Q/4\pi\lambda)G(\tau,\eta,\alpha,\beta), \quad \dots \quad (20)$$

with  $\alpha = \lambda/\lambda_i$ ,  $\beta = C_i/C$ , and  $\tau = 4at/R^2$ . The functions  $G_i$  and  $G$  are in dimensionless form :

$$G_i = \ln \tau - \gamma + 2\eta - 2\alpha \ln (r/R) + c_{i,-1}\tau^{-1} + 0(\tau^{-2} \ln^2 \tau), \quad \dots \quad (21)$$

with

$$c_{i,-1} = 2(1 - \beta)(\ln \tau - \gamma) + 2 - 4\eta\beta - 2\alpha\beta + \alpha\beta r^2/R^2, \quad \dots \quad (22)$$

and

$$G = \ln \tau - \gamma + c_{-1}\tau^{-1} + c_{-2}\tau^{-2} + 0(\tau^{-3} \ln^3 \tau), \quad \dots \quad (23)$$

with

$$c_{-1} = 2(1 - \beta)(\ln \tau - \gamma) + 2 - 2\eta\beta - \alpha\beta, \quad \dots \quad (24)$$

$$c_{-2} = -3(1 - \beta)^2(\ln \tau - \gamma)^2 + (-1 - 4\beta + 6\beta^2 + 2\alpha\beta - 3\alpha\beta^2 + 4\eta\beta - 8\eta\beta^2)(\ln \tau - \gamma) + \frac{1}{2}\pi^2(1 - \beta)^2 + \frac{3}{2} - 4\beta + 3\alpha\beta^2 - \frac{3}{4}\alpha^2\beta^2 + 8\eta\beta^2 - 3\eta\alpha\beta^2 - 4\eta^2\beta^2. \quad \dots \quad (25)$$

It can be easily checked that  $G_i$  and  $G$  reduce to (5) for  $\alpha = \beta = 1$  and  $\eta = 0$ . For a probe of infinite conductivity ( $\alpha = 0$ ) the expression for  $G_i$  reduces to those given by Blackwell (1954) and Jaeger (1956) for large  $\tau$ . Terms of the order  $\tau^{-2}$  were not computed for  $G_i$  because of the unwieldiness of the resulting expressions. However, in the applications considered below it is safe to assume that the term with  $\tau^{-2}$  in  $G_i$  is of the same order of magnitude as that in  $G$ . A further discussion of these equations is given in Section IV.

Equations (19) and (20) hold for the heating branch only. For the cooling branch ( $t > t_1$ ) we have, analogous to equation (4a) :

$$T_i(r,t) - T_0 = (Q/4\pi\lambda)[G_i(\tau,\eta,\alpha,\beta,r/R) - G_i(\tau - \tau_1,\eta,\alpha,\beta,r/R)], \quad \dots \quad (26)$$

$$T(R,t) - T_0 = (Q/4\pi\lambda)[G(\tau,\eta,\alpha,\beta) - G(\tau - \tau_1,\eta,\alpha,\beta)]. \quad \dots \quad (27)$$

(c) *Miscellaneous Factors*

In the interpretation of a thermal conductivity experiment we must also consider systematic errors arising from various factors apart from the finite radius of the probe. These factors are : finite length of the probe, finite dimensions of the sample, the inertia of the temperature-measuring system, and factors connected with moisture movement and an uneven distribution of moisture in the sample. The moisture problems will be discussed in a separate paper ; a brief discussion of the other factors is given in this subsection.

The influence of the finite probe length can be made arbitrarily small, of course, by a proper choice of the probe dimensions. Blackwell (1956) has derived an upper limit for the relative error in the slope of the  $T_i$  against  $\ln t$  curve due to axial heat flow on the assumptions that the probe is a good conductor

and that heat is generated uniformly throughout the probe. His equation for this upper limit reads in our notation

$$1 - \frac{\partial T_i(R,t)/\partial \ln t}{Q/4\pi\lambda} = \pi^{-1/2} \left[ \tau_l^{-1/2} + \frac{2(1-\alpha\beta)}{\alpha\tau_l^{3/2}} (\ln \tau - \gamma + 2\eta) \right] \exp(-\tau_l^{-1}), \dots (28)$$

with  $\tau_l = 4at/l^2$ , where  $l$  is half the probe length. Once the probe diameter and the heating time are fixed a proper value of  $l$  can be derived from equation (28). As we shall see in the following section, this does not lead to excessive probe lengths.

The influence of the finite dimensions of the soil sample can also be made sufficiently small without difficulty by a proper choice of the dimensions of the sample container. We propose the following simple criterion for this purpose: the amount of heat passing through the walls of the container must be small in comparison with the heat input at the source during the time of heating. Since we are only concerned with its order of magnitude a sufficiently accurate estimate of this amount of heat can be obtained by application of the solution for an infinite line source. For a long cylindrical sample container with radius  $R_1$  this leads to the inequality

$$\exp(-R_1^2/4at_1) \ll 1. \dots\dots\dots (29)$$

The temperature is usually measured near the centre of the source by a thermocouple in connexion with a galvanometer. The inertia of this system will be negligible for the type of temperature variation encountered here when the time at which the first reading is taken is large in comparison with the period of the galvanometer. In our experiments the former was not less than 10 sec and the latter was 0.2 sec. In this case the error in the measured deflection is less than 1 per cent.

IV. NUMERICAL AND EXPERIMENTAL DATA

In this section we shall apply the preceding theory in discussing the various current methods for measuring the thermal conductivity of soils by means of heated cylindrical test bodies. The methods used previously by one of us (de Vries 1952a, 1952b) are treated in detail and new experimental results are presented to illustrate the argument. A brief discussion of probes described by other authors is given in subsection (d).

Thermal properties of various materials used in the construction of probes are listed in the upper part of Table 1, whilst those of soil materials and soils are given in the lower part of this table. The former were taken from handbooks of physical constants, the latter were derived from various sources (see de Vries (1952b) and de Vries and de Wit (1954)).

(a) *Single Wire*

The simplest form of a linear heat source is that of a straight metal wire heated by an electric current. We have used constantan and manganin wires with diameters of 0.01 to 0.02 cm for this purpose. The temperature was measured by means of a thermojunction as close as possible to the centre

of the wire. The thermocouple was made of copper and constantan wires with a diameter of 0.01 cm. All wires were enamelled to secure good electrical insulation between the heating circuit and the temperature-measuring circuit. This method was employed previously by one of us (de Vries 1952*b*) for laboratory determinations of the thermal conductivity of a sand at various temperatures and moisture contents.

TABLE 1  
THERMAL PROPERTIES OF VARIOUS MATERIALS AT 20 °C

Material	Density, $\rho$ (g cm <sup>-3</sup> )	Specific Heat, $c$ (cal g <sup>-1</sup> °C <sup>-1</sup> )	Vol. Heat Capacity, $C$ (cal cm <sup>-3</sup> °C <sup>-1</sup> )	Thermal Conductivity, $\lambda$ (cal cm <sup>-1</sup> sec <sup>-1</sup> °C <sup>-1</sup> )	Thermal Diffusivity, $a$ (cm <sup>2</sup> sec <sup>-1</sup> )
Copper ..	8.89	0.092	0.82	0.92	1.12
Manganin	8.50	0.097	0.82	0.15	0.18
Constantan	8.88	0.099	0.88	0.054	0.061
Monel ..	8.90	0.098	0.87	0.052	0.060
Glass* ..	2.6	0.2	0.5	$2.6 \times 10^{-3}$	$5 \times 10^{-3}$
Paraffin ..	0.89	0.69	0.61	$0.6 \times 10^{-3}$	$1.0 \times 10^{-3}$
Air .. ..	0.0012	0.24	0.00029	$0.062 \times 10^{-3}$	0.21
Quartz ..	2.65	0.175	0.46	0.020	0.043
Many soil minerals*	2.65	0.175	0.46	0.007	0.015
Soil organic matter* ..	1.3	0.46	0.60	$0.6 \times 10^{-3}$	$1.0 \times 10^{-3}$
Soil, mineral, dry* ..	1.50	—	0.26	$0.5 \times 10^{-3}$	$1.9 \times 10^{-3}$
Soil, mineral, saturated*	1.93	—	0.69	$5 \times 10^{-3}$	$7 \times 10^{-3}$
Soil, organic, dry* ..	0.13	—	0.060	$0.08 \times 10^{-3}$	$1.2 \times 10^{-3}$
Soil, organic, saturated*	1.03	—	0.96	$1.2 \times 10^{-3}$	$1.3 \times 10^{-3}$

\* Approximate average values.

For a heating wire of sufficient length the temperature will lie between that given by equations (19) and (20) for the heating branch and equations (26) and (27) for the cooling branch, where in (19) and (26) we must take  $r=R$ . Strictly, we should apply similar equations derived for heat production throughout the wire instead of at its centre, but the differences are negligible here because the conductivity of the wire is large in comparison with that of the soil.

From the data given in Table 1 it can be easily checked that for  $R$ -values of 0.01 cm or less the terms of order  $\tau^{-1}$  and higher negative order in equations (21) and (23) are negligible for  $t > 10$  sec, unless  $\eta$  is very large, say greater than 10. Such large values of  $\eta$  are unlikely when soil is packed around the wire, and a numerical example given below leads to a value of the order of unity.

During our experiments the heating time,  $t_1$ , is usually 180 sec. It follows from equation (28) that the relative error due to axial flow is less than 1 per cent. for  $l > 5$  cm, the actual  $l$ -value being usually 7.5 cm. Finally  $R_1$  in (29) was 5 cm or more, which, with  $a \leq 10^{-2}$  cm<sup>2</sup> sec<sup>-1</sup> and  $t_1 = 180$  sec, leads to  $\exp(-R_1^2/4at_1) \leq 0.03$ .

Results of an experiment conducted with a dry coarse quartz sand (particle sizes ranging from 0.060 to 0.085 cm) and with a manganin heating wire of 0.01 cm diameter are presented in Figure 1 (line *a*). The density of the sand was 1.54 g cm<sup>-3</sup>, its temperature 20 °C. It will be noted that deviations from

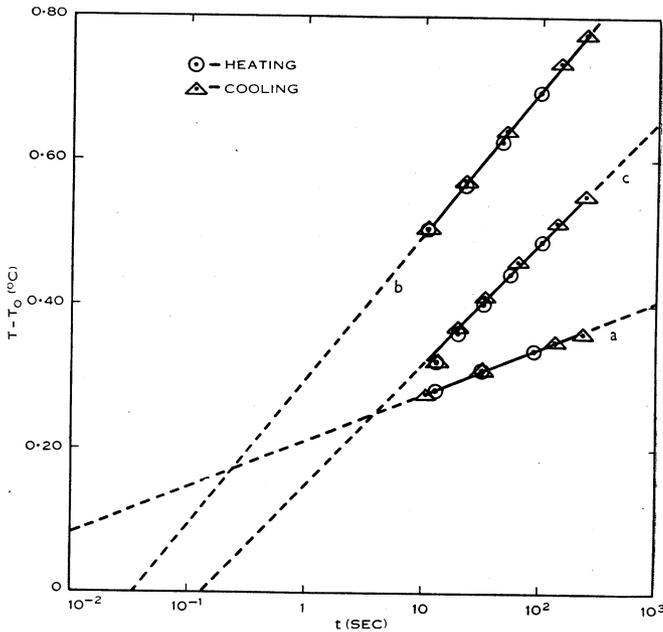


Fig. 1.—Temperature rise,  $T - T_0$ , against time,  $t$ , for thermal conductivity experiments. For the cooling branch time is counted from end of heating (180 sec) onwards and points are corrected for the influence of previous heating (see Section III (*a*)). Line *a*, single heating wire, heat input  $Q = 2.3 \times 10^{-4}$  cal cm<sup>-1</sup> sec<sup>-1</sup>; line *b*, heating wire in glass capillary,  $Q = 8.0 \times 10^{-4}$  cal cm<sup>-1</sup> sec<sup>-1</sup>; line *c*, probe,  $Q = 6.5 \times 10^{-4}$  cal cm<sup>-1</sup> sec<sup>-1</sup>.

the logarithmic relation are negligible. The value of  $\lambda$  following from this experiment was  $0.67 \times 10^{-3}$  cal cm<sup>-1</sup> sec<sup>-1</sup> °C<sup>-1</sup>. Each of the three examples given under (*a*), (*b*), and (*c*) in this section are typical of several runs taken on a sample which gave the same results within the experimental accuracy ( $\pm 5$  per cent. for  $\lambda$ ).

From equation (21) with  $r = R$  it follows that the intercept of an extrapolated line on the horizontal axis is

$$t = (R^2/4a)e^{\gamma-2\eta}, \dots\dots\dots (30)$$

whilst according to equation (24) this intercept becomes

$$t = (R^2/4a)e^{\gamma}. \dots\dots\dots (31)$$

The observed intercept will lie somewhere between these two theoretical values, probably close to the former. When  $\eta=0$  the two intercepts coincide. In our example we find, with  $\eta=0$ ,  $C=0.27 \text{ cal cm}^{-3} \text{ }^\circ\text{C}^{-1}$ , and  $t=5.15 \times 10^{-4} \text{ sec}$ , an  $R$ -value of  $0.0017 \text{ cm}$ . This is less than the actual value of  $0.005 \text{ cm}$ . Substituting the latter in (30) we obtain  $\eta=1.08$ .

A contact resistance will be due to an air gap between the heating wire and the medium. For an annular gap of thickness  $\delta$  we have

$$\eta = (\lambda/\lambda_{\text{air}}) \ln (R + \delta)/R. \quad \dots \dots \dots (32)$$

In the present example this leads to a  $\delta$ -value of  $5 \times 10^{-4} \text{ cm}$ , which is small in comparison with the soil particle size and with the diameter of the wire.

Apparently large positive or even negative values of  $\eta$  have been observed when the galvanometer received a leakage current from the heating circuit. Such a leakage is easily detected by reversing the direction of the heating current.

The methods used in the experiments given in subsections (a), (b), and (c) were previously described in detail by one of us (de Vries 1952a, 1952b). Here we shall by way of example give detailed information on the experiment with the single heating wire.

The resistance per unit length of the wire was determined at  $0.610 \pm 0.005 \text{ } \Omega \text{ cm}^{-1}$ , the heating current was measured at  $0.0400 \pm 0.0005 \text{ A}$ , which leads to a  $Q$ -value of  $(2.33 \pm 0.08) \times 10^{-4} \text{ cal cm}^{-1} \text{ sec}^{-1}$ . The time of heating was 180 sec.

The temperature rise was measured by means of a copper-constantan thermocouple connected to a micro-Moll galvanometer. The sensitivity of this system was  $0.282 \pm 0.003 \text{ }^\circ\text{C}$  per cm deflection on the galvanometer scale. The time was measured with a stopwatch when a line-shaped light mark coincided with a millimetre division on the scale. The following results were obtained:

	Heating			Cooling			
Deflection (mm)	10	11	12	3	2	1	0.7
Time (sec)	13	33	90	191	213	315	420

The accuracy of the time readings was better than  $\pm 0.5 \text{ sec}$ . Towards the end of the cooling run the movement of the light mark became so slow that the uncertainty in the reading of the coincidence of the mark with a scale division was of the order of  $\pm 5 \text{ sec}$ . However, such an error represents only a small distance on the logarithmic time scale for the times concerned.

From a graph of  $T - T_0$  against  $\log t$  the slope of the resulting line is determined at  $0.064 \pm 0.001 \text{ }^\circ\text{C}$ , from which  $\lambda$  is found at  $(0.67 \pm 0.03) \times 10^{-3} \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ }^\circ\text{C}^{-1}$ .

#### (b) Heating Wire in Glass Capillary

A thin heating wire is not always suitable for use in moist soils owing to the fact that relatively large values of the temperature gradient occur close to the wire. This can lead to a strong migration of moisture near the wire. We

therefore also use a heating wire fitted in a glass capillary with an external diameter of about 0.05 cm and internal diameter slightly greater than the wire. In that case the steepest temperature gradients are found in the glass instead of in the soil sample. The thermojunction is located outside the capillary as close as possible to its outer surface.

For  $R$ -values of about 0.025 cm and  $t > 10$  sec the terms of order  $\tau^{-1}$  and  $\tau^{-2}$  are still negligible with mineral soils unless  $\eta$  is large. However,  $\eta$  will be smaller in this case than with the single wire and  $\eta$  will be negligibly small when the diameter of the probe is of the same magnitude as the sizes of the larger soil particles, assuming that the soil is well packed around the probe.

An exception must be made for organic soils of low density at low moisture contents (see Table 1), where the terms with  $\tau^{-1}$  and  $\tau^{-2}$  in (23) are not negligible in comparison with  $\ln \tau - \gamma$ , mainly due to the large value of  $\beta$ . In these loosely packed dry soils large values of  $\eta$  can also be expected.

Errors due to axial flow are again negligible for  $l > 5$  cm. Equation (28) is not strictly applicable in this case because the conductivity of the probe is not necessarily large in comparison with that of the soil. However, this equation will still give a fair estimate of the error because in this case the dominant term is  $\pi^{-\frac{1}{2}} \tau_l^{-\frac{1}{2}} \exp(-\tau_l^{-1})$ . This term is due to deviations from radial flow in the (infinite) medium surrounding the probe and is independent of its diameter and thermal properties.

As an example results of an experiment with a heater of this type on the same coarse sand as mentioned above is given in Figure 1 (line *b*). The density of the dry sand was 1.58 g cm<sup>-3</sup>, its temperature 17 °C. The measured conductivity was 0.73 cal cm<sup>-1</sup> sec<sup>-1</sup> °C<sup>-1</sup>. From the intercept on the horizontal axis (0.035 sec) and equation (31) we find  $R = 0.014$  cm, which is smaller than the actual value, 0.030 cm. Substitution of the latter in (30) leads to  $\eta = 0.38$  and from (32) to  $\delta = 1.0 \times 10^{-3}$  cm.

### (c) *The Thermal Conductivity Probe*

A probe for measuring thermal conductivity of soil *in situ* was developed by one of us (de Vries 1952*a*, 1952*b*). Its construction and dimensions are shown in Figure 2. An automatic recording apparatus to be used in connexion with the probes was described by van Duin and de Vries (1954).

The diameter of the probe is about 0.11 cm. Average values of  $\lambda_i$  and  $C_i$  will be used in finding  $\alpha$  and  $\beta$ . From the composition of the probe its average volumetric heat capacity is found to be 0.63 cal cm<sup>-3</sup> °C<sup>-1</sup>. The average radial conductivity of the probe outside the heating wire is estimated at  $1.0 \times 10^{-3}$  cal cm<sup>-1</sup> sec<sup>-1</sup> °C<sup>-1</sup>. It will be noted that the thermal resistance is mainly located in the paraffin. The distance of the thermojunction from the centre of the heat source is approximately 0.025 cm. Values of  $\eta$  will be negligibly small when there is a good contact between the soil and the probe, except for very coarse sand or soil of very low density.

It can be easily checked that for  $t > 10$  sec terms of order  $\tau^{-1}$  and of higher negative powers of  $\tau$  in equations (21) and (22) are still small in comparison with

In  $\tau-\gamma$  for mineral soils and for wet organic soils. A numerical example is given below. For very loose and dry organic soils these equations can no longer be applied, because the various terms become of the same order of magnitude. Apart from experiments with the latter soils (which are very rare), the theory of the line source can be applied to a probe of this construction to a sufficient approximation.

As an example we give the results obtained with a probe in the same coarse sand as mentioned before at a dry density of  $1.54 \text{ g cm}^{-3}$  and at  $20^\circ \text{C}$  (Fig. 1, line *c*). Slight deviations from the linear relationship occur with the first few points ( $t=13$  and  $20$  sec), which are of the expected magnitude. On such

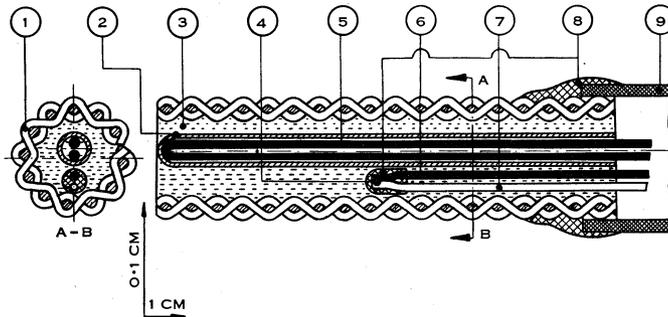


Fig. 2.—Radial and longitudinal cross sections of probe. 1, Monel gauze (filled with paraffin wax); 2, glass capillary; 3, paraffin wax; 4, thermojunction; 5, heating wire; 6, constantan wire; 7, copper wire; 8, insulating cover; 9, plastic socket.

occasions more weight is given to the points at higher  $t$  in drawing the line. In this case the value of  $\lambda$  was found to be  $0.71 \times 10^{-3} \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ }^\circ\text{C}^{-1}$ . With  $C=0.27 \text{ cal cm}^{-3} \text{ }^\circ\text{C}^{-1}$  this leads to  $a=2.63 \times 10^{-3} \text{ cm}^2 \text{ sec}^{-1}$ ,  $\alpha=0.71$ , and  $\beta=2.33$ . At  $t=13$  sec we find  $\tau=45.1$ .

From equation (21) with  $\eta=0$  and the observed intercept on the horizontal axis ( $t=0.13$  sec) we find  $r=0.021$  cm, which is sufficiently close to the estimated value (0.025 cm) to support the assumption  $\eta=0$ . Using equation (21) with  $t=13$  sec we have:  $\ln \tau-\gamma-2\alpha \ln r/R=4.607$  and  $c_{i,-1}\tau^{-1}=-0.214$ . The relative deviation from the straight line due to the term  $c_{i,-1}\tau^{-1}$  is therefore  $-0.214/4.607=-0.046$ . In comparison, the observed value at  $t=13$  is  $-0.045 (\pm 0.005$  due to a possible random error in reading  $t$ ); this includes, of course, the effect of terms with higher negative powers of  $\tau$ . The values of the various terms on the right-hand side of equation (23) are in this case:  $\ln \tau-\gamma=3.234$ ,  $c_{-1}\tau^{-1}=-0.172$ , and  $c_{-2}\tau^{-2}=8.8 \times 10^{-5}$ . With increasing  $\tau$ , i.e. for longer times or for larger  $a$  (moister or denser soil), the deviations from the theory of the line source become even smaller.

According to equation (21) the intercept for  $T=T_0$  is

$$t=(R^2/4a)(r/R)^{2\alpha}e^{\gamma-2\eta} \dots \dots \dots (33)$$

From this equation a check can be obtained on the approximate value of  $\eta$ . This is of special importance in field experiments, where it would otherwise be impossible to know whether or not a sufficiently good thermal contact between the probe and the soil exists. A value of  $a$  based on an estimated value of  $C$  will suffice for this purpose. Such an estimate can be obtained from the dry density of the soil (measured on samples taken when placing the probes) and its approximate moisture content which follows from the measured conductivity.

When  $\eta$  is not negligible its influence on the value of  $\lambda$  can be assessed from equation (21). The term  $2\eta$  has no influence on the slope of the  $T-T_0$  against  $\ln t$  curve. The contribution of  $\eta$  to the correction term of first order is

$$-\frac{4\eta\beta}{\tau} = -\frac{C_i R^2}{\lambda_{air} t} \ln\left(1 + \frac{\delta}{R}\right) \approx -\frac{C_i R \delta}{\lambda_{air} t}, \dots\dots\dots (34)$$

which is independent of the thermal properties of the medium. If, for instance,  $\delta=0.01$  cm we have  $C_i R \delta / \lambda_{air} t = 1.12/t$  and the first order term in  $G_i$  due to  $\eta$  becomes  $-0.112$  and  $-0.0063$  after 10 and 180 sec respectively. The resulting relative error in  $\lambda$  is therefore  $-0.106/\ln 18 = -0.037$ . The contribution of the terms proportional to  $\eta\tau^{-2}$  will be much smaller (see equation (25)).

The possible error in the  $\lambda$ -values following from the probe experiments was discussed by one of us (de Vries 1952*b*); it was found to be about  $\pm 5$  per cent. It was shown in the same paper that the  $\lambda$ -values for a dry quartz sand found from probe experiments were in close agreement with results on similar materials obtained with more conventional stationary methods by Smith and Byers (1938) and Kersten (1949). In addition, it was demonstrated that a theoretical calculation of the thermal conductivity from the composition of the sand led to values which were in good agreement with the experimental ones.

We therefore conclude that the probe method can be applied with confidence as an absolute method for measuring thermal conductivity. This conclusion receives further support from the work of others who compared probe results with those obtained by the guarded hot plate method (see d'Eustachio and Schreiner 1952; Mann and Forsyth 1956). Moreover, the material presented in this section shows that a detailed theoretical interpretation can be given of all aspects of the observed temperature curves.

(d) *Other Probes, General Remarks*

It will be clear from the foregoing examples that it is desirable to keep the probe diameter as small as possible to secure large values of  $\tau$  and thereby small deviations from the simple logarithmic relationship between  $T-T_0$  and  $t$ . These deviations decrease, of course, with increasing time, but large values of  $t$  are themselves undesirable. They necessitate the use of a long probe and large sample containers. In addition, with moist soils the amount of water that moves away from the probe increases with time.

Deviations from the simple theory due to the thickness of the probe can be further reduced by choosing  $\alpha$  small and  $\beta$  close to unity. These requirements

cannot be met entirely in a single probe over the range of  $\lambda$  and  $C$  values found in soils. In some instances a hollow probe will be preferable to a solid one.

Various characteristics of a number of cylindrical probes designed for measuring the conductivity of soils and thermal insulating materials are listed in Table 2 in chronological order of publication. In most cases application of the simple theory will be permissible with these probes, except that of Buettner

TABLE 2  
DIMENSIONS, THERMAL PROPERTIES, AND TIME INTERVALS FOR CYLINDRICAL PROBES

Authors	Diameter, $2R$ (cm)	Length, $2l$ (cm)	Radial Thermal Conductivity, $\lambda_i$ (cal cm <sup>-1</sup> sec <sup>-1</sup> °C <sup>-1</sup> )	Vol. Heat Capacity, $C_i$ (cal cm <sup>-3</sup> °C <sup>-1</sup> )	Time of First Reading $t_0$ (sec)	Time of Heating $t_1$ (sec)
Skeib (1950)	0.11	25	$12 \times 10^{-3}$	0.38	50	100
Hooper and Lepper (1950) ..	0.48	46	$0.08 \times 10^{-3}$	0.37	180	420
de Vries (1952 <i>a</i> , 1952 <i>b</i> ) ..	0.11	13	$1.0 \times 10^{-3}$	0.63	10	180
d'Eustachio and Schreiner (1952)† ..	0.076	10	$0.6 \times 10^{-3}$ *	0.60*	—	—
Mason and K u r t z (1952)† ..	0.63	60	—	—	60	1500
Buettner (1955 <i>a</i> ) ..	0.07	2.5	$1.0 \times 10^{-3}$ *	0.60*	1	10
Bullard <i>et al.</i> (1956) ..	0.086	6.34	—	—	5	600
Mann and Forsyth (1956) ..	0.14	10	$1.0 \times 10^{-3}$ *	0.60*	20	120
Makowski and Mochlinski (1956) ..	0.48	46	$0.08 \times 10^{-3}$	0.37	40	1200

\* Value uncertain.

† Heater in the form of a coil.

(1955*a*), which is too short. Buettner (1955*b*) based his design on the theory for a probe of infinite conductivity ( $\alpha=0$ ). However, his numerical values are in error (cf. Jaeger 1956) and, in addition, the approximation  $\alpha=0$  is not permissible for the smaller  $\tau$ -values in his experiments. This probably is the reason for his use of an empirical calibration constant to be obtained by calibrating the probe in materials with a  $C$ -value close to that of soil.

We wish to emphasize that the determination of more than two points on the  $T-T_0$  against  $\ln t$  curve is highly desirable. A procedure in which the slope of a line is determined from two points only (e.g. Hooper and Lepper 1950; Skeib 1950) provides no clues as to the occurrence of systematic or large random errors. On the other hand a reasonable safeguard is obtained by adopting the following procedure:

- (1) lines are plotted for both the heating and the cooling branches of the temperature curve, which should be found to coincide within the expected experimental error,
- (2) a check is made on the magnitude of a possible contact resistance by the method set out in subsection (c).

For certain applications it is impossible to achieve sufficiently large values of  $\tau$  to apply the simple theory or large time approximations such as equations (21) and (23). This is the case, for instance, with measurements of rock conductivity in boreholes where the probe diameter must be rather large, and in measuring the conductivity of liquids where convection sets in after a certain time. It is then necessary to apply analytical solutions, which so far have only been derived and calculated numerically for infinite conductivity of the probe. In the results reported by van der Held, Hardebol, and Kalshoven (1953) the probe conductivity is not sufficiently great in comparison with that of the measured liquids for the approximation  $\alpha=0$  to hold. This might account for the systematic error reported by these authors.

An interesting modification of the probe method for application to liquids was published recently by Hill (1957). In this method the thermocouple wires also serve to carry the heating current, the latter being applied during a fraction of a second only.

#### (e) *Measuring Thermal Diffusivity*

It has been suggested by some authors (Skeib 1950; Misener 1952) that the thermal diffusivity can also be obtained from a probe experiment, whereas others (Buettner 1955*b*; Beck, Jaeger, and Newstead 1956) have pointed to the difficulties of doing so.

From equation (21) it will be clear that  $a$  can only be found when  $\lambda_i$ ,  $r$ ,  $R$ , and  $\eta$  are known.  $R$  is easy to measure, whilst  $\lambda_i$  and  $r$  could be determined for each probe by calibration in two materials of known conductivity and diffusivity. The value of  $\eta$  depends on the contact between the probe and the soil, which during field experiments may change in the course of time. A determination of  $a$  only seems possible when it can reasonably be expected that  $\eta$  is negligibly small. But even then the accuracy in the value of  $a$  would be small, as it can be easily seen from the lines in Figure 1 that a small relative error in  $\lambda$  would cause a much larger relative error in the intercept on the horizontal axis and thereby in the measured diffusivity. With line  $c$  in Figure 1, for instance, a relative error of 3 per cent. in  $\lambda$  would result in a relative error of about 20 per cent. in  $a$ , assuming that the observed point at  $t=100$  sec is correct.

## V. CONCLUSIONS

It is shown theoretically that the thermal conductivity of soils and materials of similar thermal properties can be accurately measured by the cylindrical probe method. Experimental results in close agreement with the theory are presented.

The outer diameter of the probe should preferably be of the order of 0.1 cm or less, its length of the order of 10 cm. The volumetric heat capacity of the probe should not be large in comparison with that of the observed material, its thermal conductivity should preferably not be small in comparison with that of the material.

An estimate of the contact resistance coefficient can be obtained from the observed temperature rise and an estimate of the volumetric heat capacity of the material. The influence of the contact resistance on the value of the thermal conductivity can then be assessed from the theory; it is negligible when the soil is well packed around the probe.

The thermal diffusivity can be found when the contact resistance coefficient is negligibly small or accurately known, but no high degree of accuracy is to be expected.

## VI. REFERENCES

- BECK, A., JAEGER, J. C., and NEWSTEAD, G. (1956).—*Aust. J. Phys.* **9**: 286–96.  
 BLACKWELL, J. H. (1954).—*J. Appl. Phys.* **25**: 137–44.  
 BLACKWELL, J. H. (1956).—*Canad. J. Phys.* **34**: 412–7.  
 BUETTNER, K. (1955a).—*Trans. Amer. Geophys. Un.* **36**: 827–30.  
 BUETTNER, K. (1955b).—*Trans. Amer. Geophys. Un.* **36**: 831–7.  
 BULLARD, E. C., MAXWELL, A. E., and REVELLE, R. (1956).—*Advanc. Geophys.* **3**: 153–81.  
 CARSLAW, H. S., and JAEGER, J. C. (1948).—“Conduction of Heat in Solids.” p. 221. (Clarendon Press: Oxford.)  
 CHUDNOWSKII, A. F. (1946).—*Proc. Lenin. Acad. Agric. Sci. U.S.S.R.* **1946**: 38–43.  
 CHUDNOWSKII, A. F. (1954).—*Pochvovedenie* **1954**: 64–71.  
 VAN DUIN, R. H. A., and DE VRIES, D. A. (1954).—*Neth. J. Agric. Sci.* **2**: 168–75.  
 D'EUSTACHIO, D., and SCHREINER, R. E. (1952).—*Heat. Pip. Air Condit.* **1952**: 113–7.  
 GILLAM, D. G., and LAMM, O. (1955).—*Acta Chem. Scand.* **9**: 657–60.  
 GILLAM, D. G., ROMBÉN, L., NISSEN, H. E., and LAMM, O. (1955).—*Acta Chem. Scand.* **9**: 641–56.  
 VAN DER HELD, E. F. M., and VAN DRUNEN, F. G. (1949).—*Physica* **15**: 865–81.  
 VAN DER HELD, E. F. M., HARDEBOL, J., and KALSHOVEN, J. (1953).—*Physica* **19**: 208–15.  
 HILL, R. A. W. (1957).—*Proc. Roy. Soc. A* **239**: 476–86.  
 HOOPER, F. C. (1952).—Highway Research Board, Special Rep. No. 2: 57–9.  
 HOOPER, F. C., and LEPPER, F. R. (1950).—*Heat. Pip. Air Condit.* **1950**: 129–35.  
 JAEGER, J. C. (1956).—*Aust. J. Phys.* **9**: 167–79.  
 KERSTEN, M. S. (1949).—Bull. Univ. Minn. Engng. Exp. Sta. No. 28.  
 MAKOWSKI, M. W., and MOCHLINSKI, K. (1956).—*Proc. Instn. Elect. Engrs.* **A 103**: 453–70.  
 MAKOWSKI, M. W., and MOCHLINSKI, K. (1957).—*Nature* **179**: 778–9.  
 MANN, G., and FORSYTH, F. G. (1956).—*Mod. Refrig.* **59**: 188–91.  
 MASON, V. V., and KURTZ, M. (1952).—*Trans. Amer. Inst. Elect. Engrs.* **71**: 570–7.  
 MISENER, A. D. (1952).—Highway Research Board, Special Rep. No. 2: 51–7.  
 SCHLEIERMACHER, A. L. E. F. (1888).—*Wied. Ann. Phys.* **34**: 623.  
 SKEIB, G. (1950).—*Z. Met.* **4**: 32–9.  
 SMITH, W. O., and BYERS, H. G. (1938).—*Proc. Soil Sci. Soc. Amer.* **3**: 13–9.  
 STÅLHANE, B., and PYK, S. (1931).—*Tekn. Tidskr.* **61**: 389–93.  
 DE VRIES, D. A. (1952a).—*Soil Sci.* **73**: 83–9.

- DE VRIES, D. A. (1952b).—*Meded. Landbhogesch., Wageningen* **52**: 1-72.
- DE VRIES, D. A. (1953).—*Neth. J. Agric. Sci.* **1**: 115-21.
- DE VRIES, D. A. (1956).—*Nature* **178**: 1074.
- DE VRIES, D. A., and PECK, A. J. (1958).—On the cylindrical probe method of measuring thermal conductivity with special reference to soils. II. Analysis of moisture effects. *Aust. J. Phys.* **11** (3). (In press.)
- DE VRIES, D. A., and DE WIT, C. T. (1954).—*Met. Rdsch.* **7**: 41-5.
- WEBB, J. (1956).—*Nature* **178**: 1074.
- WEBB, J. (1957).—*Nature* **179**: 779.
- WEISHAAPT, J. (1940).—*ForschArb. IngWes.* **11**: 20-35.