ELECTRON EXCITATION OF COLLECTIVE NUCLEAR TRANSITIONS

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Summary

The inelastic scattering of high energy electrons with excitation of collective nuclear transitions is treated using a simple hydrodynamical model in which the collective nuclear motion is assumed to be irrotational and incompressible. The effects of nuclear compressibility are discussed. Using the Born approximation, scattering form factors are calculated for several charge distributions for electric quadrupole transitions, and the sensitivity of the scattering to the form of the nuclear charge distribution is examined.

I. Introduction

In a previous paper (Tassie 1956) the inelastic scattering of high energy electrons by nuclei was considered using a modified liquid drop model of the nucleus which allows for non-uniform nuclear charge and mass density distributions. The collective nuclear motion was assumed to be irrotational and incompressible, and the Born approximation was used to calculate the electron scattering using several simple forms for the nuclear charge distribution.

It is the purpose of the present paper to discuss the incompressibility assumption, to extend the calculations of the inelastic electron scattering using more realistic forms for the nuclear charge distribution, and to examine how sensitive this scattering is to the choice of the nuclear charge distribution.

II. General Theory

The differential cross section for the inelastic scattering of high energy electrons with excitation of a nuclear electric 2^L-pole transition is (Schiff 1954)

$$\mathrm{d}\sigma/\mathrm{d}\omega=(\mathrm{d}\sigma/\mathrm{d}\omega)_p\,|\,\mathscr{F}_L\,|^2, \qquad (1)$$

where $(\mathrm{d}\sigma/\mathrm{d}\omega)_p=\frac{1}{4}Z^2(e^2/\hbar c)^2k^{-2}\cos\frac{1}{2}\theta\,\mathrm{cosec}^4\,\frac{1}{2}\theta$ is the point charge scattering cross section, and

$$\mathscr{F}_L=4\pi(2L+1)^{\frac{1}{2}}\int \mathrm{j}_L(Qr)Y_{L,0}\,\rho_{\mathrm{trans}}\mathrm{d}V \qquad (2)$$

is the nuclear form factor. $Ze\rho_{\mathrm{trans}}$ is the transition charge density of the nucleus, $\hbar k$ is the momentum of the incident electron, and $q=2k\sin\frac{1}{2}\theta$.

For the nuclear model used here, equation (2) becomes (Tassie 1956)

$$\mathscr{F}_L=2\pi^{\frac{1}{2}}Q_{L,0}I_L/Ze(2L+1)^{\frac{1}{2}}\overline{r^{2(L-1)}} \qquad (3)$$

where

$$Q_{L,0}=Ze\int r^LY_{L,0}\rho_{\mathrm{trans}}\mathrm{d}V \qquad (4)$$

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is the nuclear transition 2^L-pole moment, and

$$I_L = 4\pi q \int_0^\infty \mathrm{j}_{L-1}(qr) r^{L+1} \rho(r) \mathrm{d}r. \quad \ldots\ldots\ldots\ldots \quad (5)$$

$Ze\rho(r)$ is the static charge density of the nucleus. The angular dependence of $\mathscr{F}_L$ is then given completely by I_L.

For elastic scattering by a spherically symmetrical nucleus, the cross section is given by equation (1) with

$$\mathscr{F}_{\text{elastic}} = 4\pi \int_0^\infty \mathrm{j}_0(qr) \rho(r) r^2 \mathrm{d}r. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Then equation (5) can be written

$$I_L = (-1)^{L+1} q^{L-1} \left(\frac{\partial}{\partial q} q^{-1} \right)^{L-2} \frac{\partial}{\partial q} \mathscr{F}_{\text{elastic}}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

and in particular, for electric quadrupole ($E2$) transitions we have

$$I_2 = -q \partial \mathscr{F}_{\text{elastic}} / \partial q. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

As $q \to 0$,

$$I_2 \to \tfrac{1}{3} q^2 \langle r^2 \rangle, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

where

$$\langle r^2 \rangle = 4\pi \int_0^\infty \rho(r) r^4 \mathrm{d}r. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

The values of $\langle r^2 \rangle$ are required so that the transition electric quadrupole moments can be determined by comparing I_2 with experimental results for the electron scattering form factors.

The above results are obtained by using the Born approximation, and the reliability of this must be examined. The plane wave of the incident electron is distorted by the Coulomb field of the nucleus arriving at the nucleus looking very much like a plane wave with modified amplitude and wave number and slightly curved wave fronts. Downs, Ravenhall, and Yennie (1957) have considered the inelastic scattering using a perturbation method which includes the effect of this distortion, and their results are similar to the Born approximation results obtained using a slightly modified q and with a partial filling in of the Born approximation diffraction minima. A change in q in equations (5) or (6) is equivalent to a change in the size of the nucleus, so that the effect of the modification of q can be taken into account by correcting nuclear size parameters which have been derived from experiment using the Born approximation. This procedure has been used by Fregeau (1956) and by Helm (1956).

III. Nuclear Compressibility

The effects of nuclear compressibility in the liquid drop model have been investigated assuming the nucleus has a sharp surface (Woeste 1952). However, since the nucleus has a diffuse surface (Hofstadter 1956), the energy density of the nuclear fluid must depend on the derivatives of the nuclear density (Swiatecki

1950), and this leads to complicated hydrodynamical equations. The problem is considerably simplified by neglecting compressibility, so that

$$\nabla^2\Phi=0, \qquad (11)$$

where Φ is the velocity potential of the nuclear fluid, and the results of Section I are then obtained.

Another approach is to rely on the sharp-edged liquid drop model for an estimate of the effect of compressibility, and to use this estimate for a nucleus with a diffuse edge. Then compressibility can be neglected if the velocity of the fluid is everywhere small compared to the velocity of " sound " in the nucleus. For quadrupole vibrations satisfying (**11**), the maximum velocity is given by

$$(v_{\max.}/C_s)^2=(5/3)\hbar\omega/AmC_s^2, \qquad (12)$$

and occurs at the edge of the nucleus. $\hbar\omega$ is the one-phonon energy and C_s is the velocity of sound in the nuclear fluid. Using the hydrodynamical estimates of $\hbar\omega$ given by Bohr and Mottelson (1953) and the value given by Rosenfeld (1948) for C_s, we obtain

$$(v_{\max.}/C_s)^2\simeq 6\cdot 5A^{-3/2}. \qquad (13)$$

Thus, for very light nuclei the incompressible approximation is not justified, but a collective treatment is not strictly applicable to light nuclei. For heavy nuclei, equation (**11**) gives some justification to the use of the incompressible approximation, e.g. for $A=16$, $(v_{\max.}/C_s)^2\simeq 0\cdot 1$.

The possible indirect effect of compressibility on the scattering form factor by its effect on the static charge distribution of the nucleus is discussed at the end of Section IV.

IV. Calculations

The following functional forms have been used for the nuclear charge distribution :

Fermi :

$$\rho_F(r)=\rho_{0F}/\{\exp\,[(r-c_F)/z_F]+1\}, \qquad (14)$$

Trapezoidal :

$$\left.\begin{aligned}\rho_T(r)&=\rho_{0T}, & r&\leqslant c_T-z_T,\\ &=\rho_{0T}(c_T+z_T-r)/2z_T, & c_T-z_T&\leqslant r\leqslant c_T+z_T,\\ &=0, & r&\geqslant c_T+z_T.\end{aligned}\right\} \qquad (15)$$

Modified Gaussian :

$$\rho_G(r)=\rho_{0G}/\{\exp\,[(r^2-c_G^2)/z_G^2)]+1\}. \qquad (16)$$

Three-parameter :

$$\rho_M(r)=\rho_{0M}[1+(wr^2/c_M^2)]/\{\exp\,[(r-c_M)/z_M]+1\}. \qquad (17)$$

Hahn, Ravenhall, and Hofstadter (1956) have used these forms for the charge distribution in analysing elastic scattering.

Blankenbecler (1957) has obtained an approximate expression for the Born approximation elastic scattering form factor for a Fermi charge distribution. The correction to this can be obtained as a power series in exp $(-c_F/z_F)$, so that we finally obtain

$$\begin{aligned}\mathscr{F}_{eF}=&4\pi^2\rho_{0F}c_Fz_Fq^{-1}\operatorname{cosech}\pi qz_F\\ &\times(\pi z_Fc_F^{-1}\sin qc_F\operatorname{cotanh}\pi qz_F-\cos qc_F)\\ &+8\pi\rho_{0F}z_F^3\sum_{n=1}^{\infty}(-1)^{n-1}n\exp[-nc_F/z_F]/(n^2+q^2z_F^2)^2, \quad \ldots\ldots\ldots (\mathbf{18})\end{aligned}$$

$$\rho_{0F}=(3/4\pi c_F^3)\{1+(\pi z_F/c_F)^2-\tfrac{1}{3}(z_F/c_F)^3\sum_{n=1}^{\infty}(-1)^{n-1}\exp[-nc_F/z_F]/n^3\}^{-1}. \quad \ldots (\mathbf{19})$$

Using equation (**8**), we obtain

$$\begin{aligned}I_{2F}=&4\pi^2\rho_{0F}q^2c_F^5\\ &\times\{(z_F/c_F)\operatorname{cosech}\pi qz_F[(qc_F)^{-3}(\pi z_Fc_F^{-1}\sin qc_F\operatorname{cotanh}\pi qz_F-\cos qc_F)\\ &-(qc_F)^{-2}(\sin qc_F(1-\pi^2z_F^2c_F^{-2}-2\pi^2z_F^2c_F^{-2}\operatorname{cosech}^2\pi qz_F)\\ &+2\pi z_Fc_F^{-1}\operatorname{cotanh}\pi qz_F\cos qc_F)]\\ &+8\pi^{-1}z_F^5c_F^{-5}\sum_{n=1}^{\infty}(-1)^{n-1}n\exp[-nc_F/z_F](n^2+q^2z_F^2)^{-3}\}. \quad \ldots\ldots (\mathbf{20})\end{aligned}$$

Also*

$$\begin{aligned}\langle r^2\rangle=&4\pi\rho_{0F}c_F^5\{[3+10(\pi z_F/c_F)^2+7(\pi z_F/c_F)^4]/15\\ &+120(z_F/c_F)^2\sum_{n=1}^{\infty}(-1)^{n-1}n^{-5}\exp[-nc_F/z_F]\} \quad \ldots (\mathbf{21})\\ \simeq&3c_F^2\{1+7(\pi z_F/c_F)^2/3\}/5. \quad \ldots\ldots\ldots\ldots (\mathbf{22})\end{aligned}$$

The corresponding expressions for the trapezoidal charge distribution are easily obtained:

$$\rho_{0T}=3\{4\pi c_T^3[1+(z_T/c_T)^2]\}^{-1}, \quad \ldots\ldots\ldots\ldots (\mathbf{23})$$

$$\begin{aligned}I_{2T}(q)=&4\pi\rho_{0T}(z_Tq^4)^{-1}\\ &\times\{-(qz_T)^2\sin qz_T\sin qc_T\\ &+\sin qz_T[(8-(qc_T)^2)\sin qc_T-5qc_T\cos qc_T]\\ &+qz_T\cos qz_T[2qc_T\cos qc_T-5\sin qc_T]\}, \quad \ldots (\mathbf{24})\end{aligned}$$

$$\langle r^2\rangle=(4\pi\rho_{0T}c_T^5/15)[1+3(z_T/c_T)^2][3+(z_T/c_T)^2]. \quad \ldots (\mathbf{25})$$

The three distributions, equations (**12**), (**13**), (**14**), reduce to the uniform charge distribution for $z=0$ and then

$$I_2(q)=3\mathrm{j}_2(qc). \quad \ldots\ldots\ldots\ldots (\mathbf{26})$$

The mean square radii, $\langle r^2\rangle$, are given in Table 1, and Figures 1 and 2 show the values of I_2 for the Fermi and trapezoidal distributions respectively. The

* A derivation of equation (**21**) is given by Elton (1958), but the expression itself is stated incorrectly. The correction of this mistake slightly alters the nuclear radius parameters obtained by Elton but does not affect his conclusions (Elton, personal communication).

ratio of the inelastic scattering to point charge scattering, which is proportional to $|I_2|^2$, will consist of a series of diffraction maxima separated by zeros. The main effect of increasing the thickness of the nuclear surface is to decrease the higher order diffraction maxima.

The zeros of $|I_2|^2$ are due to the use of the first Born approximation. In a more accurate treatment, in which I_2 is complex, these zeros would become diffraction minima (Downs, Ravenhall, and Yennie 1957). In general, the excited state of the nucleus will not be completely described by collective excitation, but will also consist of some single-particle excitation as in the Bohr-Mottelson collective model (Bohr and Mottelson 1953). This admixture

TABLE 1

$$\langle r^2\rangle = 4\pi\int_0^\infty \rho(r)r^4\mathrm{d}r$$

	Fermi Charge Distribution (Equation (**14**))					
z_F/c_F	0	0·08	0·16	0·2060	0·1287	0·0839
$\langle r^2\rangle/c_F^2$	0·6000	0·6884	0·9537	1·1856	0·8289	0·6971
	Trapezoidal Charge Distribution (Eqn. (**15**))			Modified Gaussian Distribution (Eqn. (**16**))	Three-Parameter Distribution (Eqn. (**17**))	
z/c	0·2373	0·4745	0·7504	0·5657	0·1010	
w	—	—	—	—	0·64	
$\langle r^2\rangle/c^2$	0·6764	0·8822	1·2261	0·8501	0·8204	

of single-particle excitation can also be expected to fill in the diffraction minima to some extent. However, the corrections to the Born approximation and the admixture of single-particle excitation should not greatly affect the magnitude of the diffraction maxima, and, for this reason, a useful quantity for comparison with experiment should be the ratio (*second maximum of* $|\mathrm{I}_2|^2$)/(*first maximum of* $|\mathrm{I}_2|^2$)$=(\mathrm{M}_2/\mathrm{M}_1)^2$. M_2 is the first non-zero minimum of I_2, and M_1 is the first maximum of I_2.

M_2/M_1 has been obtained by using the above results for I_2 for the Fermi and trapezoidal charge distributions, and is shown in Figure 3 as a function of $(z/c)^2$. Hahn, Ravenhall, and Hofstadter (1956) have used a parameter t, the surface thickness, which is defined as the distance between the points where ρ has 0·9 and 0·1 of its maximum value. The scales of $(z_F/c_F)^2$ and $(z_T/c_T)^2$ in Figure 3 have been chosen so that the figure also shows M_2/M_1 as a function of $(t/c)^2$. For this purpose the relations, $t=1\cdot 60z_T$, $t=(4\ \ln 3)z_F=4\cdot 40z_F$,* have been used.

* To be exact

$$t=\{4\ln 3+\ln [1-(80/9)(10+\exp (c_F/z_F))^{-1}]\}z_F.$$

The scale of $(t/c)^2$ for the Fermi distribution has a maximum error of 2 per cent.

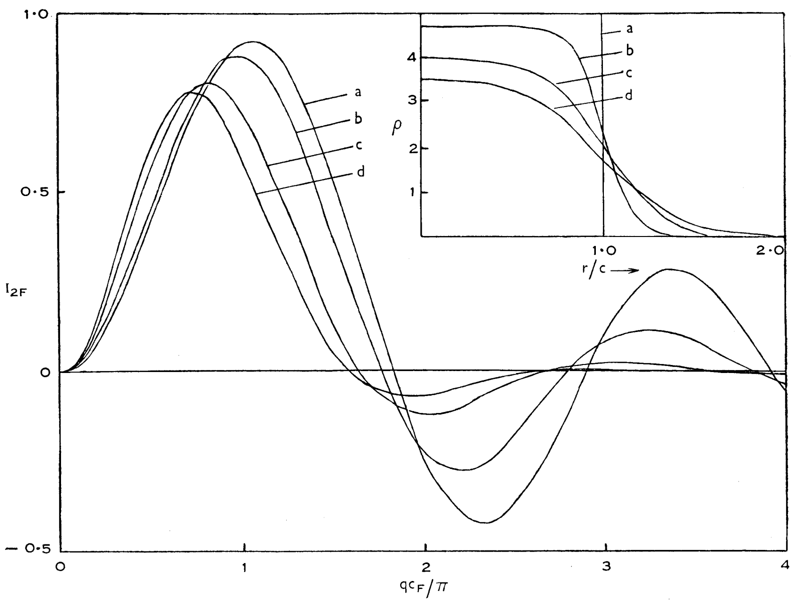

Fig. 1.—$I_{2F}(q)$, equation (**5**), for the Fermi charge distribution, equation (**14**), for the following values of z_F/c_F: (*a*) 0, (*b*) 0·08, (*c*) 0·16, (*d*) 0·206. The charge density distributions for these values of z_F/c_F are also shown. For $z_F/c_F=0$, the Fermi distribution becomes the uniform charge distribution and then $I_{2F}(q)=3j_2(qc_F)$, curve *a*.

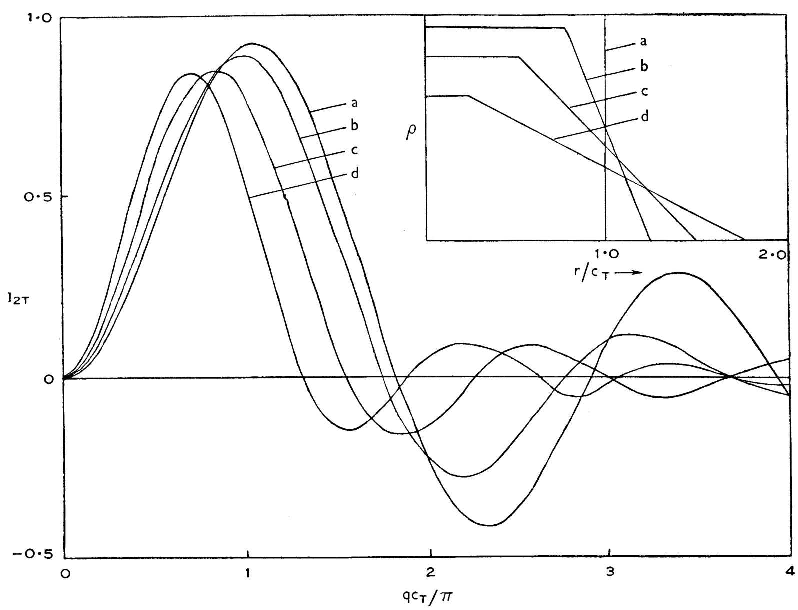

Fig. 2.—$I_{2T}(q)$, equation (**5**), for the trapezoidal charge distribution, equation (**15**), for the following values of z_T/c_T: (*a*) 0, (*b*) 0·2373, (*c*) 0·4745, (*d*) 0·7504. The charge density distributions for these values of z_T/c_T are also shown. For $z_T/c_T=0$, the trapezoidal distribution becomes the uniform charge distribution and then $I_{2T}(q)=3j_2(qc_T)$, curve *a*.

The figure indicates that I_2 is to some extent sensitive to the functional form of ρ, more especially for large values of t. It is seen that it is impossible to obtain small values of M_2/M_1 for the trapezoidal charge distributions.

I_2 for the modified Gaussian distribution, equation (**16**), and the three-parameter distribution, equation (**17**), have been obtained by numerical integration of equation (**5**). Figure 4 shows I_{2G} for $(z_G/c_G)^2=0\cdot32$ compared with the $I_{2F}(z_F/c_F=0\cdot1287)$ which has the same value of M_2/M_1. The scale of qc_F/π has been chosen to match the second zeros of the two curves. Thus, in

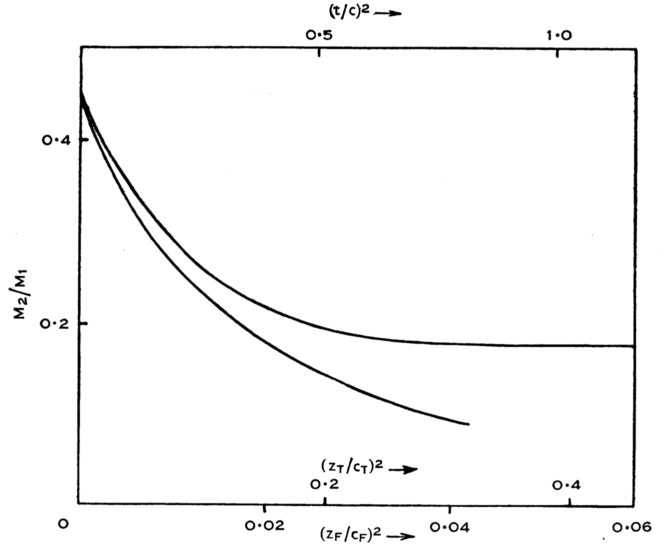

Fig. 3.—M_2/M_1 for the Fermi charge distribution, equation (**14**), as a function of $(z_F/c_F)^2$, and for the trapezoidal charge distribution, equation (**15**), as a function of $(z_T/c_T)^2$. The scales of $(z_F/c_F)^2$ and $(z_T/c_T)^2$ are chosen so that the figure gives M_2/M_1 as a function of $(t/c)^2$. t is the surface thickness, which is defined as the distance between the points where ρ has $0\cdot9$ and $0\cdot1$ of its maximum value. M_2 is the first non-zero minimum of I_2. M_1 is the first maximum of I_2.

Figure 4 the scattering form factor of a nucleus with a modified Gaussian charge distribution and a transition quadrupole moment Q_{L0} is compared with the scattering form factor for a nucleus with a Fermi charge distribution with radius $c_F=c_G/0\cdot95$, and a quadrupole moment $Q_{L0}/(0\cdot95)^3$.

I_{2M} for the three-parameter charge distribution for $z_M/c_M=0\cdot1010$ and $w=0\cdot64$ is compared in Figure 5 with I_{2F} for the Fermi distribution with $z_F/c_F=0\cdot0839$, taking $c_F=1\cdot052c_M$. Hahn, Ravenhall, and Hofstadter (1956) have made accurate calculations of the elastic scattering by these two charge distributions with these values of the parameters and show that the experimental elastic scattering cannot distinguish between these two charge distributions. From Figure 5 it is seen that it would be very difficult to distinguish between these two charge distributions by measurement of the collective electric quadrupole inelastic scattering.

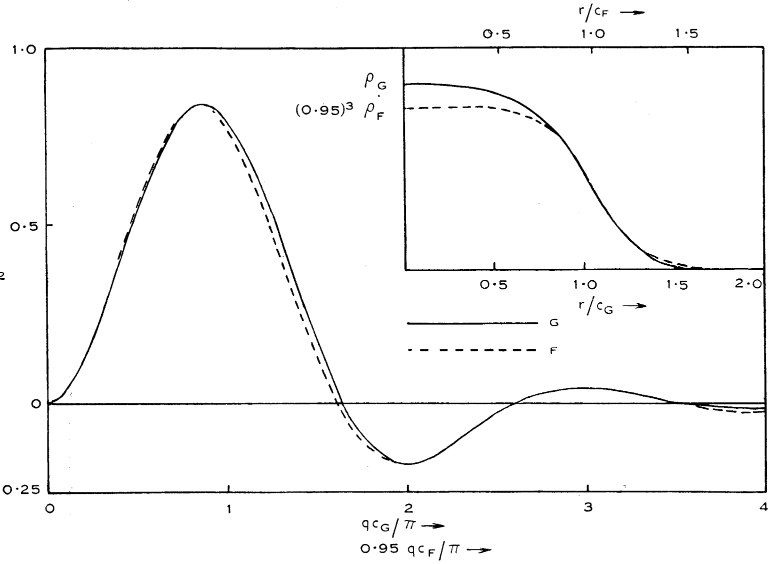

Fig. 4.—$I_{2G}(q)$, equation (**5**), for the modified Gaussian charge distribution, equation (**16**), $(z_G/c_G)^2=0\cdot32$, curve G, compared with $I_{2F}(q)$ for the Fermi charge distribution, equation (**14**), with $z_F/c_F=0\cdot1287$. The two charge distributions are also shown.

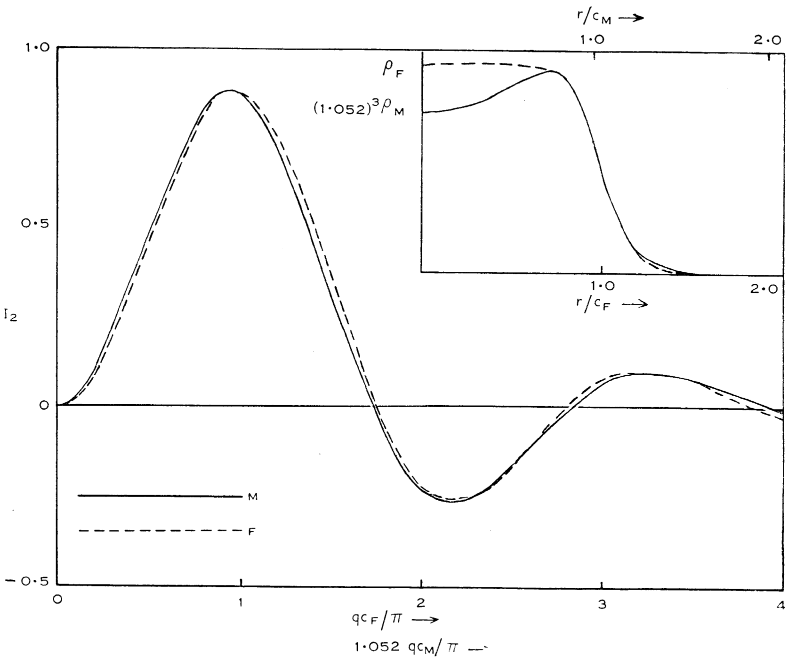

Fig. 5.—$I_{2M}(q)$, equation (**5**), for the three-parameter charge distribution, equation (**17**), with $z_M/c_M=0\cdot1010$ and $w=0\cdot64$, curve M, compared with $I_{2F}(q)$ for the Fermi charge distribution, equation (**14**), with $z_F/c_F=0\cdot0839$. The two charge distributions are also shown.

The situation described by the three-parameter charge distribution, with the charge density tending to be smaller in the middle of the nucleus, is expected to arise from the effects of nuclear compressibility (Woeste 1952). Since this static effect of nuclear compressibility causes only a small modification to $I_2(q)$, it seems safe to neglect this effect of compressibility at least for the range of qc considered here.

The mean square radii of the various charge distributions are given in Table 1.

V. Discussion

The inelastic scattering form factor is not sensitive to the form of the charge density distribution except for nuclei with very diffuse surfaces, i.e. for large values of the surface thickness. The most satisfactory test of the theory given here would be the experimental determination of (M_2/M_1), the ratio of the second and first peaks in $|\mathscr{F}_2|$. Helm (1956) has performed experiments on the electron excitation of several electric quadrupole $(0^+ \rightarrow 2^+)$ nuclear transitions, but his results are not sufficiently comprehensive to allow the determination of $(M_2/M_1)^2$. Other effects, such as the possibility of some single-particle excitation, must be considered in more detail before comparing the theoretical and experimental angular distributions of the inelastically scattered electrons.

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