RADIATION TRANSFER AND THE POSSIBILITY OF NEGATIVE ABSORPTION IN RADIO ASTRONOMY

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Summary

Stimulated transitions are relatively enormously more probable at radio than at optical frequencies and it is this which makes it possible for negative absorption to arise at radio wavelengths when the medium will behave like an amplifier to the incident radiation. A necessary condition for the existence of this phenomenon is that the kinetic energy distribution $F(\eta)$ of the radiating electrons be markedly non-thermal with an appreciable excess of high energy electrons such that $\partial F/\partial\eta$ is positive over a finite range of the kinetic energy η. However, this condition is not sufficient, since it is shown that an electron gas in which free-free transitions provide the dominant radiation process can never exhibit negative absorption whatever the form of $F(\eta)$, and it is further necessary that the stimulated transition probability should have a maximum at some finite value of the kinetic energy, the most favourable case occurring when this maximum is a sharp one at the value of η at which $\partial F/\partial\eta$ has a positive maximum. These conditions can both be met in principle for the cases in which the dominant radiation process is due (*a*) to Cerenkov effect, (*b*) to gyro radiation by non-relativistic electrons, (*c*) to synchrotron-type radiation by highly relativistic electrons, and it is shown that negative absorption can arise in all these cases; the relevance of these results to radio astronomy is discussed briefly.

I. Introduction

The fundamental equations governing the transfer of radiation in an ionized medium are of the same general form at both radio and optical wavelengths. However, the physical nature of the solution in a given case depends upon the specific processes governing the emission, absorption, and scattering of radiations; since these are very different at optical and at radio wavelengths, one may expect to find corresponding differences between the characteristics of radiation transfer.

Thus at optical wavelengths the transfer of radiation is appreciably affected by the radiative transitions between bound states which give rise to the characteristic spectral lines. However, at radio wavelengths the number of discrete lines such as the 1420 Mc/s hyperfine line of atomic hydrogen is extremely limited. Furthermore, the few lines that do exist are likely only to be observable in interstellar gas clouds in which the density is so low that the refractive index differs negligibly from unity. Under these conditions the effect of these spectral lines on the transfer of radiation is an extremely simple case of the effect encountered at optical wavelengths and will not be discussed here. Instead, we shall deal only with the case where radiation is emitted or absorbed in free

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electron transitions in which the available electron energy is much greater than $h\nu$, the energy carried by a single quantum of radiation. We shall further assume that the radiation is incoherent in origin in the sense that the probability of a radiative transition may depend upon the *average* density of matter and radiation but not upon any time-dependent variation in these. Hence we shall not consider the case where the radiation is produced by plasma oscillations, a coherent process, but only processes such as those involved in free-free transitions, Cerenkov radiation, gyro radiation, and synchrotron radiation.*

II. Comparison between the Optical and Radio Cases

If the density of a radiation field at a frequency ν is equal to that of a black-body radiation field at temperature Θ_ν, then, in a typical stellar atmosphere, the dimensionless ratio $k\Theta_\nu/h\nu$ lies usually between 1 and $0\cdot1$ at optical wavelengths. However, at a frequency of 10^8 c/s, which lies in the metre wave radio band, $k\Theta_\nu/h\nu \sim 10^8$ for the quiet Sun and may be as high as 10^{14} in the neighbourhood of a solar burst; similarly, near the intense radio sources Cassiopeia-A and Cygnus-A, $k\Theta_\nu/h\nu \sim 10^8$ and 10^9 respectively for $\nu \sim 10^8$ c/s, and even in interstellar space, not in the immediate neighbourhood of a radio source, $k\Theta_\nu/h\nu \sim 10^4$ at this frequency. One consequence of this is that stimulated emission is relatively much more important at radio than at optical wavelengths, since, in the simple case in which the refractive index μ of the medium may be taken as unity, it follows immediately from the radiation laws† that γ, the ratio of the probability of stimulated to spontaneous emission, is given by

$$\gamma = [\exp(h\nu/k\Theta_\nu) - 1]^{-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

At optical wavelengths it is usually the case that

$$\gamma \sim \exp(-h\nu/k\Theta_\nu) < 10^{-1},$$

but at radio wavelengths, where in general

$$\gamma \sim k\Theta_\nu/h\nu,$$

stimulated transitions can be enormously more probable than spontaneous transitions. In the non-equilibrium case, in which the density distribution over the accessible energy states of the matter field differs, by definition, from the thermal one, the relatively large probability for stimulated transitions may have important consequences, as we shall show below.

Equation (**1**), which is valid when $\mu = 1$, may reasonably be expected to apply under the conditions prevalent in the discrete radio sources or in interstellar space, but not under the conditions prevalent in a stellar corona. In the latter case, not only will μ be different from unity but, in the presence of magnetic fields such as those associated with sunspots, the medium will be birefringent, in strong contrast to the conditions at optical wavelengths when it is valid to take $\mu = 1$ even in the stellar photosphere.

* By gyro radiation we mean radiation at the fundamental or at the first few harmonics of the gyro frequency by weakly relativistic electrons rotating in a magnetic field. By synchrotron radiation we mean radiation by strongly relativistic electrons at high harmonics of the gyro frequency.

† See e.g. Fowler (1936), Chapter XIX.

A further characteristic of the radio case is that the emission and absorption of radiation is often markedly anisotropic. This is especially the case for synchrotron radiation at or near the critical frequencies in a plasma permeated by a magnetic field ; in the optical case, on the other hand, the emission at least can almost always be taken as isotropic.

Fortunately, these features in which the radio case is more complex than the optical are more than balanced by others for which the reverse is true. In the first place, the effects of incoherent microscopic scattering, which are so important in the transfer of optical radiation, can almost always be completely ignored in the radio case (Smerd, personal communication 1958), when appreciable scattering will only arise from irregular macroscopic variations in the electron density. Secondly, the kinetic energy of a typical particle of the matter field is so much greater than $h\nu$ that the emission or absorption of a radio-frequency quantum has a negligible effect upon the overall energy distribution of the matter field ; a fact that provides the principal justification for the use of a classical radiation theory in calculating the transition probabilities at radio wavelengths.

III. Emission and Absorption Coefficients at Radio Frequencies

When the refractive index of the medium μ_ν is not equal to unity the equation of radiation transfer can be written in the form (Woolley 1947)

$$\mu_\nu^2 \mathrm{d}(I_{(\nu)}/\mu_\nu^2)/\mathrm{d}\mathbf{s} = j_\nu - \mathbf{K}_\nu I_{(\nu)}, \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

where $I_{(\nu)}$ is the intensity of radiation in the direction $\mathbf{s}$ and where j_ν, $\mathbf{K}_\nu$ are the volume coefficients of emission and absorption appropriate to the direction $\mathbf{s}$.

When the medium is birefringent a separate transfer equation must be set up for each mode. In the general case, in which macroscopic scattering must be allowed for, these two transfer equations will be coupled by terms in the absorption coefficient which represent scattering from one mode to the other. However, we shall ignore macroscopic scattering in what follows, and in this case, as has been pointed out by Martyn (1948), $\mathbf{K}_\nu$ simply represents the difference between the effects of stimulated absorption and stimulated emission in a single mode.

(a) *The Absorption Coefficient*

Accordingly, if n_{r2}, n_{r1} are the numbers of electrons in the r_2 and r_1 states respectively, if the energy difference between these states is $h\nu$, and if $B_{r,12}I(\nu)\mathrm{d}\nu$, $B_{r,21}I(\nu)\mathrm{d}\nu$ are the stimulated absorption and emission probabilities respectively for radiation with intensity $I(\nu)$ in unit solid angle in the direction $\mathbf{s}$, then

$$\mathbf{K}_\nu = \sum_r (n_{r1}B_{r,12} - n_{r2}B_{r,21})h\nu. \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

The Einstein coefficients $B_{r,12}$ and $B_{r,21}$ are not independent but are related by the equation

$$\tilde{\omega}_{r2}B_{r,21} = \tilde{\omega}_{r1}B_{r,12}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where $\tilde{\omega}_{r2}$, $\tilde{\omega}_{r1}$ are the statistical weights of the r_1 and r_2 states respectively.

In the optical case one can nearly always assume that some sort of local equilibrium has been reached and that the radiative transitions between states

r_1 and r_2 are in balance with the transitions induced by collisions or other means. This provides the justification for calling the stellar radiation at optical wavelengths " thermal " and a similar assumption is valid for the " thermal " radio sources such as the H II regions and the undisturbed solar corona. However, for the non-thermal radio sources this assumption is surely invalid since the radiation is produced in regions of extremely low density where a disturbed non-equilibrium state can exist for a comparatively long time. This is particularly the case for the discrete radio sources which, it is thought, are formed by clouds of very high energy electrons trapped by weak magnetic fields (Alfvén and Herlofsen 1950). The lifetime of such an electron can be many thousands of years (Hoyle 1954, Twiss 1954) and it is extremely improbable that such a distribution of relativistic electrons can in any sense be regarded as in local equilibrium. Similarly, in the solar corona it is quite possible for a transient non-equilibrium electron energy distribution to last for times appreciably longer than the duration of the violent outbursts of radio noise which can often be measured in seconds (Pawsey and Bracewell 1955).

Under these circumstances the value of $\mathbf{K}_\nu$ can be very different from that of the thermal case and it is at least conceivable that conditions could temporarily arise in which $\mathbf{K}_\nu$ was negative, when the medium would behave like an amplifier. In the laboratory this condition can be achieved in the so-called maser (Gordon, Zeiger, and Townes 1955). The possibility that a similar effect could arise in nature is examined in Section IV below.

(b) *The Emission Coefficient*

If $A_{r,21}\,d\nu$ is the probability of a spontaneous transition with the emission of a quantum of given polarization into unit solid angle in the direction $\mathbf{s}$, so that

$$j_\nu=\sum_r n_{r2}A_{r21}h\nu, \qquad (5)$$

then, in the case $\mu=1$, it is a well-known consequence of the radiation laws that

$$A_{r,\,21}/B_{r,\,21}=h\nu^3/c^2. \qquad (6)$$

However, when μ is different from unity this simple relation is no longer valid. Admittedly we know from Kirchhoff's laws that

$$\frac{j_\nu}{\mathbf{K}_\nu}=\frac{\mu^2(\nu)\nu^2h\nu}{c^2\,[\exp\,(h\nu/k\Theta_\nu)-1]}, \qquad (7)$$

when the matter field is in thermal equilibrium,* but we are not justified in concluding automatically from this that

$$A_{r,\,21}/B_{r,\,21}=\mu^2(\nu)h\nu^3/c^2, \qquad (8)$$

since, for one thing, $\mu(\nu)$ may depend upon the actual form of the energy distribution.

Accordingly, in the general case, which applies to the disturbed stellar corona, it is necessary to calculate the spontaneous and stimulated transition

* This ratio is smaller by a factor 2 than that normally quoted since we are dealing with each polarization separately in this paper.

probabilities separately and both of these may be appreciably different from their equilibrium values, especially at the critical levels at which $\mu^2(\nu)$ is near to infinity or to zero.

IV. Conditions for Negative Absorption

In the ideal case in which monochromatic radiation traversing a medium is only absorbed or emitted in transitions between a single pair of non-degenerate states, the necessary and sufficient conditions for negative absorption is that the higher energy states be more densely populated than the lower. However, in most cases of astronomical interest the initial and final states will possess both continuous and overlapping energy distributions, and in some cases these states will possess different degeneracies; under these conditions the simple criterion for negative absorption is no longer valid. Clearly a *necessary* condition is that at least some of the higher energy states be over-populated, but it may be that negative absorption will not arise however widely the distribution may depart from that of thermal equilibrium. One example of this is provided by a fully ionized medium in which radiation is assumed to be emitted and absorbed only by free-free transitions; this and other cases will be considered in detail below, but we shall first consider the problem in general terms, without specific reference to the mechanism of radiation, for the idealized case in which the energy of a state is the only dynamic variable that need be considered specifically. With this last simplification we may write equation (3) in the form

$$\mathbf{K}_\nu = h\nu\int_0^\infty [n(\eta-h\nu)B_{12}(\nu,\eta)-n(\eta)B_{21}(\nu,\eta)]\,\mathrm{d}\eta, \quad \ldots\ldots \quad (9)$$

where η is the energy of the state.

Let us now put

$$\left.\begin{aligned} F(\eta) &= n(\eta)/\tilde{\omega}(\eta), \\ Q(\eta) &= \tilde{\omega}(\eta)B_{21}(\eta), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where $\tilde{\omega}(\eta)$ is the statistical weight of the state of energy η. Then from equation (4) we can write

$$\mathbf{K}_\nu = -h\nu\int_0^\infty h\nu\frac{\partial F}{\partial\eta}Q(\eta)\,\mathrm{d}\eta, \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

since, in any case of radio-astronomical interest, $\eta \gg h\nu$ for all η for which $Q(\eta)$ differs appreciably from zero.

The quantities $F(\eta)$, $Q(\eta)$ may be regarded as the effective energy distribution function and the effective cross section for stimulated transitions respectively. By definition both of these quantities are non-negative, that is,

$$F(\eta) \geqslant 0, \quad Q(\eta) \geqslant 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

while $F(\eta)$ must also satisfy the boundary condition

$$F(\infty) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

which is imposed by the requirement that no physically realizable state can have infinite energy.

If $F(\eta)$, $Q(\eta)$ are sufficiently well-behaved functions we can integrate equation (**11**) by parts to give

$$\mathbf{K}_\nu = (h\nu)^2 \left\{ F(0)Q(0) + \int_0^\infty F(\eta)\frac{\partial Q}{\partial \eta}\mathrm{d}\eta \right\}. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{14})$$

From equation (**14**) and the inequality (**12**) it follows immediately that $\mathbf{K}_\nu$ is positive, no matter what the form of $F(\eta)$, if $Q(\eta)$ is a monotonically *increasing* function of energy ; similarly, it follows from equation (**11**) that $\mathbf{K}_\nu$ is positive no matter what the form of $Q(\eta)$ if $F(\eta)$ is a monotonically *decreasing* function of η.

However, if $Q(\eta)$ *decreases* monotonically with η, then $\mathbf{K}_\nu$ will be negative if the energy distribution function $F(\eta)$ has a large enough peak at some value greater than zero or if $F(0)=0$.

A case of particular interest is that in which $Q(\eta)$ has a strong resonance such that it is only appreciably different from zero over a small range of η centred in $\eta=\eta_0$. Under these circumstances it follows from equation (**11**) that $\mathbf{K}_\nu$ will be negative if

$$(\partial F/\partial \eta)_{\eta=\eta_0} > 0.$$

These general results will now be applied to a number of specific cases.

(*a*) *Free-free Transitions*

In the classical Lorentz theory of propagation in an ionized medium the effect of the collisions of the free electrons with the ions and neutral molecules is allowed for by introducing a damping term,

$$m\mathbf{v}_1\nu(\mathbf{v}_0),$$

into the equations of motion of an electron, where $\mathbf{v}_1$ is the velocity induced by the electromagnetic field and $\mathbf{v}_0$ is the undisturbed electron velocity. The quantity $\nu(\mathbf{v}_0)$, the collision frequency appropriate to an electron with velocity $\mathbf{v}_0$ which is calculated from kinetic theory, is inherently positive so that the effect of collisions is always to *abstract* energy from the electromagnetic field at a rate proportional to

$$\tfrac{1}{2}m\int \nu(\mathbf{v}_0)\mathbf{v}_1^2(\mathbf{v}_0)n(\mathbf{v}_0)\mathrm{d}\mathbf{v}_0,$$

where $n(\mathbf{v}_0)\mathrm{d}\mathbf{v}_0$ is the contribution to the electron density from electrons with velocities in the range $\mathbf{v}_0$ to $\mathbf{v}_0+\mathrm{d}\mathbf{v}_0$. Hence on the *macroscopic* theory free-free transitions always involve a positive absorption of energy from the electromagnetic field, whatever the actual form of the electron velocity distribution function.

This conclusion can also be reached by considering the microscopic absorption and emission processes in the ionized gas as follows. We will restrict ourselves to the idealized case in which the ions can be taken as infinitely massive, a simplification that does not involve any significant loss of generality ; we will also assume that the medium is not acted upon by an external magnetic field.

The theory of free-free transitions has been extended to a medium with refractive index $\mu(\nu)$ by Smerd and Westfold (1949), who showed that the spontaneous transition probabilities must be multiplied by a factor $\mu(\nu)$, while the stimulated transition probabilities must be multiplied by a factor $\mu^{-1}(\nu)$ if results consistent with thermodynamics are to be obtained. In this treatment it was assumed that the transition probabilities for a given electron are *only* affected by the presence of the other electrons in so far as these contribute to $\mu(\nu)$, a purely macroscopic effect, and the same assumption will be made here.

Let us assume that the number of electrons per unit volume with kinetic energies in the range η to $\eta+\mathrm{d}\eta$ is given by

$$N_0 G(\eta)\eta^{\frac{1}{2}}\mathrm{d}\eta,$$

where N_0 is the total number of electrons per unit volume. In the case where the distribution is Maxwellian

$$G(\eta)=\frac{2\pi\ \exp\ (-\eta/k\Theta)}{(\pi k\Theta)^{3/2}}, \qquad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

where Θ is the equilibrium temperature.

Besides the energy η the orbit of an electron with respect to a given scattering centre will be characterized by the impact parameter b, where b is the perpendicular distance between the scattering centre and an asymptote to the orbit. Then, following Fowler (loc. cit.), it can be shown that the number of encounters per scattering centre, in unit time, with $(\eta_1,\ b_1)$ electrons is

$$N_0 2\pi b_1 \mathrm{d}b_1 (2/m)^{\frac{1}{2}} G(\eta_1)\eta_1 \mathrm{d}\eta_1,$$

if we make the physically plausible assumption that the distribution over the accessible impact parameter states is purely random.

Now the probability of a transition from an $(\eta_1,\ b_1)$ electron to an $(\eta_2,\ b_2)$ electron with the absorption of a quantum of energy $h\nu$, where

$$h\nu=\eta_2-\eta_1$$

may be taken to be

$$\beta_{12}(\eta,\ b)I(\nu)\mathrm{d}\nu,$$

so that the number of transitions of $(\eta_1,\ b_1)$ electrons to $(\eta_2,\ b_2)$ electrons per scattering centre per unit time with absorption of a quantum of energy $h\nu$ is

$$N_0(2/m)^{\frac{1}{2}}G(\eta_1)\eta_1\mathrm{d}\eta_1 2\pi b_1 \mathrm{d}b_1 \beta_{12}(\eta,\ b)I(\nu)\mathrm{d}\nu.$$

Similarly the number of transitions of $(\eta_2,\ b_2)$ electrons to $(\eta_1,\ b_1)$ electrons with the emission of a quantum of energy $h\nu$ may be written

$$N_0(2/m)^{\frac{1}{2}}G(\eta_2)\eta_2\mathrm{d}\eta_2 2\pi b_2 \mathrm{d}b_2[\beta_{21}(\eta,\ b)I(\nu)\mathrm{d}\nu+\alpha_{21}(\eta,\ b)\mathrm{d}\nu],$$

where $\alpha_{21}(\eta,\ b)$ denotes the spontaneous transition probability.

To find the relation between β_{12}, β_{21}, and α_{21} we use the fact that detailed balancing must apply under the conditions of thermal equilibrium when $G(\eta)$ is given by equation (**15**) and $I(\nu)$ is given by

$$I(\nu)=\frac{\mu^2(\nu)\nu^2h\nu}{c^2[\exp\ (h\nu/k\Theta_\nu)-1]}, \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{16})$$

remembering that we are dealing with a polarized radiation field.

From this we obtain the relations

$$\eta_2 b_2\beta_{21}(\eta,\ b)\mathrm{d}\eta_2\mathrm{d}b_2=\eta_1 b_1\beta_{12}(\eta,\ b)\mathrm{d}\eta_1\mathrm{d}b_1, \quad \ldots\ldots\ldots \quad (\mathbf{17})$$

and

$$\alpha_{21}(\eta,\ b)/\beta_{21}(\eta,\ b)=(h\nu^3/c^2)\mu^2(\nu), \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{18})$$

so that equation (**8**) *is* valid for the case of free-free transitions.

From equation (**17**) it follows that the degeneracy of the state is proportional to the energy ; a result that is to be expected when there is no preferred spatial direction.

If we substitute from equations (**17**), (**18**) in equation (**3**), replace sums by integrals, and integrate over the allowable values of η and b, we get

$$\mathbf{K}_\nu=\left(\frac{2}{m}\right)^{\frac{1}{2}}N_0^2h\nu 2\pi\int_0^\infty \mathrm{d}\eta\int_0^{b_{\max}}\mathrm{d}b\eta b[G(\eta-h\nu)-G(\eta)]\frac{c^2}{h\nu^3\mu^2(\nu)}\alpha_{21}(\eta,\ b), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{19})$$

where $b_{\max.}$ is determined by the distance beyond which a given scattering centre is screened by the electronic cloud.

If the results given by Westfold (1950) are written in m.k.s. units, and in the form appropriate to a single state of polarization, we get

$$\alpha_{21}(\eta,\ b)=\frac{e^2}{3\pi^2c^3m\varepsilon_0}\ \frac{\eta\mu(\nu)}{1+(8\pi\varepsilon_0 b\eta/Ze^2)^2}\ \frac{1}{h\nu}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{20})$$

so that

$$\mathbf{K}_\nu=-\frac{1}{\nu^2\mu(\nu)}\left(\frac{2}{m}\right)^{\frac{1}{2}}\frac{e^2N_0^2}{3\pi cm\varepsilon_0}\left(\frac{Ze^2}{8\pi\varepsilon_0}\right)^2\int_0^\infty \mathrm{d}\eta\frac{\partial G}{\partial\eta}\ln\left[1+\left(\frac{8\pi\varepsilon_0 b_{\max.}\eta}{Ze^2}\right)^2\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{21})$$

where Z is the ionic charge.

This equation is of the general form of equation (**11**) with

$$F(\eta)\sim G(\eta)\ \text{and}\ Q(\eta)\sim\log\ [1+(8\pi\varepsilon_0 b_{\max.}\eta/Ze^2)^2], \quad \ldots\ldots \quad (\mathbf{22})$$

so that in this case $Q(\eta)$ is a monotonically increasing function of η. It follows from the general discussion given above that $\mathbf{K}_\nu$ is positive, whatever the form of $G(\eta)$, in agreement with the predictions of the macroscopic theory. This conclusion, however, does not hold for other radiation mechanisms, as we shall now demonstrate.

(b) Cerenkov Radiation

The spontaneous emission probability for Cerenkov radiation is proportional to

$$e^2\frac{v_0}{c}\nu d\nu\left[1-\frac{c^2}{v_0^2\mu^2(\nu)}\right] \quad \text{for } v_0>\frac{c}{\mu(\nu)}, \qquad \ldots\ldots\ldots\ldots (23)$$

where v_0 is the velocity of the particle charge e traversing a medium with refractive index $\mu(\nu)$. In terms of the total energy E this may be written

$$\left.\begin{aligned} P(E)&\sim e^2\nu d\nu\left(1-\frac{m_0^2c^4}{E^2}\right)^{\frac{1}{2}}\left[1-\frac{E^2}{\mu^2(\nu)(E^2-m_0^2c^4)}\right] && \text{for } E>\frac{\mu(\nu)m_0c^2}{\sqrt{\{\mu^2(\nu)-1\}}} \\ &=0 && \text{for } E<\frac{\mu(\nu)m_0c^2}{\sqrt{\{\mu^2(\nu)-1\}}}, \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (24)$$

which is zero below a certain critical value of the total energy E and thereafter increases monotonically with E. However, it does not follow from this and the general discussion given above that negative absorption is inherently impossible in this case, since Cerenkov radiation is not emitted isotropically but in a direction making an angle φ with the path of the charged particle where

$$\cos\varphi=c/v_0\mu(\nu). \qquad \ldots\ldots\ldots\ldots (25)$$

Hence, if we have a stream of fast particles moving, in a given direction, with a small spread in transverse velocities, the emission in a direction making an angle θ with the direction of the stream will be obtained from electrons with a narrow range of energies centred on $E=E(\theta)$, where

$$E(\theta)=\frac{m_0c^2}{[1-\sec^2\theta/\mu^2(\nu)]^{\frac{1}{2}}}. \qquad \ldots\ldots\ldots\ldots (26)$$

If the energy distribution function $G(E)$ is such that

$$(\partial G/\partial E)_{E=E(\theta)}>0, \qquad \ldots\ldots\ldots\ldots (27)$$

the absorption coefficient will be negative for propagation in this direction.

This condition is similar to that for growing longitudinal waves in an ionized plasma, which was first derived by Bohm and Gross (1949). These workers considered a unidimensional plasma with a velocity distribution function $H(v)$ and showed that a plasma wave with phase velocity v_0 would be amplified if

$$[\partial H(v)/\partial v]_{v=v_0}>0. \qquad \ldots\ldots\ldots\ldots (28)$$

However, there is one important difference between the two cases in that the growing plasma wave, which arises under conditions equivalent to a negative Landau damping (Landau 1946), is a coherent phenomenon while the negative Cerenkov absorption considered above is incoherent, so that, in this second case, the radiated power would tend to zero with the electronic charge e when the total current was kept constant.

(c) *Gyro Radiation*

If we consider an electron spiralling under the action of an external magnetic field with kinetic energy η then in the frame of reference of the rotating electron the frequency spectrum of the emitted radiation is confined to narrow bands centred on integral multiples of the gyro frequency

$$\nu_H = eB_0/2\pi m, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

where B_0 is the flux density of the magnetic field in weber/metre^2 and m is the relativistic transverse mass of the electron defined by

$$m = m_0(1 - v_0^2/c^2)^{-\frac{1}{2}}.$$

In the case where

$$m - m_0 \ll m_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

which we shall assume in the present section, appreciable radiation will only take place at the fundamental and at the first few harmonics of the gyro frequency. The width of these spectral lines measured in the frame of reference in which the axial velocity of the electrons is zero is determined essentially by the lifetime of an excited state, which is in turn affected by collisions with the background plasma ions and by diffusion into regions with slightly different magnetic fields. We shall assume that this fractional line width is very much less than the separation between successive harmonics, a condition which will certainly be valid in practice as long as the inequality (**30**) is satisfied. Under the conditions occurring in nature, the fast spiralling electrons will have different axial momenta so that the radiated frequency spectrum seen by a fixed observer will be Doppler broadened. However, in the present paper, in which we are primarily concerned to show that negative absorption is possible in principle, we shall only consider the idealized case in which the fast electrons all have zero axial momentum.

The presence of the external magnetic field removes the spin and angular momentum degeneracy of the electron, so that the weights of the states between which radiative transitions can take place are equal to unity.

Under these conditions we can apply the general results given in equations (**9**)-(**14**) above to write

$$\mathbf{K}_\nu = -(h\nu)^2 \int_0^\infty Q(\eta, \theta, \nu) \frac{\partial F}{\partial \eta} \mathrm{d}\eta, \quad \ldots\ldots\ldots\ldots \quad (31)$$

where $F(\eta)$ is now simply the electron kinetic energy distribution function; $Q(\eta, \theta, \nu)$ is the stimulated transition probability in the direction θ for electrons of kinetic energy η.

When there is no Doppler broadening, $Q(\eta, \theta, \nu)$ is only appreciably different from zero for values of η close to the values

$$(\eta_r + m_0c^2) = eB_0c^2/2\pi r\nu, \quad r = 1, 2 \ldots, \quad \ldots\ldots\ldots \quad (32)$$

where r is the order of the harmonic of the gyro frequency so that we may write

$$Q(\eta, \theta, \nu) = \sum_{r=1}^{\infty} Q(\eta_r, \theta, \nu)\delta(\eta - \eta_r). \quad \ldots\ldots\ldots\ldots \quad (33)$$

If the energy of the fast electrons is sufficiently high, appreciable contributions to the power radiated at a given frequency ν can come from electrons with energies such that equation (**32**) is satisfied by one of several values of r. However, when the inequality (**30**) is valid, appreciable radiation at a given frequency will only occur for a single value of r, and in this case the associated value of $\mathbf{K}_\nu$ will be negative as long as

$$(\partial F/\partial \eta)_{\eta=\eta_r} > 0, \quad \text{(34)}$$

a condition that might arise in a practical case if a well-defined group of fast electrons was superimposed upon the thermal plasma electrons.

(*d*) *Synchrotron Radiation*

In this case where

$$m - m_0 \gg m_0,$$

the fast electrons are highly relativistic and appreciable energy is radiated over a very large number $\sim (m/m_0)^3$ of the harmonics of the gyro frequency. In the extreme case, which we shall consider here, the harmonics will be so closely spaced that the power spectrum $P(E, \varphi, \nu)$ for spontaneous emission may be assumed to be a smoothly continuous function given (Schwinger 1949) for the case $\mu=1$ to which we shall confine ourselves by

$$P(E, \varphi, \nu) = \frac{\sqrt{3}.\mu_0 c}{2} \cdot \nu_{H0} \cdot \frac{\nu}{\nu_c}\int_{\nu/\nu_c}^{\infty} K_{5/3}(\zeta)d\zeta \text{ watts/cycle}, \quad \text{(35)}$$

where $K_\sigma(\zeta)$ is the modified Hankel function of order σ,

$\nu_c = 1 \cdot 5\nu_{H0}(E/m_0c^2)^2 \sin\varphi$,

$\nu_{H0} = eB_0/2\pi m_0$,

B_0 is the flux density in weber/m^2 of the external magnetic field.

φ is the angle between the direction of motion of the electron and the magnetic field.

The orbit of a given electron is characterized by four parameters :

E, the total energy ;

p_z, the axial momentum ;

r_0, ψ_0, the cylindrical coordinates of the axis of the orbit.

In processes involving the emission or absorption of a quantum $h\nu$ in a direction making an angle θ to the magnetic field the changes δE and δp_2 in the energy and axial momentum are given, from the conservation laws, by

$$\left.\begin{aligned} \delta E &= \pm h\nu, \\ \delta p_z &= \pm h\nu \cos\theta/c, \end{aligned}\right\} \quad \text{(36)}$$

the sign being taken positive for energy absorption and negative for energy emission. The coordinates of the axis of the orbit can also suffer changes which must be allowed for in a quantum calculation of the spontaneous emission probabilities (Judd *et al.* 1952 ; Olsen and Wergeland 1952), but there is no need to take explicit account of these in the present discussion since they have no influence on the form of the absorption coefficient.

The energy and momentum of the particle are related by the equation

$$cp_z = (E^2 - m_0^2c^4)^{\frac{1}{2}} \cos \varphi. \quad \ldots \quad (37)$$

From this it can be shown that the change $\delta\varphi$ in the inclination of the electron orbit after absorption or emission of a quantum of energy $h\nu$ is given by

$$\delta\varphi \approx \pm \frac{h\nu}{\sin \varphi \sqrt{(E^2 - m_0^2c^4)}} \left[\frac{E \cos \varphi}{\sqrt{(E^2 - m_0^2c^4)}} - \cos \theta \right]. \quad \ldots \quad (38)$$

Now let us assume that $N(E, \varphi)$, the combined distribution function for the total energy and orbital inclination of the relativistic electrons, has first-order derivatives with respect to both E and φ. Let

$$I(\nu, \theta)\beta_{12}(E, \varphi, \nu, \theta) \text{ and } I(\nu, \theta)\beta_{21}(E, \varphi, \nu, \theta) \quad \ldots \quad (39)$$

be the stimulated absorption and emission probabilities for transitions between (E, φ) and $(E+\delta E, \varphi+\delta\varphi)$ states involving quanta of energy $h\nu$ emitted or absorbed in the θ-direction, where $I(\nu, \theta)$ is the intensity of radiation polarized so that the electric vector is perpendicular to the external magnetic field.* Then, from the fact that the electron states are non-degenerate, we have

$$\beta_{12}(E, \varphi, \nu, \theta) = \beta_{21}(E, \varphi, \nu, \theta), \quad \ldots \quad (40)$$

and

$$\mathbf{K}_\nu(\theta) = h\nu \int_{m_0c^2}^{\infty} \mathrm{d}E \int_0^{\pi} \mathrm{d}\varphi \beta_{21}(E, \varphi, \nu, \theta)[N(E, \varphi) - N(E+\delta E, \varphi+\delta\varphi)]. \quad \ldots \quad (41)$$

Let $\alpha_{21}(E, \varphi, \nu, \theta)$ be the spontaneous emission probability in unit solid angle in the direction θ, so that

$$\alpha_{21}(E, \varphi, \nu, \theta) = \frac{P(E, \varphi, \nu)}{h\nu} \frac{G(\theta)}{2\pi \sin \theta}, \quad \ldots \quad (42)$$

where $G(\theta)$, the function which determines the angular distribution of the radiation, is subject to the relation

$$\int_0^{\pi} G(\theta) \mathrm{d}\theta = 1. \quad \ldots \quad (43)$$

Since the refractive index μ of the medium has been assumed to be equal to unity, and since the radiation is emitted with a specific polarization, it follows from the basic radiation laws (Fowler loc. cit.) that

$$\beta_{21}(E, \varphi, \nu, \theta) = \frac{c^2}{h\nu^3} \alpha_{21}(E, \varphi, \nu, \theta). \quad \ldots \quad (44)$$

* This is the only polarization which interacts appreciably with relativistic electrons.

From equations (**41**)-(**44**) we see that

$$\mathbf{K}_\nu = -\frac{1}{2\pi\sin\theta}\int_{m_0c^2}^{\infty} \mathrm{d}E\int_0^{2\pi} \mathrm{d}\varphi\frac{c^2}{\nu^2}P(E,\varphi,\nu)G(\theta)\left\{\frac{\partial N}{\partial E}(E,\varphi) + \frac{1}{\sin\varphi}\left(\frac{E\cos\varphi}{\sqrt{(E^2-m_0^2c^4)}}-\cos\theta\right)\frac{\partial N(E,\varphi)}{\partial\varphi}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{45})$$

Now the radiation from highly relativistic electrons is effectively confined to a narrow cone, with the axis tangential to the instantaneous orbit, so that $G(\theta)$ will only depart appreciably from zero when $\varphi\approx\theta$. In this case $\delta\varphi$, as given by equation (**38**), is effectively zero, as, therefore, is the coefficient of $\partial N(E,\varphi)/\partial\varphi$ in equation (**45**). If we make the further assumption that $\partial N(E,\varphi)/\partial E$ does not vary appreciably over the resonance of $P(E,\varphi,\nu)$ we may put

$$P(E,\varphi,\nu)G(\theta)\simeq P(E,\varphi,\nu)\delta(\theta-\varphi), \quad \ldots\ldots\ldots\ldots (\mathbf{46})$$

when from equation (**45**)

$$\mathbf{K}_\nu = \frac{-c^2}{2\pi\nu^2\sin\theta}\int_{m_0c^2}^{\infty} \mathrm{d}EP(E,\theta,\nu)\frac{\partial N(E,\theta)}{\partial E}, \quad \ldots\ldots\ldots\ldots (\mathbf{47})$$

which is of the form of equation (**11**).

The function $P(E,\theta,\nu)$ is effectively zero for values of E such that

$$E\ll E_{\mathrm{max.}}\approx(\nu/\nu_{H0}\sin\theta)^{\frac{1}{2}}\,m_0c^2, \quad \ldots\ldots\ldots\ldots (\mathbf{48})$$

rises to a maximum at $E_{\mathrm{max.}}$, and thereafter decreases monotonically to zero as E tends to infinity.

From the general discussion given above in the introduction to Section IV it follows that $\mathbf{K}_\nu$ is certainly negative, whatever the nature of $N(E,\theta)$ in the region $E>E_{\mathrm{max.}}$, as long as $N(E,\theta)$ is zero for all $E<E_{\mathrm{max.}}$. The most favourable case for negative absorption, for propagation in the θ-direction, arises when $N(E,\theta)$ is zero except for a narrow range of energies centred around the value at which $\partial P(E,\theta,\nu)/\partial E$ has a maximum, but this optimum is not critical since the maximum is not a sharp one.

Even if negative absorption is present in a medium it will only be important in regions in which the difference between the stimulated emission and absorption is at least comparable with the spontaneous emission. In the ideal case, where the relativistic electrons all have the same energy E, it can be shown quite simply that $\gamma(\theta)$, the ratio between the excess stimulated emission and the spontaneous emission, is given by

$$\gamma(\theta)=\frac{k\Theta_\nu(\theta)}{E}\;\frac{2\alpha(\partial X/\partial\alpha)}{X(\alpha)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{49})$$

where $\Theta_\nu(\theta)$ is the effective black-body temperature of the radiation field at frequency ν in the θ-direction, where α is given by

$$\alpha=\frac{2\nu}{3\nu_{H0}\sin\theta}\left(\frac{m_0c^2}{E}\right)^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{50})$$

and where $X(\alpha)$ is the function defined by

$$X(\alpha)=\alpha\int_{\alpha}^{\infty} K_{5/3}(\zeta)\mathrm{d}\zeta, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (51)$$

which has been tabulated by Vladimirsky (1948).

The quantity

$$2\alpha(\partial X/\partial\alpha)/X(\alpha)$$

is a slowly varying function of α, for values of E at which appreciable radiation can be emitted at frequency ν, of the order of magnitude of unity so that even under ideal conditions appreciable negative absorption can only arise when

$$k\Theta_{\nu}\simeq\bar{E}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52)$$

where $\bar{E}$ is the average energy of the relativistic electrons which contribute appreciably to the radiation flux at frequency ν in the θ-direction.

V. Discussion and Conclusions

The main result of this paper has been to show that negative absorption or, in other words, amplification, can occur under certain circumstances in an electron gas.

A necessary condition for this is that the electron energy distribution function $F(\eta)$ be markedly non-thermal with an appreciable excess of high energy electrons such that the derivative of $F(\eta)$ is positive over a range of values of the kinetic energy η. However, this condition is not sufficient since, as we have shown, negative absorption can never occur if the dominant radiation process is that of free-free transitions, whatever the form of the electron energy distribution function. On the contrary, a further necessary condition is that the absorption cross section, in the direction of propagation, associated with the dominant radiation process should have a maximum at some finite value of the electron energy, and the most favourable case is when this maximum is a sharp one occurring at the value of η at which $\partial F/\partial\eta$ has its maximum positive value. More specifically, we have shown that negative absorption can occur in principle when the basic radiation process is (*a*) Cerenkov, (*b*) gyro, (*c*) synchrotron, and the relevance of this to radio astronomy will now be discussed briefly.

(*a*) *Cerenkov Radiation from the Auroral Rays*

A charged particle moving with velocity $c\beta$ through a medium will produce Cerenkov radiation in a particular mode and at given frequency ν in the direction θ for which

$$v_{\varphi}=c\beta\cos\theta,$$

where v_{φ} is the phase velocity of the associated electromagnetic wave in the medium. Since, for a material particle, $\beta<1$, this condition can be met only for $v_{\varphi}<c$ and in an ionized medium this occurs only in the extraordinary mode at frequencies below the gyro frequency. Such radiation cannot escape from the Sun nor indeed from any galactic radio source (Pawsey and Bracewell 1955) at frequencies which can be transmitted through the Earth's ionosphere. It therefore appears that this effect cannot play any role in the production of radio

waves received from outside the Earth. However, it is conceivable that such radiation might be produced by fast electrons in the auroral rays (Ellis 1957) and, as long as the direction of these electrons is confined to a narrow solid angle, the discussion given in Section IV (*b*) suggests that the Cerenkov radiation generated in a certain solid angle will be amplified if the burst of fast electrons should have a sufficiently narrow energy spread.

(*b*) *Gyro Radiation and the Type I Solar Disturbances*

It still seems likely that the mechanism underlying the major outbursts of non-thermal radio emission from the solar corona is connected with the presence of organized plasma oscillations. However, it is possible that the strongly circularly polarized noise storms and solar bursts of spectral type I (Wild 1951) are due to harmonic gyro radiation (Twiss and Roberts 1958). If this is indeed the case one must assume the presence of negative absorption, or at least of anomalous radiation transfer, since for reasons connected with the circular polarization and lack of harmonic structure of the bursts the rotational energy of the fast gyrating electrons cannot be more than a few thousand electron-volts. This corresponds to a kinetic temperature of 10^7–10^8 °K, while the brightness temperature associated with the observed flux may be as high as 3×10^{10} °K. A discussion of this possibility must be based on a detailed analysis of the experimental evidence which will be published elsewhere (Twiss, in preparation); all we would like to emphasize here is that many of the objections to the gyro theory for type I disturbances can be resolved on the assumption that the medium is exhibiting negative absorption.

(*c*) *Synchrotron Radiation in the Discrete Radio Sources*

The principal objection to the theory which attributes the emission from the discrete radio sources to synchrotron type radiation is that one must assume that very large energies are stored both in the relativistic electrons and in the trapping magnetic field. These difficulties would be appreciably reduced if negative absorption were playing a significant role at least at frequencies below a few hundred megacycles. However, it seems virtually certain that this cannot be the case, as will now be shown.

Thus let us consider the brilliant discrete radio source Cygnus-A, which has an effective brightness temperature Θ_ν given approximately by the law

$$\Theta_\nu \simeq 10^9(\nu/10^8)^{-3}\ ^\circ\mathrm{K}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (53)$$

over the frequency range

$$3\times10^7<\nu<10^{10}\ \mathrm{c/s}.$$

Now, as shown in Section IV, in the case most favourable for negative absorption, the energy of the fast electrons is given approximately by

$$E_{\mathrm{opt}} \simeq m_0c^2\left(\frac{\nu}{1\cdot5\nu_{H0}\sin\theta\times0\cdot05}\right)^{\frac{1}{2}}, \quad \ldots\ldots\ldots\ldots \quad (54)$$

and for negative absorption to be significant it is then necessary that

$$E_{\mathrm{opt}} \simeq k\Theta_\nu. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (55)$$

The frequency at which this last condition is met is linearly proportional to $\nu_{H0}^{\frac{1}{2}}$, where $\nu_{H0}=eB_0/2\pi m_0$, is the gyro frequency of a non-relativistic electron in the magnetic trapping field. The average value of B_0 in Cygnus-A, which has been identified with two colliding galaxies, is probably within an order of magnitude of 10^{-8} weber/m^2 and it is almost certainly less than 10^{-6} weber/m^2, since otherwise the energy stored in the magnetic trapping field would be prohibitively large. Hence $\nu_{H0}<10^5$ c/s and equation (**55**) will not be satisfied for frequencies above 2×10^7 c/s. Below this frequency the effective brightness temperature of Cygnus-A increases less rapidly than ν^{-3} (Lamden and Lovell 1956), while the opposite would be the case if negative absorption were starting to be effective. Admittedly, this effect might be partially marked by absorption in H II regions within our own galaxy, but, since we have taken very optimistic assumptions for the electron energy distribution and since Cygnus-A is an exceptionally brilliant source, the condition most favourable for stimulated emission, one is probably justified in concluding that negative absorption is always unimportant in the discrete radio sources, at frequencies of interest to radio astronomy, if the radio emission from these is indeed synchrotron-type radiation.

On the contrary, it is to be expected that the relativistic electron gas will exhibit positive absorption at low frequencies, as has been suggested earlier (Twiss 1954) and this phenomenon may be connected with the observed low-frequency cut-off in the spectrum of the discrete radio sources (Lamden and Lovell 1956).

VI. References

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