THERMAL TRANSIENTS ASSOCIATED WITH NATURAL CONVECTION

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[*Manuscript received November* 6, 1959]

Summary

An electrical circuit, equivalent to the thermal processes concerned in the act of setting up a system of natural convection in a fluid, has been proposed. This circuit consists of parallel elements, one attributed to convection alone and one to conduction alone. The former consists of a resistance and an inductance in series, the latter of a resistance and a shunt capacity. The nature of the transients associated with each circuit has been derived by Heaviside analysis. Functions, at the most involving no more than two independent variables, are found connecting easily measured characteristics of the transients and the circuit parameters. These functions are, it is proposed, to be used to examine experimental thermal transients and through them to examine their circuit parameters.

I. Introduction

The way in which the distribution of temperature settles down after a sudden thermal shock is ordinarily described by the Fourier equation. In the Fourier equation a single characteristic property of the system is referred to the thermal diffusivity, which in turn is equal to the ratio of the measures of two properties of the medium in which heat flow occurs, namely, the conductivity and the capacity per unit volume. For a system of given geometrical dimensions the diffusivity gives rise to a single time constant, namely, the ratio of the thermal capacitance to the thermal conductance.

The representation described by the Fourier equation can accordingly be expressed, at least in principle, by an equivalent electrical circuit involving only resistances and capacitances. Systems which involve convective as well as conductive flow may be more complicated. Frank-Kamenetskii (1938), for example, claimed that two time constants are associated with thermal convection. The nature of the initial heat flow associated with convection, especially in liquid systems, is formally quite different from that from a resistance-capacity circuit, and one of the authors (Bosworth 1946, 1948) has suggested that these systems exhibit a phenomenon properly called thermal inductance. These phenomena are connected with the temperature changes which follow the sudden steady application of heat to a wire immersed in the fluid. The heat flow from the wire to the fluid was constant, while the temperature difference between the wire and the fluid, which was initially zero, finally settled down to a steady value after a sufficient time of heating. In the phenomenon that one of the authors described as thermal inductance the temperature difference at first rose to a value higher than the final value and approached it from above.

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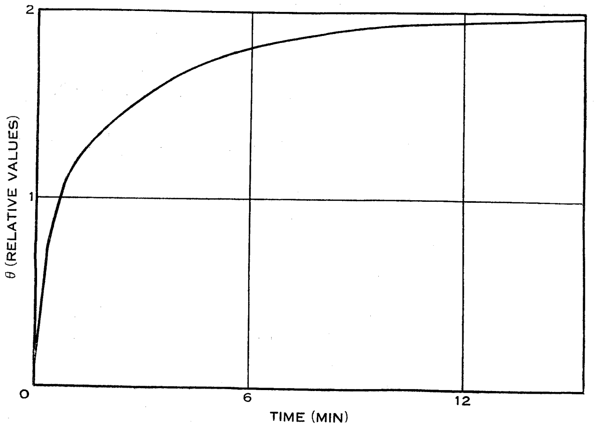
2
1
0
θ (RELATIVE VALUES)
6
12
TIME (MIN)

Fig. 1

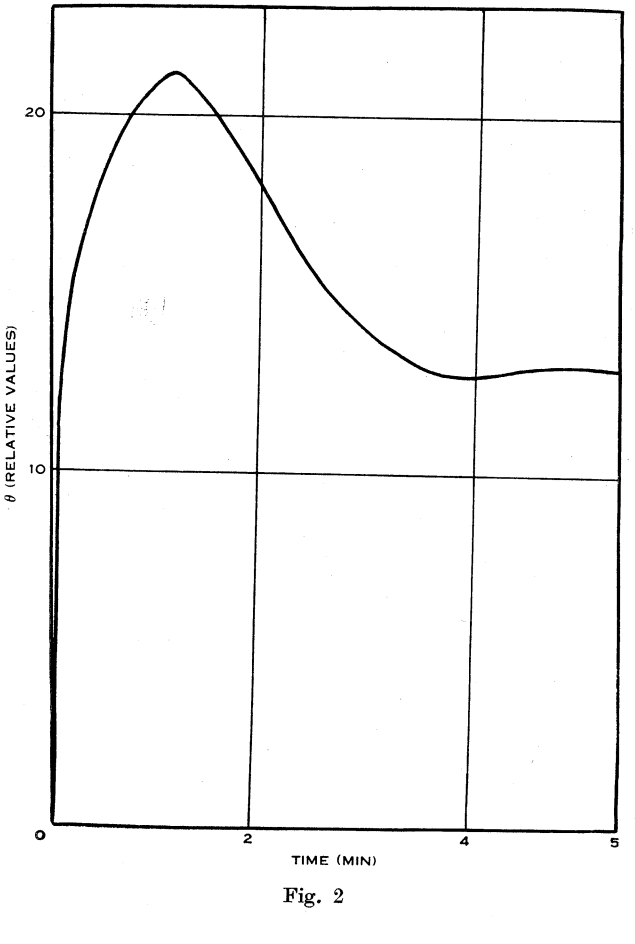
20
10
0
θ (RELATIVE VALUES)
2
4
5
TIME (MIN)

Fig. 2

In certain fluids, notably in gases and in very viscous liquids, the initial rise in the temperature difference above the final value was not realized; in other words, the final temperature value was approached from below. Such a transient may be regarded as of a purely capacitive type. On the other hand the transient which approaches its full value from above could be described as an inductive type. Thermal transients of both types have been described by Ostroumoff (1956) and interpreted in a purely mechanical manner. A more exhaustive study has shown that moderately viscous liquids appear to give a capacitive-type transient at a low heating rate and an inductive-type transient at a higher heating rate.

Typical transients for each type are shown by a hot wire of 30 S.W.G. immersed in an 81·2 per cent. (supersaturated) sucrose solution at 18·0 °C. Figure 1 shows the thermal transient for a heating current of 0·52 A and Figure 2 shows that for a heating current of 2·65 A. The former appears to be of the capacitive type, the latter of the inductive type.

II. An Assumed Equivalent Circuit

On the application of the flow of heat from the wire to the fluid an initial effect results in setting up a temperature difference between the wire and the outer regions of the fluid, and later also a temperature difference occurs between the fluid in close contact with the wire and the more distant regions. At the same time, there exists initially no circulating system of convection currents and the thermal resistance opposing heat flow is initially high. When the system of convection currents has been fully developed the resistance to thermal flow is considerably reduced. The equivalent circuit representing the heat flow thus must involve two resistances in parallel, one of which r is always operative and the other R only becomes operative after the system of convection currents has been developed. In the equivalent circuit we represent R as in series with an inductance L. The whole system has an additional capacitance C which operates in parallel to r and $R+L$. This takes into account both the heating up processes and the action of setting up a steady temperature gradient through the fluid. The capacitance C is expected to include the thermal capacity of the wire and of the fluid adjacent to it.

The overall suggested equivalent circuit is represented in Figure 3. The circuit, as an electrical one, has been discussed by Carter (1944) in connexion with the transient voltage across the Tirrill regulator contacts on a generator. Carter expressed the input impedance in terms of Heaviside operators and interpreted the operators in terms of exponential functions. For certain ranges of the variables (r, R, C, and L) the exponents become complex quantities and the transient takes the form of a damped oscillation. When the exponents are both real the disturbance is overdamped and there are no oscillations.

By matching actual values of the thermal transient to the theoretical values of the calculated transient with selected values of R, C, L, and r, it is, at least in principle, possible to derive all these associated circuit characteristics. In practice it is relatively easy to match the circuit characteristics from the oscillatory transient. Points may be selected from values of the temperature which pass through the asymptotic value and which attain one or more turning points. However, the derivation of such characteristics as these from the theoretical expressions for the circuit diagram is not particularly convenient in the form given by Carter, and an equivalent, but more convenient, expression may be derived.

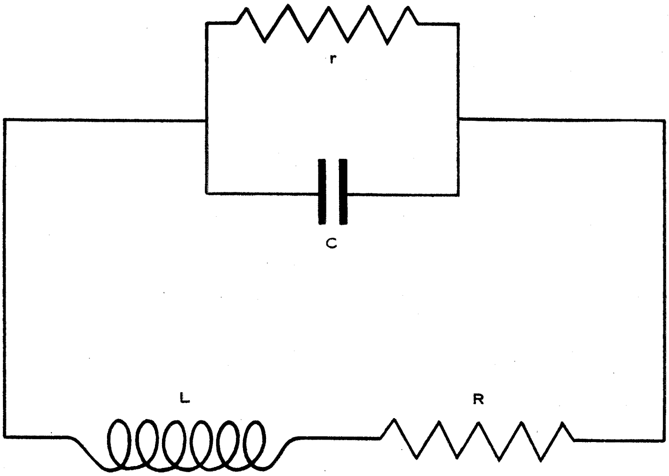

Fig. 3

III. The Characteristics of the Equivalent Circuit

The overall formal impedance Z of Figure 3 may readily be seen to be

$$Z=\frac{r(R+Lp)}{CLrp^2+RCrp+Lp+r+R}, \qquad \ldots\ldots\ldots\ldots \quad (1)$$

where p is the Heaviside operator

$$p^{-1}=\int \mathrm{d}t.$$

Let the current take the form of the Heaviside function, namely

$$I=0, \text{ when } t<0,$$
$$I=1, \text{ when } t\geqslant 0.$$

The potential for any positive value of t then takes the form ZI, where

$$Z=\frac{rR}{r+R}\left\{1-\exp-\left(\frac{L+RCr}{2rCL}\right)t\left[\cosh\frac{\sqrt{\{(L-RCr)^2-4r^2CL\}}}{2rCL}t\right.\right.$$
$$\left.\left.-\frac{2rL/R+L-RCr}{\sqrt{\{(L-RCr)^2-4r^2CL\}}}\sinh\frac{\sqrt{\{(L-RCr)^2-4r^2CL\}}}{2rCL}t\right]\right\}.$$

$$\ldots\ldots\ldots\ldots \quad (2)$$

This equation may be written in a more concise form by means of the following substitutions:

$$X=(L-RCr)/2r\sqrt{(LC)}, \qquad \text{(2a)}$$

$$Y=(1/R)\sqrt{(L/C)}, \qquad \text{(2b)}$$

and

$$\tau=t/\sqrt{(LC)}, \qquad \text{(2c)}$$

Equation (2) then becomes

$$Z=\frac{rR}{r+R}\left\{1-\exp-\left(X+\frac{1}{Y}\right)\tau\left[\cosh\sqrt{(X^2-1)}\tau-\frac{X+Y}{\sqrt{(X^2-1)}}\sinh\sqrt{(X^2-1)}\tau\right]\right\}. \qquad \text{(3)}$$

When the modulus of X is less than unity the arguments of the hyperbolic function become complex and may be replaced by trigonometrical functions,

$$Z=\frac{rR}{r+R}\left\{1-\exp-\left(X+\frac{1}{Y}\right)\tau\left[\cos\sqrt{(1-X^2)}\tau-\frac{X+Y}{\sqrt{(1-X^2)}}\sin\sqrt{(1-X^2)}\tau\right]\right\}, \qquad \text{(4)}$$

valid when $-1\leqslant X\leqslant 1$.

The oscillatory type of transient is represented by equation (4). A completely overdamped transient is represented by equation (3) with the value X negative and numerically greater than unity. The positive values of X that are greater than unity show no true oscillatory motion. However, the expression in the square brackets in equation (3) can always change from a positive to a negative value with an increasing τ; this means that the transient value of Z overshoots the asymptotic value and finally approaches that value from above and not from below as in the region for which $-1<X<+1$.

Experimentally we may expect a capacitive type of transient to appear when $X<-1$ and an inductive-type transient when $X>-1$. In the limited region $-1\geqslant X\geqslant 1$ the transient may show a series of damped waves. However, in all thermal flow circuits we expect the damping to be fairly severe and any wave other than the first may well be swamped by the noise level.

As indicated earlier the four most readily attainable characteristics of experimental "inductive-type" transients are the value t_0 of the time required to first attain a value equal to the asymptotic value, and the value t_1 to attain the first stationary point and the corresponding impedances Z_0, the asymptotic value, and Z_1, the value at the first maximum. Since four variables are necessary to describe the transient, the four easy experimental values are, at least in principle, sufficient to fix the nature of the transient.

The value of t_1 from equation (3) takes the value

$$\frac{t_1}{\sqrt{(LC)}}=\frac{1}{\sqrt{(X^2-1)}}\operatorname{arc\,tanh}\frac{\sqrt{(X^2-1)}}{X} \qquad \text{(5)}$$

$$=\frac{1}{\sqrt{(X^2-1)}}\operatorname{arc\,cosh} X. \qquad \text{(5a)}$$

From equation (4) the corresponding value reads

$$\frac{t_1}{\sqrt{(LC)}}=\frac{1}{\sqrt{(1-X^2)}} \text{ arc cos } X. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The value of t_0 from equation (3) takes the value

$$\frac{t_0}{\sqrt{(LC)}}=\frac{1}{\sqrt{(X^2-1)}} \text{ arc tanh } \frac{\sqrt{(X^2-1)}}{X+Y}, \qquad \ldots\ldots\ldots (7)$$

and from equation (4)

$$\frac{t_0}{\sqrt{(LC)}}=\frac{1}{\sqrt{(1-X^2)}} \text{ arc tan } \frac{\sqrt{(1-X^2)}}{X+Y}. \qquad \ldots\ldots\ldots (8)$$

The ratio of these two times, which we shall take as equal to A, becomes, in the trigonometrical region,

$$\frac{t_1}{t_0}=\frac{\text{arc cos } X}{\text{arc tan }\{\sqrt{(1-X^2)}/(X+Y)\}}=A>1.$$

We now make a substitution for X, putting

$$X=\cos T,$$

and obtain

$$Y=\frac{\sin (1-1/A)T}{\sin T/A}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

In the hyperbolic region a corresponding substitution,

$$X=\cosh T,$$

gives

$$Y=\frac{\sinh (1-1/A)T}{\sinh T/A}. \qquad \ldots\ldots\ldots\ldots\ldots (9a)$$

Let the ratio of the corresponding two impedances be

$$Z_1/Z_0=B.$$

The trigonometrical region then gives

$$B=1+Y\exp -\left\{\frac{X+1/Y}{\sqrt{(1-X^2)}} \text{ arc cos } X\right\}, \qquad \ldots\ldots (10)$$

and, in the hyperbolic region,

$$B=1+Y\exp -\left\{\frac{X+1/Y}{\sqrt{(X^2-1)}} \text{ arc cosh } X\right\}. \qquad \ldots\ldots (10a)$$

Substituting T for X we get

$$\ln Y/(B-1)=\frac{T}{\sin T}\left(\cos T+\frac{1}{Y}\right), \qquad \ldots\ldots\ldots\ldots (11)$$

or

$$\ln Y/(B-1)=\frac{T}{\sinh T}\left(\cosh T+\frac{1}{Y}\right), \qquad \ldots\ldots (11a)$$

in the respective region.

Elimination of Y between equation (**9**) and (**11**) now gives

$$\ln (B-1)=\ln \left[\frac{\sin (1-1/A)T}{\sin T/A}\right]-T \cot (1-1/A)T, \quad \ldots\ldots \quad (\mathbf{12})$$

or

$$\ln (B-1)=\ln \left[\frac{\sinh (1-1/A)T}{\sinh T/A}\right]-T \coth (1-1/A)T, \quad \ldots \quad (\mathbf{12a})$$

in which one of the required combinations of the circuit characteristics (namely, $T=\text{arc} \cos X$ or $\text{arc} \cosh X$) are implicitly expressed in terms of measurable characteristics of the transients, namely, A and B.

IV. Evaluation of Circuit Parameters from the Characteristics of the Transient

The relationship between X and A and B can be obtained from tables, such as Table 1, derived from equations (**12**) and (**12a**). Reading off from values of A and B (interpolated, if necessary) it is possible to derive X and thus $(L-RCr)/2r\sqrt{(LC)}$ from Table 1.

Table 1 in effect gives a relationship between the parameters A and B at a series of constant values of X. A more convenient tool is a relationship between A and X at a series of fixed values of B. Such data, derived from Table 1, are shown in Figure 4. It will be noted that the curves showing A versus X at constant B are practically rectilinear at the higher values of X. For limiting high values beyond the range covered by Table 1 or Figure 4, equation (**12a**) may be written in the approximate form

$$\ln (B-1)=\ln \frac{e^{(1-1/A)T}}{e^{T/A}} -T,$$

or

$$\begin{aligned} B-1&=e^{-2T/A} \\ &=(2X)^{-2/A}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{13}) \end{aligned}$$

Under similar conditions the limiting value of Y becomes

$$Y=X^{2(1-2/A)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{13a})$$

so that the apparently linear relationship between A and X does not apply to still higher ranges.

V. The Ratio of Conductive to Convective Resistance

By using Figure 4 and equations (**9**) (or (**9a**)) it is possible to derive both X and Y from the known values of A and B. Now X and Y are connected from their primary definitions in equations (**2a**) and (**2b**) and, in fact,

$$2X/Y+1/Y^2=R/r. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{14})$$

Accordingly, the ratio R/r thus depends only on the parameters A and B. The relationship proves to be a particularly simple one. Substituting equation (**9**) for equation (**14**) gives

$$\frac{2 \cos T \sin T/A}{\sin (T-T/A)}+\frac{\sin^2 T/A}{\sin^2 (T-T/A)}=n,$$

TABLE 1

VALUES OF B FOR GIVEN X AND A, COVERING BOTH TRIGONOMETRICAL AND HYPERBOLICAL REGIONS

X \ A	1·2	1·3	1·4	1·5	1·8	2·0	2·5	3·0	3·5	4·0	4·5
—1·0	1·004+	1·029	1·082	1·163	1·575	2·000	3·776	7·135	13·248	24·141	43·275
—0·9	1·002+	1·015	1·046	1·093	1·324	1·539	2·276	3·297	4·619	6·258	8·237
—0·8	1·002—	1·012	1·037	1·077	1·264	1·435	1·991	2·706	3·567	4·563	5·687
—0·7	1·001+	1·011	1·032	1·066	1·228	1·373	1·831	2·397	3·051	3·779	4·573
—0·6	1·001+	1·0095	1·028	1·059	1·203	1·330	1·726	2·199	2·732	3·312	3·933
—0·5	1·001+	1·009	1·026	1·054	1·184	1·298	1·648	2·059	2·513	3·000	3·513
—0·4	1·001+	1·008	1·024	1·049	1·169	1·273	1·588	1·953	2·351	2·773	3·213
—0·3	1·001—	1·007	1·022	1·046	1·156	1·252	1·540	1·870	2·226	2·600	2·987
—0·2	1·001—	1·007	1·021	1·043	1·146	1·235	1·501	1·803	2·126	2·463	2·811
—0·1	1·001—	1·006	1·0195	1·040	1·137	1·221	1·468	1·747	2·044	2·352	2·669
0	1·001—	1·006	1·018	1·038	1·129	1·208	1·440	1·699	1·975	2·260	2·551
0·1	1·001—	1·006	1·017	1·036	1·122	1·197	1·415	1·659	1·916	2·181	2·452
0·2	1·001—	1·005	1·017	1·034	1·116	1·187	1·393	1·623	1·865	2·114	2·368
0·3	1·001—	1·005	1·016	1·033	1·111	1·178	1·374	1·592	1·821	2·056	2·294
0·4	1·001—	1·005	1·015	1·031	1·106	1·170	1·357	1·564	1·781	2·004	2·231
0·5	1·001—	1·005	1·0145	1·030	1·101	1·163	1·342	1·539	1·746	1·959	2·175
0·6	1·001—	1·005	1·014	1·029	1·097	1·157	1·328	1·517	1·715	1·918	2·124
0·7	1·001—	1·004	1·013	1·028	1·094	1·151	1·315	1·497	1·687	1·882	2·078
0·8	1·001—	1·004	1·013	1·027	1·090	1·145	1·304	1·479	1·661	1·848	2·038
0·9	1·001—	1·004	1·0125	1·026	1·087	1·140	1·293	1·462	1·638	1·817	2·002
1·0	1·0005	1·004	1·012	1·025	1·085	1·135	1·282	1·446	1·616	1·792	1·968
1·2	1·0005—	1·0035	1·011	1·023	1·079	1·127	1·266	1·418	1·579	1·742	1·909
1·4	1·0004+	1·0035	1·011	1·022	1·074	1·120	1·251	1·395	1·546	1·700	1·857
1·6	1·0004+	1·003	1·010	1·021	1·070	1·113	1·237	1·374	1·518	1·664	1·814
1·8	1·0004—	1·003	1·0095	1·020	1·067	1·107	1·225	1·356	1·493	1·633	1·775
2·0	1·0004—	1·003	1·009	1·019	1·064	1·102	1·215	1·339	1·470	1·605	1·741
2·5	1·0003+	1·0026	1·0081	1·017	1·057	1·091	1·193	1·305	1·424	1·547	1·672
3·0	1·0003+	1·0024	1·0073	1·015+	1·051	1·083	1·175	1·279	1·388	1·502	1·617
3·5	1·0003—	1·0022	1·0067	1·014	1·047	1·076	1·161	1·257	1·359	1·465	1·573
4·0	1·0003—	1·0020	1·0061	1·012+	1·043	1·070	1·149	1·239	1·335	1·435	1·537
4·5	1·0002+	1·0019	1·0057	1·012—	1·040	1·065	1·139	1·224	1·314	1·409	1·506
5·0	1·0002+	1·0017	1·0053	1·011—	1·037	1·060	1·130	1·211	1·297	1·387	1·480
5·5	1·0002+	1·0016	1·0049	1·010+	1·035	1·057	1·123	1·199	1·282	1·368	1·458
6·0	1·0002—	1·0015	1·0046	1·010—	1·033	1·053	1·116	1·189	1·268	1·352	1·438
6·5	1·0002—	1·0014	1·0043	1·009—	1·031	1·050	1·110	1·180	1·256	1·337	1·420
7·0	1·0002—	1·0014	1·0041	1·008+	1·029	1·048	1·105	1·172	1·246	1·324	1·405
7·5	1·0002—	1·0013	1·0039	1·008+	1·028	1·045	1·100	1·165	1·236	1·312	1·390
8·0	1·0002—	1·0012	1·0037	1·008—	1·026	1·043	1·096	1·159	1·228	1·301	1·377
8·5	1·0001+	1·0012	1·0035	1·007+	1·025	1·041	1·092	1·153	1·220	1·291	1·366
9·0	1·0001+	1·0011	1·0034	1·007—	1·024	1·040	1·089	1·147	1·213	1·282	1·355
9·5	1·0001+	1·0011	1·0032	1·007—	1·023	1·038	1·085	1·142	1·206	1·274	1·345
10·0	1·0001+	1·0010	1·0031	1·006	1·022	1·037	1·082	1·138	1·200	1·267	1·336

with n the ratio of the resistances. This expression may readily be reduced to

$$\sin^2 T=(1-n)\sin^2 T(1-1/A), \qquad \text{(15)}$$

with a corresponding expression in the hyperbolic region.

The quantity n may take values from 0 to ∞ but may never be negative. The upper limit corresponds to a type of heat flow in which the contribution by the convective transfer has vanished. The lower limit corresponds to a condition at which the convective contribution is infinitely large in comparison with the conductive transfer. The only solution of equation (**15**) pertinent to practical problems for which n vanishes is the particularly simple form

$$A=T/(2T-\pi), \qquad \tfrac{1}{2}\pi<T<\pi. \qquad \text{(16)}$$

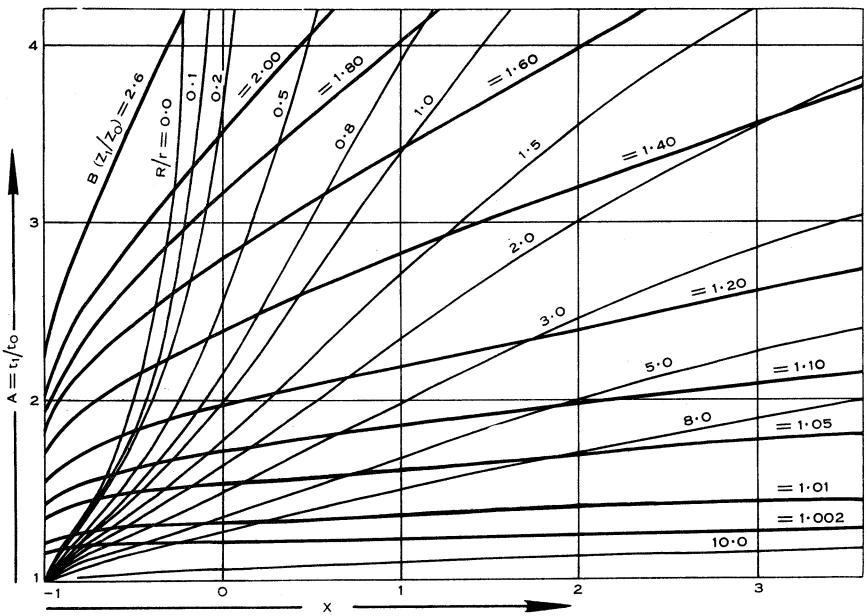

Fig. 4

The points where the curve of $R/r=0$ on Figure 4 cross the lines of constant B are solutions of equation (**16**). The region to the right of this line corresponds to finite values of R/r, that to the left to unattainable values of R/r. Any experimental value of A and B lying to the left of this line thus corresponds to a condition which is inconsistent with the equivalent circuit postulated in Figure 3.

Values for A at a range of values of T (or X) may be obtained from equation (**15**) for selected values of n (0·1, 0·2, etc.). The lines at selected values of n are drawn as a series of contours crossing the constant B lines in Figure 4. Accordingly, from this figure we can read off, given a known pair of values for A and B, the corresponding values for X and for the ratio R/r.

VI. Treatment of Experimental Results

Experimental results are obtained in the form of a transient curve plotting the temperature excess (or alternatively the quotient of the temperature excess by the heat flow rate) against the time. The peak temperature and the time to attain that peak are recorded, together with the final temperature and the time first to cross that amplitude. From these measurements the ratio of temperatures (B) and the ratios of the times (A) can be read off. From Figure 4 we may now read off the value X and also the ratio R/r. Alternatively, from the X value we may compute the value of Y using equation (**9**) or (**9a**)). From Y and X the value of R/r can also then be derived by use of equation (**14**). From the final temperature

$$Z_0=Rr/(R+r), \qquad \textbf{(17)}$$

it is then possible to obtain values for both R and r.

Table 2

VALUES OF THE RATIO $\sqrt{(LC)}/t_1$ AS FUNCTION OF X

X	$\sqrt{(LC)}/t_1$	X	$\sqrt{(LC)}/t_1$	X	$\sqrt{(LC)}/t_1$
—1·0	0·0000	0·6	0·8627	5·5	2·263
—0·9	0·1620	0·7	0·8978	6·0	2·387
—0·8	0·2402	0·8	0·9324	6·5	2·510
—0·7	0·3044	0·9	0·9665	7·0	2·630
—0·6	0·3613	1·0	1·0000	7·5	2·749
—0·5	0·4135	1·2	1·066	8·0	2·866
—0·4	0·4624	1·4	1·130	8·5	2·983
—0·3	0·5086	1·6	1·193	9·0	3·098
—0·2	0·5529	1·8	1·255	9·5	3·211
—0·1	0·5954	2·0	1·315	10·0	3·324
0	0·6366	2·5	1·462	12·0	3·763
0·1	0·6766	3·0	1·605	14·0	4·191
0·2	0·7155	3·5	1·743	16·0	4·608
0·3	0·7534	4·0	1·877	18·0	5·018
0·4	0·7906	4·5	2·008	20·0	5·415
0·5	0·8270	5·0	2·137		

Again from equation (**2b**) it is possible to obtain a direct measurement of $\sqrt{(L/C)}$,

$$\sqrt{(L/C)}=RY.$$

A complementary expression may be obtained for the quantity $\sqrt{(LC)}$. The ratio $t_1/\sqrt{(LC)}$ using equations (**5a**) and (**6**), may be expressed as a function of X. Numerical data for this function (covering the range $-1 \leqslant X < 10$) are given in Table 2.

From this table the values of $\sqrt{(LC)}$ may be derived from t, knowing the value of X. With a very high value of X the expression takes a limiting form

$$\sqrt{(LC)}=Xt_1/2{\cdot}303 \log 2X. \qquad \textbf{(18)}$$

From known values of $\sqrt{(L/C)}$ and $\sqrt{(LC)}$, independent values of L and C may be obtained, and the identification of all experimental parameters is now complete. Examples illustrating these methods will be given in the following paper (Bosworth 1960).

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