THE GENERAL THEORY OF THE MOTIONS OF IONS AND ELECTRONS IN GASES

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[*Manuscript received June* 20, 1960]

Summary

In this paper the more important general formulae for the drift velocities and diffusion coefficients of ions and electrons in gases are derived by the application of dynamical principles. These formulae agree with those already established by a proper application of the method of free paths. Formulae for the distribution of speeds of agitation are also derived.

I. Introduction

There is an extensive literature on the theory of the motions of ions and electrons in gases but the derivations of the formulae for drift velocities, coefficients of diffusion of ions and electrons, and the conductivities of weakly ionized gases in direct or alternating electric fields accompanied or unaccompanied by a magnetic field, that find most frequent practical use, are not readily accessible.

In what follows the chief formulae are established in a general form by the application of dynamical principles. Although the treatment is purposely elementary it does not lack rigour.

The aim is to provide a compact and uniform summary of the theory of the subject in a useful form (general references : Allis 1956 ; Margenau 1958).

II. Electronic Motion in Gases

(*a*) *General*

When electrons move freely among the molecules of a gas in the absence of an electric field they interchange energy and momentum with molecules in collisions and their steady state of random agitational motion is one in which at any instant the directions of their velocities **c** are distributed isotropically and their speeds c are distributed according to Maxwell's formula which states that the proportion of a group of n electrons whose speeds exceed c but do not exceed $c+\mathrm{d}c$ is

$$\frac{\mathrm{d}n_c}{n}=\frac{4}{\alpha^3\sqrt{\pi}}\exp\left(-c^2/\alpha^2\right).c^2\mathrm{d}c, \quad \ldots\ldots\ldots\ldots \quad (1)$$

where α is the most probable speed. In addition a condition of equipartition of energy prevails in which the mean kinetic energy of agitation $\frac{1}{2}m\overline{c^2}$ of an electron is equal to that, $\frac{1}{2}M\overline{C^2}$, of a molecule.

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This motion becomes modified in important respects when electrons move in a steady state of motion in a gas in the presence of a uniform and constant electric field **E**. It now comprises a steady drift velocity **W** of the centroid of the group with a superimposed random motion of agitation such that the speed W is much smaller (a few per cent.) than the mean speed of agitation $\bar{c}$.

The speeds c do not, in general, conform to Maxwell's distribution formula and the mean kinetic energy of agitation $\frac{1}{2}m\overline{c^2}$ of an electron exceeds that, $\frac{1}{2}M\overline{C^2}$, of a molecule by a factor k (Townsend's energy factor) which is a function (specific to each gas) of the ratio E/N of the electric field strength to the number of molecules N in unit volume of the gas. When the measurements are referred to a standard temperature (15 °C) the parameter E/p is more commonly employed than E/N, p being the pressure of the gas.

These and other aspects of electronic motion in gases are considered in greater detail in what follows.

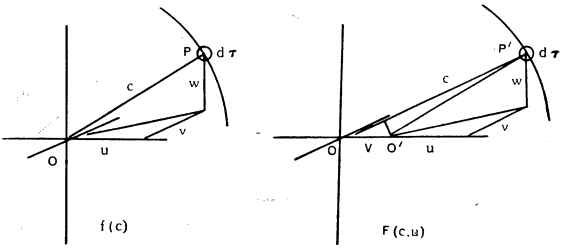


Fig. 1

(b) Nature of the Distribution Function in a Steady State of Motion

Consider a general isotropic distribution of the speeds c represented by a function $f(c)$ with the meaning that the proportion of electrons of a group n whose speeds exceed c but do not exceed $c+\mathrm{d}c$ is $\mathrm{d}n_c/n=4\pi f(c)c^2\mathrm{d}c$.

This function is represented in velocity space by a spherical distribution of points, such that the number of points contained within an element $\mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w$ of velocity space at a distance $c(u,v,w)$ from the origin is $n\,f(c)\mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w$. Consider the unsymmetrical distribution $F(c,u)$ obtained from $f(c)$ by increasing the u component of every velocity c by an amount $V(c)$ as shown in Figure 1. The centre of symmetry has been displaced from O to O' through a distance V, thus leaving unaltered the number of points in the element of space $\mathrm{d}\tau$ at P and P' respectively, but the velocity c increases from $c=OP$ on the left to $c=OP'$ on the right. Let θ be the angle between OP' and the u-axis and suppose that $V\ll c$, then $\cos\theta=u/c$ and $O'P'\doteq c-V\cos\theta=c-Vu/c$.

The number of representative points in $\mathrm{d}\tau$ on the right is

$$nF(c,u)\mathrm{d}\tau=nf\left(c-\frac{u}{c}V\right)\mathrm{d}\tau,$$

whence

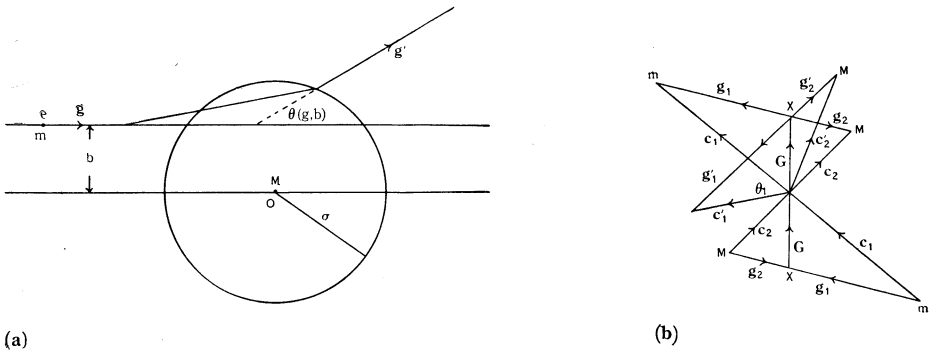
$$F(c,u)\simeq f(c)-\frac{Vu}{c}\frac{\mathrm{d}f}{\mathrm{d}c}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

It remains to determine the value of V and the form of the function $f(c)$ associated with steady electronic motion in a gas under the influence of a steady and uniform electric field **E**, it being assumed throughout that the number of electrons in unit volume is much smaller than the number of molecules in unit volume so that mutual interactions of electrons are unimportant.

(*c*) *Specification of an Encounter*

In Figure 2 (*a*) an electron or ion e with mass m approaches a molecule O with mass M at a relative velocity $\mathbf{g}$ and is deflected to leave the vicinity of O at a velocity $\mathbf{g}'$. It is supposed that when e is sufficiently distant from a molecule its trajectory is rectilinear.

In Figure 2, b is the perpendicular distance of O from the direction of $\mathbf{g}$ and θ the angle of deflection (the angle between $\mathbf{g}$ and $\mathbf{g}'$). $\theta(b,g)$ is a function of b and g.



(a) (b)

Fig. 2

It will be assumed that there is a limiting distance σ such that if b exceeds σ the deflections $\theta(b,g)$ are zero or so small that even in aggregate they are unimportant.

The vector diagram of velocities relating to an encounter is (following Maxwell) depicted in Figure 2 (*b*). $\mathbf{c}_1$ and $\mathbf{c}_2$ are the velocities respectively of the ion (or electron) and the molecule before an encounter and $\mathbf{c}_1'$ and $\mathbf{c}_2'$ those after the encounter.

G is the velocity of the centroid X of m and M and $\mathbf{g}_1$ and $\mathbf{g}_2$ are those of m and M relative to X before the encounter and $\mathbf{g}_1'$ and $\mathbf{g}_2'$ those after it. It follows that $\mathbf{g}=\mathbf{g}_1+\mathbf{g}_2$ and $\mathbf{g}'=\mathbf{g}_1'+\mathbf{g}_2'$.

In a system in which the gas as a whole has no mass motion, and to which the velocities $\mathbf{c}$ refer, the actual deflection of e is the angle θ_1 between $\mathbf{c}_1$ and $\mathbf{c}_1'$ and in general this angle depends not only upon b and g, but when $\mathbf{c}_1$ is given, upon $\mathbf{c}_2$. However, when $m/M \ll 1$ and also $c_1 \gg c_2$, as is the case if e is an electron, then $\mathbf{g} \simeq \mathbf{c}_1$ and $\theta(b,g)=\theta(b,c_1)=\theta_1$. Also c_1' differs little from c_1 in most instances.

An important quantity is $\overline{\cos\theta_1}$ the average value of $\cos\theta_1$ taken over all encounters in which the ion or electron is travelling with speed c_1 before an

encounter. Since the velocities $\mathbf{c}_1'$ are distributed with axial symmetry about the direction of $\mathbf{c}_1$, in the case of an electron where $\theta_1=\theta(b,c_1)$,

$$\overline{\cos\theta_1}=\left[2\pi\int_0^\sigma \cos\theta(b,c_1)b\,db\right]/\pi\sigma^2=2/\sigma^2\int_0^\sigma \cos(\theta,c_1)b\,db.$$

The mean value of $\cos\theta_1$ averaged over the whole distribution of speeds c_1 would be written $\overline{\overline{\cos\theta_1}}$, that is to say it is the mean value of θ in an encounter of any kind.

(*d*) *Free Paths*

Consider a large number p of free paths $x_1, x_2, \ldots, x_n$ all traversed at the same speed c by the ion or electron, but not necessarily consecutively.

If l_{0c} is the mean free path, then

$$pl_{0c}=\sum_1^n x_k.$$

Let $\bar{g}$ be the mean speed of the ion (with speed c) relative to the molecules which move at random and let N be the number of molecules in unit volume.

The sum of the times spent in traversing the paths x_k is $t=pl_{0c}/c$, that is to say, $p=ct/l_{0c}$. But $p=\bar{g}t.N\pi\sigma^2$, consequently, if $l_0=1/N\pi\sigma^2$, $l_{0c}=c/\bar{g}.N\pi\sigma^2=(c/\bar{g})l_0$. For electrons $\bar{g}=c$; $l_{0c}=1/N\pi\sigma^2=l_0$. σ is the limiting value of the impact parameter beyond which deflection of the ions are unimportant.

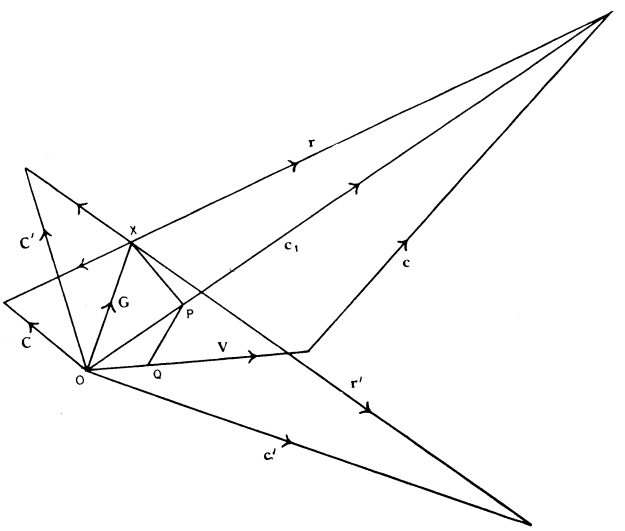


Fig. 3

(*e*) *Calculation of* V

According to Section 2 (*b*), the velocity of an electron is the vector sum $\mathbf{V}+\mathbf{c}$ of a fixed velocity $\mathbf{V}$ and velocity of agitation $\mathbf{c}$.

Consider the vector diagram of velocities of a collision of such an electron with a molecule with momentum $M\mathbf{C}$ (Fig. 3).

In Figure 3, $\mathbf{c}_1=\mathbf{V}+\mathbf{c}$; $\mathbf{G}=(m\mathbf{c}_1+M\mathbf{C})/(m+M)$ is the velocity of the centroid X of m and M, and $\mathbf{r}$ is the velocity of the electron (or ion) relative to X. These are the velocities before the encounter. After the encounter the corresponding velocities are $\mathbf{c}'$, $\mathbf{C}'$, $\mathbf{G}$, and $\mathbf{r}'$.

If no change results in the internal energy of the molecule then $r'=r$.

Consider first the case in which all directions of $\mathbf{r}'$ are equally probable when $\mathbf{C}$ and $\mathbf{c}_1$ are given, as would occur in encounters between rigid smooth spheres (Maxwell 1890).

The mean velocity of electrons and molecules ($\mathbf{c}_1$ and $\mathbf{C}$ as specified) after encounters is therefore $\mathbf{G}$. Consider next, the mean residual velocity in all encounters of electrons for which $\mathbf{c}_1$ is given but $\mathbf{C}$ is allowed to range over all possible directions. Draw XP parallel to $\mathbf{C}$ to meet $\mathbf{c}_1$ at P. Then $\mathbf{G}=\mathbf{OP}+\mathbf{PX}$. But $OP=\{m/(M+m)\}c_1$ and $XP=\{M/(M+m)\}C$, consequently P is a fixed point and XP has a fixed length when C is constant. The possible end points of $\mathbf{C}$ lie on the surface of a sphere about O and those of $\mathbf{G}$ on a sphere about P. Thus the mean projections of both $\mathbf{C}$ and $\mathbf{PX}$ on the direction of $\mathbf{c}_1$ are zero. The mean residual velocity after all such encounters is $\mathbf{OP}$. Finally $\mathbf{c}$ may assume all directions with respect to $\mathbf{V}$ and therefore $\mathbf{QP}$ does likewise. The mean residual velocity is therefore $\mathbf{OQ}=\{m/(m+M)\}\mathbf{V}$ for encounters in which the agitational speed has a fixed value c.

Suppose that all directions of $\mathbf{r}'$ are not equally probable but that $\mathbf{r}'$ is distributed with axial symmetry about $\mathbf{r}$. Since the scattering occurs with axial symmetry about the direction XQ, the mean values of the components of $\mathbf{c}'$ and $\mathbf{r}'$ normal to XQ are zero, so that the mean value of the vector $\mathbf{c}'$ when the scattering about X is not isotropic is a vector parallel to $\mathbf{r}$. It may be written $\alpha\mathbf{r}$.

But, from the triangle PXQ, $\alpha\mathbf{r}=\alpha\{M/(M+m)\}(\mathbf{c}_1-\mathbf{C})$, so that when $\mathbf{C}$ is distributed over all possible directions in space the residual mean value of $\mathbf{c}'$ is a vector parallel to $\mathbf{c}_1$ and equal to $\{\alpha M/(M+m)\}\mathbf{c}_1$. But $\mathbf{c}_1=\mathbf{V}+\mathbf{c}$, so that when $\mathbf{c}$ ranges over all possible directions the mean value of $\mathbf{c}'$ reduces to $\{\alpha M/(M+m)\}V$, where $\alpha=\overline{\cos\theta}$, where θ is an angle of deflection of the velocity $\mathbf{r}$ relative to X in an encounter when the speed of the ion or electron relative to a molecule is $g=\{(M+m)/M\}r$.

The mean residual momentum after encounters is therefore $\{(m+\alpha M)/(M+m)\}m\mathbf{V}$, which is the same as the mean momentum that the ion possesses when it enters the speed group c. The mean momentum lost in an encounter is

$$m\mathbf{V}\left(1-\frac{m+\alpha M}{M+m}\right)=\frac{mM}{M+m}(1-\alpha)\mathbf{V},$$

and the mean rate at which an ion or electron loses momentum in encounters is

$$\frac{mM}{M+m}(1-\alpha)\frac{g}{l_0}\mathbf{V},$$

where $l_0=1/N\pi\sigma^2$.

In a steady state of motion the mean momentum $m\mathbf{V}$ is the sum of the mean momentum $\{(m+\alpha M)/(m+M)\}m\mathbf{V}$ at entry and that acquired from the field,

namely, Eel_0/g. The mean momentum imparted by the field is therefore equal to the mean momentum lost in an encounter. Consequently,

$$\left.\begin{aligned} \frac{Eel_0}{g} &= \frac{mM}{m+M}(1-\alpha)V, \\ \text{or} \quad V &= Eel/mg, \\ \text{where} \quad l &= \frac{M+m}{M(1-\alpha)}l_0, \quad \alpha = \overline{\cos\theta}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

l is an equivalent mean free path and g is the mean speed of the ion or electron with agitational speed c, relative to the molecules. For electrons, $g \simeq c$, $m/M \ll 1$, $l = l_0/(1-\alpha)$. The equivalent cross section is here

$$A(c) = 1/Nl = (1-\overline{\cos\theta})/l_0 = 2\pi\int_0^\sigma \{1-\cos\theta(b)\}b\,db.$$

When electrons and ions interact as point centres of repulsive force $P = k/r^\nu$, then, if $\nu > 2$, the integral

$$2\pi\int_0^\infty \{1-\cos\theta(b)\}b\,db$$

is convergent and defines an equivalent cross section and mean free path

$$l = 1/2\pi N\int_0^\infty (1-\cos\theta(b))b\,db = 1/NA(c),$$

where $A(c) = 2\pi BA_1(\nu)/c^{4/(\nu-1)}$, in which $B = (k/m)^{2/(\nu-1)}$ and $A_1(\nu)$ is a function of ν only (Chapman and Cowling 1952, p. 171; Huxley 1957*a*, p. 125). For instance, when $\nu = 5$; $A(c) \propto 1/c$.

If the scattering is isotropic,

$$A = 2\pi\int_0^\sigma b\,db = \pi\sigma^2 = A_0.$$

In this event, $\alpha = 0$,

$$V = \frac{Ee}{mg}\,\frac{l_0(M+m)}{M} = \frac{Eel_0}{g}\left(\frac{1}{m}+\frac{1}{M}\right). \quad \ldots\ldots\ldots\ldots \quad (4)$$

In general, $l = l_0(M+m)/M(1-\alpha)$ may be written $l = l_0 + S$ (Huxley 1957, 1960), where

$$S = (m+\alpha M)l_0/M(1-\alpha). \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

(f) Formula for Drift Velocity **W** *in a Steady and Uniform Field* **E**

The mean speed of drift in the direction of **E** (parallel to Ox) of the group of electrons with speeds c is the mean value of the component u of the velocities **c** averaged over all directions in space. From equation (2) it follows that the mean value of $u = c\cos\theta$ is

$$W(c) = cf(c)\,\overline{\cos\theta} - Vc\frac{\mathrm{d}f}{\mathrm{d}c}\,.\,\overline{\cos^2\theta} = -\frac{Vc}{3}\,\frac{\mathrm{d}f}{\mathrm{d}c},$$

so that, from equation (3),

$$W(c)=\frac{Eel}{3mg}c\frac{\mathrm{d}f}{\mathrm{d}c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

The mean value of $W(c)$ taken over all speeds c is the drift speed W of the centroid of the whole group. Thus,

$$\begin{aligned} W&=-\frac{Ee}{3m}\int_0^\infty 4\pi\frac{c^2lc}{g}\frac{\mathrm{d}f}{\mathrm{d}c}\mathrm{d}c \\ &=\frac{Ee}{3m}\left\{-\left[\frac{4\pi lc^3}{g}f\right]_0^\infty+4\pi\int_0^\infty\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{lc^3}{g}\right)f.c^2\mathrm{d}c\right\} \\ &=\frac{Ee}{3m}\cdot\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{lc^3}{g}\right)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)\end{aligned}$$

where $l=l_0+S$; (equation (5)) ; g is the mean speed of an ion or electron relative to the molecules. For electrons, $m/M\ll 1$, $g=c$, equation (7) becomes

$$W=\frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)}, \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

the bar denoting an average with respect to c.

This formula was previously derived directly by a correct application of the method of free paths (Huxley 1957*a*, 1960).

Throughout, the atomic charge e is regarded as algebraically positive, that is to say, a negative value should be substituted in the case of electrons.

(*g*) *Drift Speed in an Alternating Electric Field*

Let the electric field be considered to be a rotating vector in the XOY plane, $E=E_0\exp \mathrm{i}pt$. It is assumed that the frequency $p/2\pi$ is sufficiently large that fluctuations in the mean energy $\frac{1}{2}m\overline{c^2}$ are unimportant. This assumption is also correct for the special case $p=0$.

An expression for the velocity $\mathbf{V}$ is first obtained. The momentum $m\mathbf{V}$ fluctuates in a time-dependent field because its rates of loss by encounters and of gain from the field are not equal. Thus, from Section II (*e*),

$$m\frac{\mathrm{d}V}{\mathrm{d}t}=E_0e\exp(\mathrm{i}pt)-\left(\frac{M}{M+m}-\alpha\right)\frac{g}{l_0}mV,$$

or

$$\frac{\mathrm{d}V}{\mathrm{d}t}+\frac{g}{l}V=\frac{E_0e}{m}\exp(\mathrm{i}pt), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

whence

$$V=\frac{E_0e}{m(\nu+\mathrm{i}p)}\exp(\mathrm{i}pt), \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

where $\nu=g/l$.

When the vector **V** is not directed along an axis of coordinates the appropriate form of equation (**2**) is

$$F(c,u,v,w)=f(c)-\frac{\mathbf{V}\cdot\mathbf{c}}{c}\frac{\mathrm{d}f}{\mathrm{d}c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

So that, when $E=E_0\exp(ipt)=E_x+iE_y$ and $V=V_x+iV_y$, the appropriate form of equation (**11**) is

$$F(c,u,v)=f(c)-\frac{E_0eu\exp(ipt)}{cm(\nu+ip)}\frac{\mathrm{d}f}{\mathrm{d}c}.$$

It follows, as in Section II (*f*), that, with $E=E_0\exp(ipt)$,

$$\begin{aligned}W&=-\frac{Ee}{3m}\int_0^\infty\frac{4\pi c^3}{(\nu+ip)}\frac{\mathrm{d}f}{\mathrm{d}c}\mathrm{d}c\\&=\frac{Ee}{3m}\int_0^\infty\left[c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^3}{\nu+ip}\right)\right]4\pi c^2f\mathrm{d}c\\&=\frac{Ee}{3m}\cdot\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^3}{\nu+ip}\right)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)\end{aligned}$$

(*h*) *Drift Velocity in the Presence of a Magnetic Field*

Let a magnetic field B act along the direction Oz and E along Ox and write $\omega=-Be/m$. Then equation (**9**) is to be replaced by

$$\frac{\mathrm{d}V}{\mathrm{d}t}+\nu V=\frac{E_0e}{m}\exp(ipt)+i\omega V,$$

or

$$\frac{\mathrm{d}V}{\mathrm{d}t}+(\nu-i\omega)V=\frac{E_0e}{m}\exp(ipt). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

Whence

$$V=\frac{E_0e\exp(ipt)}{m[\nu-i(\omega-p)]}, \quad \ldots\ldots\ldots\ldots \quad (14)$$

and

$$W=\frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^3}{\nu-i(\omega-p)}\right]}. \quad \ldots\ldots\ldots \quad (15)$$

Equations (**12**) and (**15**) can be derived directly by use of the method of free paths (Huxley 1957*b*).

With electrons $\nu=g/l$ becomes $\nu=c/l$.

(*i*) *Magnetic Deflection of an Electron Stream in a Gas*

In equation (**15**) let $p=0$, then

$$W=W_0\exp(i\theta)=\frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^3}{\nu-i\omega}\right)}=\frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{(\nu+i\omega)c^3}{\nu^2+\omega^2}\right]}, \quad \ldots \quad (16)$$

$$\omega=-Be/m.$$

Choose the axes of coordinates so that $+Oy$ is parallel to **B** and $+Oz$ to **E**, then it follows from equation (**16**) that

$$\left.\begin{aligned} W &= W_z + \mathrm{i}W_x = W_0(\cos\theta + \mathrm{i}\sin\theta). \\ W_z &= \frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\nu c^3}{\nu^2+\omega^2}\right)}, \\ W_x &= \frac{Ee}{3m}\cdot\omega\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^3}{\nu^2+\omega^2}\right)}, \\ \tan\theta &= \frac{W_x}{W_z} = \omega\cdot\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^3}{\nu^2+\omega^2}\right)}\Big/\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\nu c^3}{\nu^2+\omega^2}\right)}. \end{aligned}\right\} \quad \ldots\ (17)$$

In laboratory experiments in gases at pressures of a few millimetres of mercury and with fields B less than a few tens of gauss the condition $\omega^2 \ll \nu^2$ holds.

In this event, with electrons,

$$\left.\begin{aligned} W_z &\to \frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^3}{\nu}\right)} = \frac{Ee}{3m}\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)} = W, \\ W_x &\to \frac{Ee}{3m}\cdot\omega\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(l^2c)}, \\ \tan\theta &= \omega\cdot\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(l^2c)}\Big/\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)} \\ &= \omega\cdot\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(l^2c)}\cdot\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)}\Big/\left[\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)}\right]^2 = -\frac{WB}{CE}, \end{aligned}\right\} \quad \ldots\ (18)$$

where

$$C = \frac{1}{3}\left[\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)}\right]^2\Big/\overline{c^{-2}\frac{\mathrm{d}}{\mathrm{d}c}(l^2c)},$$

a dimensionless factor. Thus

$$|W| = \frac{CE}{B}|\tan\theta|. \quad \ldots\ (19)$$

Since $\tan\theta$ can be measured directly, W can be calculated if C is known. The value of C depends upon the distribution function $f(c)$ and the dependence $l \equiv l(c)$, of l upon c.

(*j*) *High Frequency Conductivity of a Weakly Ionized Gas*

An alternating electric field $E_x = X_0\cos(pt+\alpha)$, in the directions $\pm Ox$ can be resolved into two oppositely rotating vector fields as follows:

$$E_x = \tfrac{1}{2}X\exp(\mathrm{i}pt) + \tfrac{1}{2}X^*\exp(-\mathrm{i}pt),$$

where $X = X_0\exp(\mathrm{i}\alpha)$ and $X^* = X_0\exp(-\mathrm{i}\alpha)$.

It follows from equation (**15**) that the components of W are given by

$$W_x+\mathrm{i}W_y=(W_x^+ +\mathrm{i}W_y^+)+(W_x^- +\mathrm{i}W_y^-)$$
$$=\frac{e}{6m}c^{-2}\,\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^3\left[\frac{X\exp(\mathrm{i}pt)}{\nu-\mathrm{i}(\omega-p)}+\frac{X^*\exp(-\mathrm{i}pt)}{\nu-\mathrm{i}(\omega+p)}\right]\right\}. \quad \ldots (20)$$

The associated current densities are $J_x=neW_x$ and $J_y=neW_y$.

When the electric field also possesses components $E_y=Y\cos(pt+\beta)$ and $E_z=Z\cos(pt+\gamma)$ it can be seen that the complex current density $J(J_x,J_y,J_z)$ whose real parts give the physical current density is

$$\left.\begin{aligned}(J)=\begin{Bmatrix}J_x\\J_y\\J_z\end{Bmatrix}=|\,\sigma^+\,|\begin{Bmatrix}X\\Y\\Z\end{Bmatrix}\exp(\mathrm{i}pt)+|\,\sigma^-\,|\begin{Bmatrix}X^*\\Y^*\\Z^*\end{Bmatrix}\exp(-\mathrm{i}pt),\\ \text{in which}\\ |\,\sigma^+\,|=\begin{vmatrix}\sigma_{xx}^+&\sigma_{xy}^+&0\\\sigma_{yx}^+&\sigma_{yy}^+&0\\0&0&\sigma_{zz}^+\end{vmatrix}\quad\text{and}\quad|\,\sigma^-\,|=\begin{vmatrix}\sigma_{xx}^-&\sigma_{xy}^-&0\\\sigma_{yx}^-&\sigma_{yy}^-&0\\0&0&\sigma_{zz}^-\end{vmatrix},\\ \text{with } X=X_0\exp(\mathrm{i}\alpha),\ Y=Y_0\exp(\mathrm{i}\beta),\ Z=Z_0\exp(\mathrm{i}\gamma),\text{ and}\\ \sigma_{xx}^+=\sigma_{yy}^+=\mathrm{i}\sigma_{xy}^+=-\mathrm{i}\sigma_{yx}^+=(ne^2/6m)c^{-2}\,\frac{\mathrm{d}}{\mathrm{d}c}\,\frac{c^3}{\nu-\mathrm{i}(\omega-p)},\\ \sigma_{xx}^-=\sigma_{yy}^-=\mathrm{i}\sigma_{xy}^-=-\mathrm{i}\sigma_{yx}^-=(ne^2/6m)c^{-2}\,\frac{\mathrm{d}}{\mathrm{d}c}\,\frac{c^3}{\nu-\mathrm{i}(\omega+p)},\\ \sigma_{zz}^+=\sigma_{zz}^-=\sigma=\frac{ne^2}{3m}c^{-2}\,\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^3}{\nu+\mathrm{i}p}\right].\end{aligned}\right\} \quad \ldots (21)$$

III. Diffusion

(*a*) *General*

The agitational motion of the electrons or ions operates to diminish inequalities in their concentration n and to disperse a group of electrons throughout the gas. Across an elementary geometrical surface $\mathrm{d}\mathbf{S}$ at a position where grad n is not zero there is, due to diffusion, a net flux of electrons which is a function of the components of grad n. In practice $|\,\text{grad}\,n\,|/n$ is small and this flux is accurately proportional to $-\text{grad}\,n\cdot\mathrm{d}\mathbf{S}$ when no magnetic field is present. The coefficient of proportionality D is called the coefficient of diffusion. The flux is therefore $-D\,\text{grad}\,n\cdot\mathrm{d}\mathbf{S}=n\mathbf{w}\cdot\mathrm{d}\mathbf{S}$, where $\mathbf{w}$ is an equivalent convective velocity that would give the same flux across $\mathrm{d}\mathbf{S}$ were grad n equal to zero. The net transport of electrons across $\mathrm{d}\mathbf{S}$ in time $\mathrm{d}t$, being the difference between those that cross in opposite senses, is therefore

$$n\mathbf{w}\cdot\mathrm{d}\mathbf{S}\mathrm{d}t=-D\,\text{grad}\,n\cdot\mathrm{d}\mathbf{S}\mathrm{d}t.$$

(*b*) *Formula for the Coefficient of Diffusion* D

Let the direction of grad n be that of the coordinate axis $+Ox$ so that $n\mathbf{w}=n\mathbf{w}_x=-D\partial n/\partial x$, $w=-(D/n)\partial n/\partial x$. Let $w(c)$ and n_c refer to the electrons with speeds between c and $c+\mathrm{d}c$. Then $w(c)=-(D(c)/n_c)\partial n_c/\partial x$.

Consider a volume bounded by surfaces of unit area normal to Ox and a distance $\mathrm{d}x$ apart. The mean momentum of the electrons (or ions) with speeds c within this volume is $n_c m w(c)\mathrm{d}x$ and, according to equation (**3**), the rate at which momentum is destroyed within the volume is $n_c m w(c) g/l \mathrm{d}x$, where

$$l = l_0/[M/(M+m)-\alpha] = (l_0+S),$$

where S is defined in equation (**5**) and g is the mean velocity of an electron (or ion) with speed c relative to a molecule. When $\mathbf{w}(c)$ is constant this momentum is restored by transport of momentum across the boundaries. Let the unit boundary surfaces lie at positions x and $x+\mathrm{d}x$ on Ox. The mean momentum transported in time $\mathrm{d}t$ by electrons with speeds c in the direction $+Ox$ into the volume, across the unit boundary at x, is $\frac{1}{2}n_c m \overline{u^2}\mathrm{d}t = \frac{1}{6}n_c m c^2 \mathrm{d}t$. The same quantity of momentum leaves the volume in time $\mathrm{d}t$ in the direction $-Ox$ and by Newton's third law there is an equal gain of momentum to the volume in the direction $+Ox$. Thus the total gain of momentum is $\frac{1}{3}n_c m c^2 \mathrm{d}t$ in the direction $+Ox$. Similarly, the gain of momentum in the sense $+Ox$ across the boundary at $x+\mathrm{d}x$ is $-\frac{1}{3}mc^2\mathrm{d}t\{n_c+(\mathrm{d}n_c/\mathrm{d}x)\mathrm{d}x\}$.

The total gain of momentum in the sense $+Ox$ is therefore

$$-\tfrac{1}{3}mc^2(\mathrm{d}n_c/\mathrm{d}x)\mathrm{d}x\mathrm{d}t = -(\partial p_c/\partial x)\mathrm{d}x\mathrm{d}t,$$

where p_c is the partial pressure of the electrons (or ions) with speeds c. Thus the condition that $w(c)$ should not change with time is

$$-n_c m w(c) g/l - \tfrac{1}{3}mc^2 \mathrm{d}n_c/\mathrm{d}x = 0,$$

or

$$\frac{-D(c)}{n_c}\frac{\mathrm{d}n_c}{\mathrm{d}c} = w(c) = -\frac{lc^2}{3g}\frac{1}{n_c}\frac{\mathrm{d}n_c}{\mathrm{d}x},$$

from which it follows that

$$\left.\begin{aligned} D(c) &= \tfrac{1}{3}(lc^2/g) \quad \text{and} \quad D = \overline{D(c)} = \tfrac{1}{3}\overline{(lc^2/g)}. \\ \text{With electrons, } g &= c \quad \text{and} \quad D = \tfrac{1}{3}(\overline{lc}). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (22)$$

When a magnetic field B is present, the free paths of the ions or electrons are changed from straight lines between encounters to helices whose axes are parallel to B and about which they move with angular velocity $\omega = -Be/m$. The effect of the field is to reduce the coefficient of diffusion in directions normal to B to some value $D_B < D$, whereas that parallel to B retains the value D. It is necessary to obtain a formula for D_B.

Let $|\operatorname{grad} n| = \mathrm{d}n/\mathrm{d}x$, and consider the situation where $\mathbf{B}$ is parallel to $+Oy$. The Lorentz force on an electron or ion moving with velocity $\mathbf{w}$ across a magnetic field $\mathbf{B}$ is $\mathbf{F} = e\mathbf{w}\times\mathbf{B}$, consequently the direction of $\mathbf{w}$ in the presence of a magnetic field $\mathbf{B}$ does not in general coincide with that of $-\operatorname{grad} n$. In the present instance $\mathbf{w}$ has components w_x and w_z whereas $\operatorname{grad} n$ has the single component $\mathrm{d}n/\mathrm{d}x$.

The equations of dynamic equilibrium in this case become

$$\left.\begin{aligned} n_c\{-m w_z(c) g/l + w_x(c) eB\} &= 0, \\ -n_c m w_x(c) g/l - n_c w_z(c) eB - \tfrac{1}{3}mc^2 \mathrm{d}n_c/\mathrm{d}x &= 0. \end{aligned}\right\} \quad \ldots\ldots \quad (23)$$

Write, as before, $\omega=-eB/m$, then

$$w_z(c)=-w_x(c)l/g \text{ and } n_c w_x(c)=-\frac{lc^2}{3g}\cdot\frac{1}{[1+\omega^2l^2/g^2]}\frac{\mathrm{d}n}{\mathrm{d}x},$$

whence

$$D_B(c)=\frac{lc^2}{3g}\frac{1}{[1+\omega^2l^2/g^2]}=\frac{1}{3}\frac{c^2T}{[1+\omega^2T^2]},$$

where $T=l/g=1/\nu$. The coefficient of diffusion is therefore

$$D_B=\tfrac{1}{3}\overline{c^2T/[1+\omega^2T^2]}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

With electrons $g=c$ and $T=l/c=1/\nu$.

Also,

$$nw_z=-\overline{\omega T}.nw_x=\overline{\omega T}D_B\mathrm{d}n/\mathrm{d}x. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

The more general case of diffusion in the presence of a magnetic field with **B** directed along $+Oy$ and grad n arbitrarily directed is expressible in matrix notation as follows ($\omega=-Be/m$):

$$-n\begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix}=\begin{Bmatrix} D_B & 0 & \overline{\omega T}D_B \\ 0 & D & 0 \\ -\overline{\omega T}D_B & 0 & D_B \end{Bmatrix}\begin{Bmatrix} \partial n/\partial x \\ \partial n/\partial y \\ \partial n/\partial z \end{Bmatrix}. \quad \ldots\ldots \quad (26)$$

IV. The Distribution Function f(c)

Consider first the interchange of energy in a collision between an electron (or ion) and a molecule of the gas.

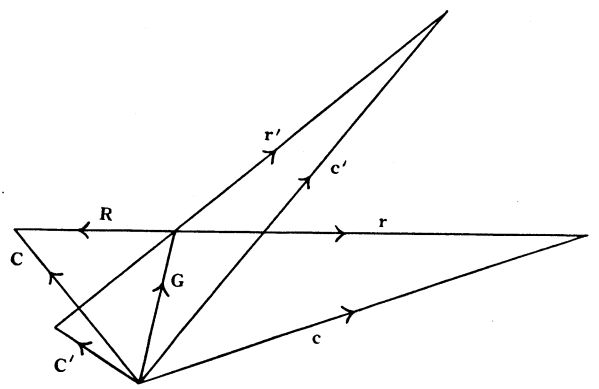


Fig. 4

(a) *Losses of Energy in Collisions*

Let **c** and **C** (Fig. 4) be the velocities respectively of an electron (or ion) and a molecule before the encounter and **c′** and **C′** the velocities following the encounter. The velocity $\mathbf{G}=m\mathbf{c}+M\mathbf{C}/(m+M)=m\mathbf{c}'+M\mathbf{C}'/(m+M)$ of the centroid X of m and M is unchanged by the encounter but the velocities **r** and **R** relative to X become **r′** and **R′**.

The following relations hold:

whence

$$\left.\begin{aligned} \mathbf{c}=\mathbf{G}+\mathbf{r}, \qquad & \mathbf{c}'=\mathbf{G}+\mathbf{r}', \\ c^2=G^2+r^2+2\mathbf{G}\cdot\mathbf{r}, \quad & c'^2=G^2+r'^2+2\mathbf{G}\cdot\mathbf{r}', \\ c^2-c'^2=2\mathbf{G}\cdot(\mathbf{r}-\mathbf{r}')+r^2-r'^2. & \end{aligned}\right\} \quad \ldots\ldots (27)$$

Further progress requires special assumptions.

(i) *Perfectly Elastic Collisions between Smooth Rigid Spheres.*—$r^2=r'^2$; all directions of $\mathbf{r}'$ about X are equally probable and the mean value of $\mathbf{G}\cdot\mathbf{r}'$ is zero. Then,

$$c^2-c'^2=2\mathbf{G}\cdot\mathbf{r}=2\left(\frac{m\mathbf{c}+M\mathbf{C}}{m+M}\right)\cdot\frac{M}{m+M}(\mathbf{c}-\mathbf{C}),$$

which, after reduction, gives

$$c^2-c'^2=\frac{2M}{(m+M)^2}[mc^2-MC^2+(M-m)\mathbf{c}\cdot\mathbf{C}].$$

Since $\mathbf{c}\cdot\mathbf{C}$ is zero in the mean, all directions of $\mathbf{C}$ being equally probable, it follows that the mean loss of energy in an encounter is, when c is given,

$$\tfrac{1}{2}mc^2-\tfrac{1}{2}mc'^2=\frac{2Mm}{(M+m)^2}[\tfrac{1}{2}mc^2-\tfrac{1}{2}M\overline{C^2}]. \quad \ldots\ldots (28)$$

(ii) *Scattering Elastic but not Isotropic.*—In this case $r'=r$, but with $\mathbf{c}$ and $\mathbf{C}$ given the mean value of the projection of $\mathbf{r}'$ on $\mathbf{r}$ is not zero but equal to $\beta\mathbf{r}$ where β is a function of r.

Thus, $c^2-c'^2=2\mathbf{G}\cdot(\mathbf{r}-\mathbf{r}')=2(1-\beta)\mathbf{G}\cdot\mathbf{r}$, and it follows as above, that

$$c^2-c'^2=\frac{2M}{(M+m)^2}(1-\beta)[mc^2-MC^2-(M-m)\mathbf{c}\cdot\mathbf{C}]$$

so that when $\mathbf{C}$ is given all directions and magnitudes

$$\tfrac{1}{2}mc^2-\tfrac{1}{2}mc'^2=(1-\overline{\beta})\frac{2Mm}{(M+m)^2}[\tfrac{1}{2}mc^2-\tfrac{1}{2}M\overline{C^2}], \quad \ldots\ldots (29)$$

in which $\overline{\beta}$ is the mean value of the projection of the velocities $\mathbf{r}'$ upon c, which, according to Section II (*e*), is the same as $\alpha=\overline{\cos\theta}$.

(*b*) *Derivation of the Distribution Function* f(c)

According to equations (2) and (3) the general form of the distribution function with a uniform and constant electric field $\mathbf{E}$ is,

$$F(c,u)=f(c)-\frac{Eel}{mg}\frac{u}{c}\frac{\mathrm{d}f}{\mathrm{d}c}. \quad \ldots\ldots (30)$$

In a steady state of motion the number of electrons with speeds between c and $c+\mathrm{d}c$ is $4\pi F(c,u)c^2\mathrm{d}c$ and the mean population of this group is constant. The

mean rate at which energy is supplied to the group is $nEeuF(c,u)\cdot 4\pi c^2 dc$ averaged over all values of u with c constant and is equal to

$$-\frac{E^2e^2lc}{3mg}\frac{df}{dc}4\pi c^2 dc.$$

The group transfers energy to the molecules at the same rate, otherwise its population would change.

Consider first the case in which the molecules are at rest ($\overline{C^2}=0$) and their encounters with electrons (or ions) resemble those between smooth rigid spheres ($\alpha=0$). The rate at which the group loses energy in encounters is

$$(4\pi c^2 f dc)n\cdot\frac{2Mm}{(M+m)^2}\cdot\tfrac{1}{2}mc^2\frac{g}{l_0},$$

with $l_0=Ml/(M+m)$ (Section II (e)).

It follows that

$$-\tfrac{1}{3}\left(\frac{Eel}{mg}\right)^2\frac{df}{dc}=-\frac{V^2}{3}\frac{df}{dc}=\frac{mc}{(M+m)}f, \quad \ldots\ldots\ldots (31)$$

whence

$$f(c)=\text{Const. exp}-\frac{m}{M+m}\int^c\frac{3cdc}{V^2}, \quad \ldots (32)$$

where $V=Eel/mg$.

In order to infer the form of $f(c)$ when $\frac{1}{2}M\overline{C^2}$ is not zero use is made of the fact that $f(c)$ reverts to Maxwell's distribution $f(c)=\exp(-3mc^2/2M\overline{C^2})$ when $V\to 0$. In this condition

$$df/dc=-(3mc/M\overline{C^2})f, \quad \ldots\ldots\ldots\ldots (33)$$

and the appropriate form of equation (31), when $\frac{1}{2}M\overline{C^2}$ is not zero, may be inferred to be

$$\tfrac{1}{3}\left(V^2+\frac{M}{M+m}\overline{C^2}\right)\frac{df}{dc}=-\frac{mc}{(M+m)}f. \quad \ldots\ldots\ldots (34)$$

Thus

$$\left.\begin{aligned} f(c)&=\text{const. exp}-\frac{3m}{(M+m)}\int^c\frac{cdc}{\left(V^2+\frac{M}{M+m}\overline{C^2}\right)}\\ &=\text{const. exp}-\int^c\frac{mcdc}{\frac{1}{3}(M\overline{C^2}+(M+m)V^2)}\\ &=\text{const. exp}-\int^c\frac{mcdc}{\varkappa T+\frac{1}{3}(M+m)V^2}. \end{aligned}\right\} \quad \ldots\ldots (35)$$

For electrons $M+m\simeq M$, $g\to c$, $V=Eel/mc$, and the expression for $f(c)$ is equivalent to that given by Chapman and Cowling (1952, p. 350).

The problem of the distribution function $f(c)$ has been considered by many investigators and references to their work will be found in the treatises of Chapman and Cowling (1952, p. 346) and of Loeb (1955, Ch. IV).

Equation (**31**) may also be derived as follows. The acceleration Ee/m in ordinary space corresponds to a constant velocity Ee/m in velocity space with a radial component $(Ee/m)(u/c)$. This radial component produces in the distribution given by equation (**2**) outward flux of representative points over the spherical surface with radius c, of amount

$$-4\pi c^2\left(\frac{E^2e^2}{3m}\,\frac{l}{g}\right)\frac{\mathrm{d}f}{\mathrm{d}c}\cdot n,$$

and in a permanent distribution this outward flux is balanced by an inward flux brought about by losses of energy in encounters. Let the encounters be similar to those between rigid smooth spheres, then the mean loss of speed $\Delta c=c-c'$ in an encounter is to be obtained from $c^2-c'^2=2Mmc^2/(M+m)^2$, it being supposed that the molecules are at rest. When $\Delta c/c\ll 1$ it follows that $\Delta c\simeq Mmc/(M+m)^2$. The inward flux over the surface with radius c is equal to the number of collisions in unit time that are made by all electrons in the velocity range c to $c+\Delta c$, namely,

$$nf\frac{g}{l_0}\cdot 4\pi c^2\cdot\frac{Mmc}{(M+m)^2}=nf\frac{g}{l}\cdot 4\pi c^2\frac{mc}{(M+m)},$$

from which it follows that

$$-\tfrac{1}{3}\left(\frac{Ee}{m}\,\frac{l}{g}\right)\frac{2\mathrm{d}f}{\mathrm{d}c}=\frac{mc}{M+m}f,$$

which is equation (**31**).

Equation (**34**) may also be derived with greater rigour as follows. As discussed above, the acceleration $\mathbf{E}e/m$ in ordinary space becomes a velocity $\mathbf{E}e/m$ of a representative point in velocity space with a radial component Eeu/mc at the surface of a sphere with radius c. This radial component is associated with an outward flux of representative points over the sphere equal to $4\pi c^2n\overline{F(c,u)Eeu/mc}$ where the average is taken with respect to u with c constant. Since

$$F(c,u)=f(c)-\left(\frac{Eel}{mg}\right)\frac{u}{c}\,\frac{\mathrm{d}f}{\mathrm{d}c},$$

it follows that the outward flux of points (in unit time) is

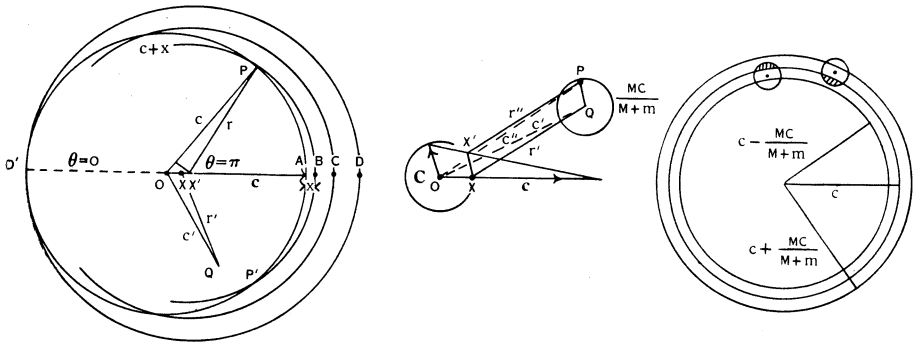
$$\frac{-4\pi c^2}{3}n\left(\frac{Ee}{m}\right)^2\frac{l}{g}\,\frac{\mathrm{d}f}{\mathrm{d}c}.$$

In a steady distribution this outward flux is cancelled by an equal and opposite inward flux that arises from the losses of speed in encounters. First suppose that the molecules are at rest. In Figure 5, $\mathbf{OA}=\mathbf{c}$ is the velocity of an electron (mass m) that collides with a molecule at rest (mass M). $\mathbf{OX}$ is the velocity $\mathbf{G}=m\mathbf{c}/(M+m)$ of the centroid of m and M, and $G\ll c$. Let the velocity of the electron after the collision relative to the centroid be $\mathbf{r}'=\mathbf{XQ}$, then the total velocity is $\mathbf{c}'=\mathbf{OQ}$. Since $r'=XA$, it follows that $c'<c$ and all representative points $\mathbf{c}'$ lie within the sphere with radius c. If the initial velocity

is **OB** with a speed $c+x$, where $x=AB$, then the representative points after collision lie on a sphere whose centre is at X', where

$$OX'=\frac{m}{M+m}(c+x)\simeq\frac{mc}{M+m},$$

and whose radius $r=X'B$. This sphere intersects the sphere with centre O and radius c in a circle on which lies the point P. Thus the representative points that lie on the spherical cap PBP' lie outside the sphere with centre O whereas those on the complementary arc lie within. Let $c^2\mathrm{d}\omega$ be an element of surface on which A lies. The number of velocity points within an element of volume $c^2\mathrm{d}\omega\mathrm{d}x$ at B is $nfc^2\mathrm{d}\omega\mathrm{d}x$ and the corresponding number of encounters



(a) (b) (c)

Fig. 5

in unit time is $p=c^2\mathrm{d}\omega nf(g/l_0)\mathrm{d}x$. Let the angle $D'OP=\theta$, then if the scattering is isotropic the number of encounters that give velocity points within the sphere of radius c is $\frac{1}{2}p(1-\cos\theta)$ in unit time. When the scattering is not isotropic this number becomes

$$p\frac{\varphi(\theta)(1-\cos\theta)}{2},\quad\text{with }\tfrac{1}{2}\int_0^\pi\varphi(\theta)\sin\theta\mathrm{d}\theta=1.$$

From the figure, $c^2=r^2+OX'^2-2OX'.r\cos\theta$ with $OX'=\{mc/(M+m)\}(c+x)$ and $r=\{M/(M+m)\}(c+x)$, and it can be deduced that

$$x\simeq\frac{Mmc}{(M+m)^2}(1+\cos\theta),\quad \mathrm{d}x=-\frac{Mmc}{(M+m)^2}\sin\theta\mathrm{d}\theta.$$

The number of encounters in unit time for which the initial velocity points lie within the element of velocity space $c^2\mathrm{d}\omega.AD$ and whose final points fall within the sphere c is therefore

$$\begin{aligned}nfc^2\mathrm{d}\omega\frac{g}{2l_0}\int_A^D\varphi(\theta)(1-\cos\theta)\mathrm{d}x&=\frac{nfc^2\mathrm{d}\omega g}{2l_0}\cdot\frac{Mmc}{(M+m)^2}\int_0^\pi(1-\cos\theta)\varphi(\theta)\sin\theta\mathrm{d}\theta\\&=\frac{(1-\overline{\cos\theta})}{l_0}g\cdot\frac{Mmc}{(M+m)^2}nfc^2\mathrm{d}\omega\\&=\frac{gmc}{l(M+m)}nfc^2\mathrm{d}\omega,\end{aligned}$$

where, in accordance with equation (3), $l=l_0(M+m)/M(1-\overline{\cos\theta})$. The inward flux of points across unit area in unit time is therefore $\{gmc/l(M+m)\}nf$. To estimate the influence of the molecular motions, consider Figure 5 (*b*). When the molecules are at rest the velocity of the centroid is $\mathbf{OX}=m\mathbf{c}/(M+m)=\mathbf{G}$; $G\ll c$. An encounter transforms $\mathbf{c}$ into $\mathbf{c}'=\mathbf{G}''\mathbf{r}'$. When the molecule has a velocity $\mathbf{C}$, $\mathbf{G}=\mathbf{OX}$ becomes $\mathbf{G}=\mathbf{OX}'$, $\mathbf{c}'\rightarrow\mathbf{c}''=\mathbf{G}'+\mathbf{r}''$, with $r''\simeq r'$. As $\mathbf{C}$ assumes all possible directions, the end points of c'' lie on the surface of a sphere about P and with radius $\{M/(M+m)\}C$. Thus, in velocity space (Fig. 5 (*c*)) the points which originally lay within an element of volume now become dispersed over a sphere centred on the volume and with radius $MC/(M+m)$. The spherical dispersal of velocity points about such elements of volume lying between two spherical surfaces with radii $c\pm MC/(M+m)$ give a flux of points inwards across the spherical surface with radius c. If $|y|<MC/(M+m)$ is the distance of an elementary volume from the surface with radius c, then it is readily shown that the flux of points inwards across unit surface of the sphere c is

$$(1-\cos\bar{\theta})n\frac{g}{2l_0}\times\int_{-MC/(M+m)}^{MC/(M+m)}(1-|y|)\left(f+y\frac{\mathrm{d}f}{\mathrm{d}c}\right)\mathrm{d}y$$
$$=\frac{(1-\overline{\cos\theta})M}{3(M+m)l_0}\frac{gMC^2}{(M+m)}n\frac{\mathrm{d}f}{\mathrm{d}c}=\frac{gMC^2}{3l(M+m)}n\frac{\mathrm{d}f}{\mathrm{d}c}.$$

The mean flux for all values of C is

$$\frac{ng}{3l}\frac{M\overline{C^2}}{(M+m)}\frac{\mathrm{d}f}{\mathrm{d}c}.$$

In a steady distribution of speeds c the total flux across any sphere in velocity space is zero, thus,

$$-\left(\frac{Ee}{m}\right)^2\frac{l}{3g}\frac{\mathrm{d}f}{\mathrm{d}c}-\frac{g}{l}\frac{m}{(M+m)}cf-\frac{gM\overline{C^2}}{3l(M+m)}\frac{\mathrm{d}f}{\mathrm{d}c}=0,$$

or

$$-\tfrac{1}{3}\left(V^2+\frac{M}{M+m}\overline{C^2}\right)\frac{\mathrm{d}f}{\mathrm{d}c}=\frac{mc}{(M+m)}f, \quad \ldots\ldots \quad (34)$$

which is equation (34) whose solution is equation (35).

The general case in which electrons make both elastic encounters and inelastic encounters in which a large proportion of their energy is lost, does not appear to lead to a simple formula for the distribution. Nevertheless, when electrons move in diatomic gases and their mean energies are not greater than about five times that of the gas molecules, the inelastic losses of energy in encounters are associated with changes in the rotational states of the molecules. Moreover, these transitions are produced by those electrons with energies appreciably greater than the mean energy. Let ΔQ be the energy absorbed by a molecule in changing from one rotational state to a neighbouring state. Suppose that in such collisions $\Delta Q/\frac{1}{2}mc^2\ll 1$. Since $c^2-c'^2=2\Delta Q/m$, $\Delta C\simeq\Delta Q/mc$. Let a proportion of encounters in which the speed is c and losses ΔQ occur be $x(c)$. It

follows that the inward flux of points over the sphere of radius c in velocity space now becomes, with $l_0=lM/(M+m)$,

$$nf\left[\frac{mc}{(M+m)}(1-x)g/l+\frac{xg}{l_0}\frac{\Delta Q}{mc}\right]4\pi c^2\mathrm{d}c=nf\left[(1-x)+\left(\frac{M+m}{m}\right)^2\frac{x\Delta Q}{Mc^2}\right]\frac{mc}{(M+m)}4\pi c^2\mathrm{d}c,$$

from which it follows that

$$-\tfrac{1}{3}\left(\frac{Eel}{mg}\right)^2\frac{\mathrm{d}f}{\mathrm{d}c}=\left[1-x+\left(\frac{M+m}{m}\right)^2\frac{x\Delta Q}{MC^2}\right]\frac{mc}{M+m}f$$
$$=\frac{1}{a^2}\frac{mc}{(M+m)}f,$$

where a^2 is the reciprocal of the quantity in the square bracket. Which, when modified to allow for the influence of the agitational motions of the molecules to give Maxwell's formula when $E=0$, becomes

$$-\tfrac{1}{3}\left[(aV)^2+\frac{M\overline{C^2}}{M+m}\right]\frac{\mathrm{d}f}{\mathrm{d}c}=\frac{mc}{M+m}f,$$

whence

$$f=\text{const. exp}-\int^c\frac{3mc\mathrm{d}c}{(M+m)(aV)^2+M\overline{C^2}}. \quad \ldots\ldots \quad (36)$$

In practice $x\ll 1$; $m/M\ll 1$; $1/a^2=1+Mx\Delta Q/(mc)^2$,

$$f=\text{const. exp}-\int^c\frac{3mc\mathrm{d}c}{M[(aV)^2+\overline{C^2}]}. \quad \ldots\ldots\ldots\ldots \quad (37)$$

Suppose that $x(c)=0$ when $c<c_1$, then

$$\left.\begin{aligned} f&=\text{const. exp}-\int^c\frac{3mc\mathrm{d}c}{M(V^2+\overline{C^2})}, \qquad c<c_1 \ldots (a),\\ f&=\text{const. exp}-\int^c\frac{3mc\mathrm{d}c}{M[(aV)^2+\overline{C^2}]}, \qquad c>c_1 \ldots (b),\end{aligned}\right\} \quad \ldots\ldots \quad (38)$$

when $c_1\gg\bar{c}$, the distribution, except for the groups with large energies, is the same as if all collisions were elastic ($a=1$, $x=0$).

When c exceeds c_1, $(M/m)x\Delta Q$ soon exceeds 1 and $a\to mc/(xM\Delta Q)^{\frac{1}{2}}$,

$$f\to\text{const. exp}-\frac{3m\Delta Q}{(Ee)^2}\int^c\frac{xc\mathrm{d}c}{l}, \quad c>c_1. \quad \ldots\ldots\ldots \quad (39)$$

(c) *Limiting Form of Equations* (35) *and* (38)

When $V^2\gg\overline{C^2}$ and $c_1\gg\bar{c}$ the speeds of most electrons are distributed according to the law

$$f(c)=\text{const. exp}-\int^c\frac{3mc\mathrm{d}c}{V^2}, \quad \ldots\ldots\ldots\ldots \quad (40)$$

where $V=Eel/mc$.

When $l(c) \propto c^r$ equation (**40**) reduces to the form

$$4\pi c^2 f(c)\mathrm{d}c = \frac{4c^2}{\alpha^3 \Gamma(3/n)} \exp(-c^n/\alpha^n)\mathrm{d}c, \quad \ldots\ldots\ldots (41)$$

where α is a speed and $n = 4 - 2r$. Also

$$4\pi \int_0^\infty f(c) c^2 \mathrm{d}c = 1.$$

It follows from the standard integral

$$\int_0^\infty \exp(-y^n) y^m \mathrm{d}y = \frac{1}{n} \Gamma\left(\frac{m+1}{n}\right)$$

that the mean value of the sth power of the speeds c is

$$\overline{c^s} = \alpha^s \Gamma\left(\frac{s+3}{n}\right) / \Gamma(3/n). \quad \ldots\ldots\ldots\ldots (42)$$

The form of the distribution function in a high frequency field is obtained if V^2 in equations (**37**) and (**40**) is given the value

$$V^2 = \left(\frac{Ee}{2m}\right)^2 \left[\frac{1}{\nu^2 + (\omega - p)^2} + \frac{1}{\nu^2 + (\omega + p)^2}\right],$$

as follows from equation (**14**).

V. Ratio W/D

From the expressions for the velocity of drift **W** and the coefficient of diffusion D (equations (**7**) and (**22**)) it follows that

$$\frac{W}{D} = \frac{Ee}{m} \overline{c^{-2} \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{lc^3}{g}\right)} \Big/ \left(\overline{\frac{lc^2}{g}}\right). \quad \ldots\ldots\ldots\ldots (43)$$

In the special cases in which the speeds c are distributed according to Maxwell's formula (equation (**1**)) the factor

$$\begin{aligned}
\overline{c^{-2} \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{lc^3}{3}\right)} &= \frac{4}{\alpha^3 \sqrt{\pi}} \int_0^\infty \mathrm{e}^{-c^2/\alpha^2} \frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{lc^3}{g}\right)\mathrm{d}c \\
&= \frac{8}{\alpha^5 \sqrt{\pi}} \int_0^\infty \mathrm{e}^{-c^2/\alpha^2} \frac{lc^4}{g}\mathrm{d}c \\
&= 2\left(\overline{\frac{lc^2}{g}}\right) \\
&= \frac{3}{\overline{c^2}}\left(\overline{\frac{lc^2}{g}}\right).
\end{aligned}$$

Consequently, in this case, equation (**43**) becomes

$$\frac{W}{D} = \frac{3}{2}\left(\frac{Ee}{\frac{1}{2}m\overline{c^2}}\right) = \frac{Ee}{\varkappa T} = \frac{EN_0 e}{RT}, \quad \ldots\ldots\ldots (44)$$

where N_0 is Avogadro's number (per mol), $\varkappa$ Boltzmann's constant, and R the gas constant.

The same formula results in the special case in which $l \propto g$ (inverse fifth power law of interaction).

When ions move in gases their agitational speeds are distributed according to Maxwell's law (unless E/p is very large) and equation (**44**) is satisfied, but the motions of electrons in an electric field in a gas do not conform to equation (**43**) in general. The complete expression in this case becomes (with $g=c$)

$$\left.\begin{aligned} &\frac{W}{D}=\left(\frac{Ee}{\tfrac{1}{2}m\overline{c^2}}\right)F, \\ \text{with}\quad &F=\overline{c^2}.\overline{c^{-2}\,\frac{\mathrm{d}}{\mathrm{d}c}(lc^2)/2(\overline{lc})}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (45)$$

The value of F is determined both by the law of distribution of the speeds c and the dependence $l \equiv l(c)$. When $l \propto c$ equation (**44**) is valid, but when the law of distribution assumes the more general form of equation (**43**) the values of F that correspond to values of $n=2$, 4, and 6 are $F=3/2$, $1{\cdot}312$, and 1.

VI. References

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