FREE CONVECTION IN A VERTICAL TUBE WITH A LINEAR WALL TEMPERATURE GRADIENT

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Summary

The paper discusses convection in a vertical tube closed at both ends in which the temperature of the walls is arranged to increase linearly with depth. For the purposes of the theory, the convective flow is assumed to be radially symmetrical about the axis of the tube.

The temperature and velocity distributions in the pipe are found to be dependent on a non-dimensional modified Rayleigh number H. Experiments conducted at values of H between 90 and 30 000 agree with theoretical predictions for values of H below 300 and above 3000. Negative temperature gradients occurring along the axis for values of H between 700 and 950 indicate that in this range the flow cannot be radially symmetric. This non-symmetric flow would develop first, as H is increased from zero, and it is suggested that the agreement between the experimental and theoretical results for $H<300$ shows that the critical value of H for the non-symmetrical flow does not differ greatly from the value (142) obtained theoretically for the symmetric regime. Presumably it is at values of H of 3000 and above that the flow takes an axially symmetric form. This higher range is appropriate to bores in the New Zealand thermal regions.

I. INTRODUCTION

Ever since the first bores were drilled at Wairakei, N.Z., for steam for power production, temperature measurements have been made using a geothermograph at 100 ft intervals down closed bores. These temperatures are one of the few physical measurements that can be made at depth in the field, but up to the present time there has been considerable doubt as to whether the temperatures measured indicate closely the temperatures of the surrounding ground at the same depth. It is probable that these temperatures will be related in some manner to the steady-state temperatures of the surrounding ground, but many man-introduced factors (e.g. bore history) may have a controlling influence.

Since a bore is at a higher temperature at the bottom, it is probable that free convection will occur within it. This convective flow will have the effect of transferring heat from one region of the surrounding ground to another at a different level, thus temperatures measured in the bore are likely to be different from those that would exist in the ground originally, i.e. the temperature isotherms will be displaced in the vicinity of a bore. It will also cause some variation of temperature with radial distance.

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II. Previous Work

Ostrach (1957) reviews most of the available literature about convection phenomena in fluids heated from below. In the bibliography at the end of this paper it is seen that there is very little previous work on convection in vertical pipes. Ostrach (1952) discusses free convection in vertical channels (between parallel plates), which enables him to introduce some simplification not available when studying tubes. The few papers in existence concerning free convection in vertical pipes have arisen because of the problem of cooling turbine blades by free convection of the fluid in channels in the blades. Of these papers, only two have dealt with the theoretical side, Lighthill (1953) and Eckert and Jackson (1950). Both these papers deal with a vertical pipe at constant wall temperature fed at the top by a reservoir of cool fluid. The work of Lighthill has been verified experimentally by Martin and Cohen (1954).

III. Assumptions

In order to build up a mathematical theory it has been necessary to make many assumptions that to some degree divorce the problem from that of a bore drilled in a thermal area. This is regrettable, but it is possible to obtain an indication of the parameters involved and the effect of the temperature and to design further experiments from which it may be possible to estimate empirically the effect in bores.

The following assumptions were made :

(1) A pipe of length h and radius a, closed at both ends, is so maintained that the temperature gradient at the wall is linear, the higher temperature being at the lower end.

(2) There is a point on the axis at which the flow velocity is zero—this point is taken as the origin of coordinates.

(3) The length of the pipe is much greater than the radius.

(4) The kinematic viscosity, the thermal diffusivity, and the coefficient of volumetric expansion are all independent of temperature.

(5) The law of variation of density with temperature is

$$\rho^{-1}=\rho_0^{-1}\{1+\beta(T-T_0)\}, \tag{1}$$

where ρ is the density at a temperature T,
ρ_0 is the density at a temperature T_0,
β is the coefficient of volumetric expansion of water at T_0,
T_0 is the temperature at the origin.

(6) The boundary layer approximations apply, i.e. the gradient of a quantity along the tube is neglected in comparison with its gradient along a radius. This may be justified by the large length/radius ratio.

(7) The flow is symmetrical about the axis. At the time when the theoretical approach was done this was felt to be a logical assumption. Some of the experimental results, however, could only be accounted for by a non-symmetric flow pattern. A. McNabb (personal communication) has since shown from stability

theory that this non-symmetric flow regime would occur, as the controlling parameter H is increased from zero, before the axially symmetric flow regime.

The non-symmetric flow regime is not discussed theoretically in this paper as it is outside the limits expected to be met in bores.

IV. Theoretical Discussion

Inside the pipe in the vicinity of the origin, the equations of conservation of mass, momentum, and heat which apply to the liquid for steady axi-symmetrical flow, and with the boundary layer approximation, are (Lighthill 1953):

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial R}+\frac{V}{R}=0, \tag{2}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial R}=-g-\frac{1}{\rho}\frac{\partial p}{\partial X}+\nu\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}\right), \tag{3}$$

$$\frac{\partial p}{\partial R}=0, \tag{4}$$

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial R}=\alpha\left(\frac{\partial^2 T}{\partial R^2}+\frac{1}{R}\frac{\partial T}{\partial R}\right), \tag{5}$$

where X is the vertical distance measured upwards from the origin,
R is the radial distance,
U is the axial velocity upwards of the liquid,
V is the radial velocity of the liquid,
g is the acceleration due to gravity,
ρ is the density of the liquid,
p is the pressure,
ν is the kinematic viscosity (viscosity/density) of the liquid,
α is the thermal diffusivity of the liquid.

The boundary conditions are:

$$\left.\begin{array}{lll} U=0, & V=0, & \text{at } R=a, \\ & U=0, & \text{at } X=0, \\ & T=T_s(X), & \text{at } R=a, \\ & T=T_0, & \text{at } X=0. \end{array}\right\} \tag{6}$$

From the boundary conditions, since at $R=a$, $U=0$, $V=0$, equation (3) reduces to

$$0=-g-\frac{1}{\rho_s}\frac{\partial p}{\partial X}+\nu\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}\right)_{R=a}; \tag{7}$$

hence allowing for the variation of density (1), and substituting for $\partial p/\partial X$,

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial R}=g\beta(T-T_s(X))+\nu\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}\right)_a^R. \tag{8}$$

It is possible to reduce the equations and boundary conditions simultaneously to a non-dimensional form by the following substitutions:

$$\left.\begin{aligned} U&=(\alpha h/a^2)u,\\ V&=(\alpha/a)v,\\ R&=ar,\\ X&=hx,\\ T-T_0&=-(\nu\alpha h/\beta g a^4)\theta,\\ T_s(X)-T_0&=-(\nu\alpha h/\beta g a^4)\theta_s(x), \end{aligned}\right\} \tag{9}$$

The equations become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}=0, \tag{10}$$

$$\frac{1}{\sigma}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)=-(\theta-\theta_s)+\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}\right)_1^r, \tag{11}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial r}=\left(\frac{\partial^2 \theta}{\partial r^2}+\frac{1}{r}\,\frac{\partial \theta}{\partial r}\right), \tag{12}$$

and the boundary conditions are

$$\left.\begin{aligned} u=0,\ v&=0, && \text{at } r=1,\\ u&=0, && \text{at } x=0,\\ \theta&=\theta_s(x), && \text{at } r=1,\\ \theta&=0, && \text{at } x=0. \end{aligned}\right\} \tag{13}$$

The equations (10), (11), (12) may be integrated over a cross section to give

$$\int_0^1 ru\mathrm{d}r=0, \tag{14}$$

$$\frac{\partial}{\partial x}\int_0^1 ru\theta\mathrm{d}r=\left(\frac{\partial \theta}{\partial r}\right)_{r=1}, \tag{15}$$

$$\int_0^1 r\theta\mathrm{d}r=\tfrac{1}{2}\left(\frac{\partial u}{\partial r}-\frac{\partial^2 u}{\partial r^2}\right)_{r=1}. \tag{16}$$

Since experimental measurements were made only at the wall of the pipe and along the central axis, equations (10), (11), and (12) may be simplified to read

$$\left(\frac{\partial^2 \theta}{\partial r^2}+\frac{1}{r}\,\frac{\partial \theta}{\partial r}\right)_{r=1}=0, \tag{17}$$

$$\left(u\frac{\partial \theta}{\partial x}\right)_{r=0}=\left(\frac{\partial^2 \theta}{\partial r^2}+\frac{1}{r}\,\frac{\partial \theta}{\partial r}\right)_{r=0}, \tag{18}$$

$$(\theta-\theta_s)_{r=0}=\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}\right)_1^0, \tag{19}$$

and these equations, (17), (18), and (19), together with (14), (15), and (16), are the equations for which a solution is required.

As throughout the pipe $\theta_s(x)=Hx$, where H is the modified Rayleigh number, $\beta g a^4(T_2-T_1)/\nu\varkappa h$, T_1 and T_2 being the temperature at top and bottom of the pipe respectively, the solution for θ and u in the vicinity of the origin must be linear with respect to x. Hence θ and u may be replaced by

$$\theta=x(\alpha_0+\alpha_1r^2+\alpha_2r^4+\alpha_3r^6)+Hx, \tag{20}$$
$$u=x(\beta_0+\beta_1r^2+\beta_2r^4+\beta_3r^6). \tag{21}$$

Then from the boundary conditions (13) and the equations (14), (16), (17), and (19),

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3=0, \tag{22}$$
$$\beta_0+\beta_1+\beta_2+\beta_3=0, \tag{23}$$
$$12\beta_0+8\beta_1+4\beta_2+3\beta_3=0, \tag{24}$$
$$12\alpha_0+8\alpha_1+4\alpha_2+3\alpha_3+12H=-12(8\beta_2+24\beta_3), \tag{25}$$
$$\alpha_1+4\alpha_2+9\alpha_3=0, \tag{26}$$
$$\alpha_0=-(16\beta_2+36\beta_3). \tag{27}$$

Solving for all the unknowns in terms of α_0, α_1, and H,

$$\theta=x\left(\alpha_0+\alpha_1r^2-\frac{9\alpha_0+8\alpha_1}{5}r^4+\frac{4\alpha_0+3\alpha_1}{5}r^6\right)+Hx, \tag{28}$$

$$\begin{aligned}u=\frac{x}{1440}\{&(-66\alpha_0+68\alpha_1+240H)+(126\alpha_0-153\alpha_1-540H)r^2\\&+(-36\alpha_0+153\alpha_1+540H)r^4+(-24\alpha_0-68\alpha_1-240H)r^6\},\end{aligned} \tag{29}$$

and substituting in (18)

$$\{5760-68(\alpha_0+H)\}\alpha_1=(-66\alpha_0+240H)(\alpha_0+H) \tag{30}$$

and in (15)

$$\begin{aligned}&\alpha_0(1\cdot239\alpha_0-2\cdot439\alpha_1-8\cdot614H)\\&+\alpha_1(-1\cdot258\alpha_0+1\cdot535\alpha_1+5\cdot414H)=-1152(3\alpha_0+\alpha_1).\end{aligned} \tag{31}$$

From (30) it is possible to obtain a unique value of α_1, in terms of α_0 and H which may be substituted in (31) to give

$$\begin{aligned}&\{68(\alpha_0+H)-5760\}^2(1\cdot239\alpha_0^2-8\cdot614\alpha_0H+3456\alpha_0)\\&+\{68(\alpha_0+H)-5760\}(66\alpha_0-240H)(\alpha_0+H)(-3\cdot697\alpha_0+5\cdot414H+1152)\\&+(66\alpha_0-240H)^2(\alpha_0+H)^2\times1\cdot535=0.\end{aligned} \tag{32}$$

Multiplying this out and substituting

$$\mu=\alpha_0/H, \tag{33}$$

then

$$\begin{aligned}&4\cdot177\mu^4+(12\cdot180-21\cdot585\times10^3H^{-1})\mu^3+(11\cdot771-10\cdot017\times10^3H^{-1}\\&+3\cdot104\times10^6H^{-2})\mu^2+(3\cdot708+22\cdot883\times10^3H^{-1}+1\cdot838\times10^6H^{-2}\\&-0\cdot115\times10^9H^{-3})\mu+(-0\cdot060+11\cdot317\times10^3H^{-1}-1\cdot593\times10^6H^{-2})=0\end{aligned} \tag{34}$$

is obtained.

From this it is possible to obtain solutions for μ for various values of H. For the axially symmetric flow regime considered, convection would have the effect of reducing the temperature gradient and hence α_0 would lie in the range $0\geqslant\alpha_0\geqslant-H$. Thus only solutions for μ in the range $0\geqslant\mu\geqslant-1$ are considered.

V. Results

Table 1 lists the value of μ, α_0, and the non-dimensional axial temperature gradient for various values of the non-dimensional wall temperature gradient H.

Table 1

THEORETICAL RELATIONSHIP BETWEEN WALL AND AXIAL TEMPERATURE GRADIENTS

All units are non-dimensional

Wall Temperature Gradient H	μ	α_0	Axial Temperature Gradient
0	0	0	0
142	0	0	142
250	—0·177	—44	206
500	—0·332	—161	339
750	—0·366	—275	475
1 000	—0·391	—391	609
1 250	—0·405	—506	744
2 000	—0·420	—840	1 160
2 500	—0·422	—1 055	1 445
5 000	—0·395	—1 975	3 025
7 500	—0·345	—2 588	4 912
10 000	—0·293	—2 930	7 070
25 000	—0·112	—2 800	22 200
50 000	—0·046	—2 280	47 700
75 000	—0·024	—1 830	73 200
100 000	—0·014	—1 350	98 700

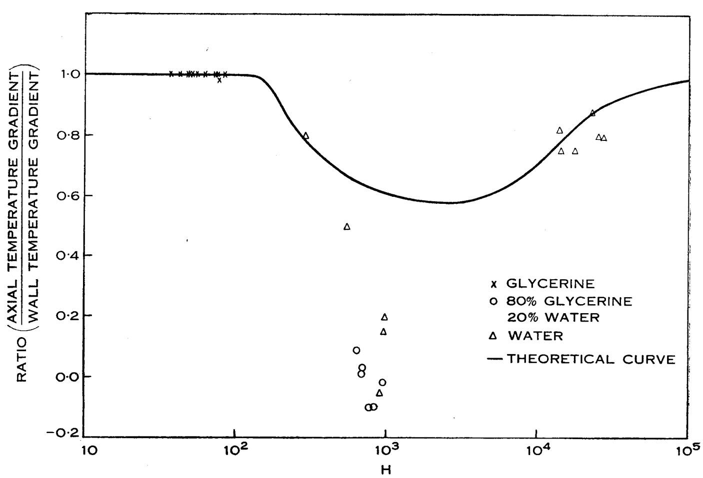

Fig. 1.—Comparison of theoretical and experimental results.

The ratio of the non-dimensional axial temperature gradient to the non-dimensional wall temperature gradient is plotted against H in Figure 1—experimental results are displayed on the same graph.

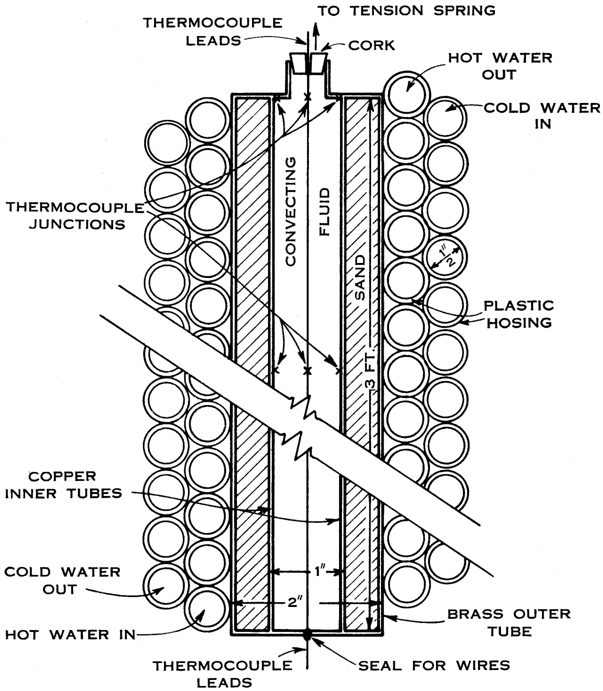

Fig. 2 (*a*).—The experimental apparatus, lengthwise central section.

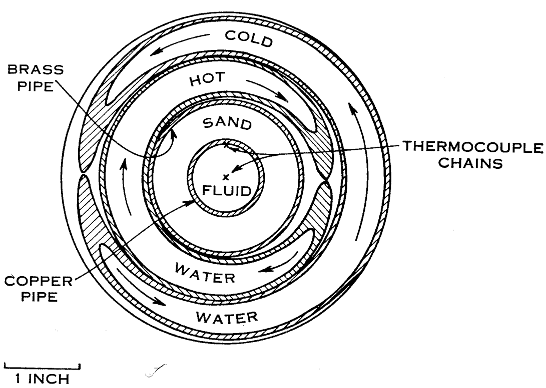

Fig. 2 (*b*).—The experimental apparatus, section perpendicular to axis.

VI. Experimental Procedure

The convection pipe used in the experiment consisted of a copper tube 36 in. long and of internal diameter 1 in., closed at both ends by a copper plate. Temperatures were measured by thermojunctions arrayed along the axis of the tube and along the wall. Holes were provided for filling the tube. The arrangement is shown in Figure 2 (*a*).

This tube was sealed within a larger brass tube, also 36 in. long, but 2 in. in diameter, the intervening space being filled with dry sand. Two layers of plastic hosing, $\frac{1}{2}$ in. internal diameter, were close-wound about the outer tube. Hot water from a constant temperature, constant flow reservoir was fed upwards through the inner coil, and cold water from another controlled reservoir was fed downwards through the outer coil. The hot water was therefore cooled

TABLE 2

EXPERIMENTALLY DETERMINED RELATIONSHIP BETWEEL WALL AND CENTRAL TEMPERATURE GRADIENTS

Convecting Fluid	Run No.	Non-dimensional Wall Temp. Grad. (H)	Non-dimensional Central Temp. Grad.	Difference
Glycerine	1	51	51	0
	2	49	49	0
	4	72	72	0
	5	37	37	0
	6	50	50	0
	8	73	73	0
	9	55	55	0
	10	43	43	0
	12	49	49	0
	13	62	62	0
	14	84	84	0
	15	78	76·5	—1·5
80% Glycerine 20% Water	18	685	7	—678
	19	685	20	—665
	20	775	—80	—855
	21	650	58	—592
	22	860	—89	—951
	23	945	—19	—963
Water	24	25 900	20 700	—5 170
	25	27 350	21 900	—5 464
	26	14 500	10 900	—3 430
	27	17 800	13 500	—4 300
	28	14 200	11 600	—2 600
	29	23 300	20 500	—2 800
	30, 31	290	233	—57
	32, 33	550	275	—275
	34, 35, 36	970	145	—835
	37, 38	990	200	—790
	39, 40	910	—45	—955

as it ascended and by this means it proved possible to create temperature gradients ranging from 0·4 degC to 24 degC per 3 ft of pipe. A cross-sectional view of the arrangement is shown in Figure 2 (*b*).

Eleven thermojunctions were set at equal spacing along the wall of the inner tube and 11 others were arranged along the axis of the inner tube, matching those in the wall.

By the use of glycerine, water, or a mixture of the two it was possible to adapt the modified Rayleigh number H to a wide range of values, between 40 and 3×10^4.

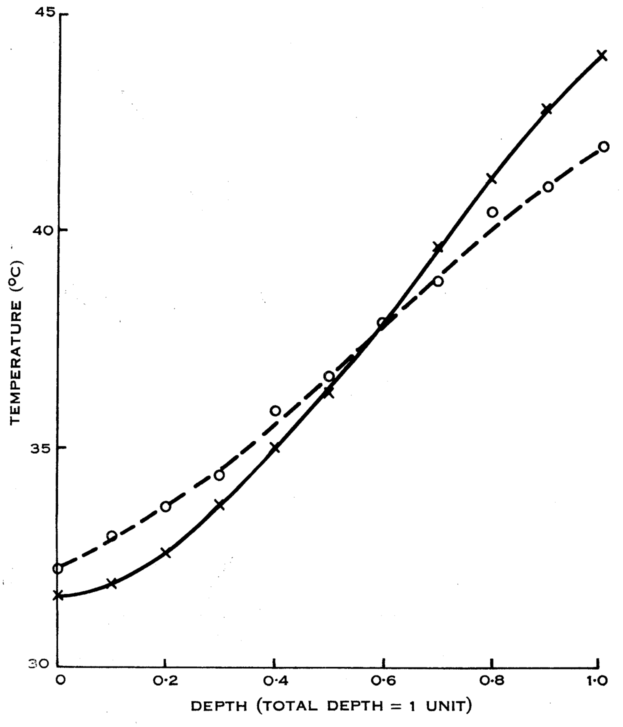

Fig. 3.—Typical profiles, —— along wall and ---- along axis for large values of H. This case is run No. 26 with $H = 14\,500$. Here the flow is believed to be axially symmetric.

VII. Experimental Results

The experimental results are given in Table 2.

The H values are the product of the temperature gradients and the physical constants of the liquid. As these physical constants have been taken at the mean temperature of the fluid, based on the thermocouple readings, the H values are only approximate. The axial temperature gradient is estimated in the central section of the pipe, around the point where the axial and wall temperatures are equal.

Figures 3 and 4 show typical results.

VIII. Conclusion

For H less than 300 and greater than 3000 the theoretical results appear to agree with the experimental ones as closely as would be expected taking into account the approximations involved. Between these values the experimental results differ considerably from those calculated from the theory. In particular at values of H between 700 and 950, negative temperature gradients occur on the axis. Figure 4 illustrates this regime. Such a flow cannot have radial symmetry. The fluid must be rising up one side of the pipe and descending on the other. As mentioned earlier, McNabb (personal communication) states

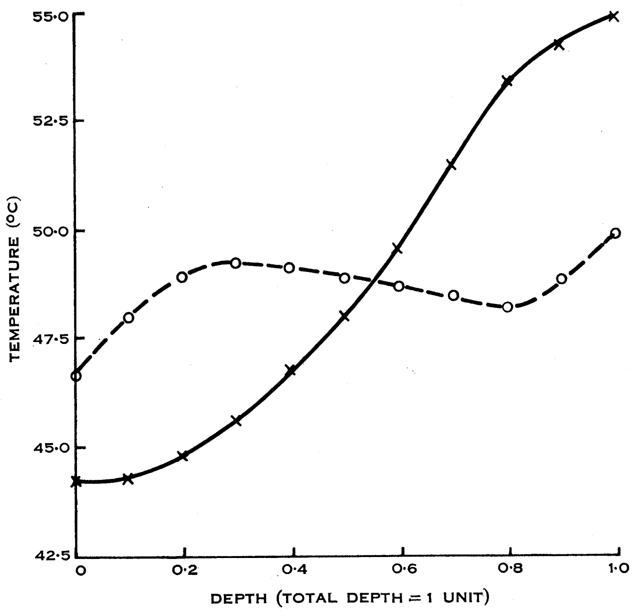

Fig. 4.—Typical profiles, —— along wall and ---- along axis for medium values of H. Here the flow is believed to be up one side of the pipe and down the other. This case is run No. 20, for $H=775$.

that, according to instability theory, this non-symmetric flow would develop first, as H is increased from zero, and that the radially symmetric regime discussed here would occur at higher values of H. It would appear that the experimental results support this view. The agreement between the experimental and theoretical results for H below 300 indicates that the critical value for this non-symmetric flow is not greatly different from that determined theoretically for the radially symmetrical flow (i.e. 142).

It is seen that the temperature effects (and thus the velocity and heat flow effects) are controlled by the value of the modified Rayleigh number

$$H=\frac{\beta g a^4(T_2-T_1)}{\nu\alpha h}.$$

Thus by determining this for the various bores we are able to determine the effects inside the bore, although not as yet in the ground surrounding it. For the bores, in almost all cases, H is greater than 10^5. Thus we see from the theory that, for the section under consideration, the temperatures measured in the bores will be a good indication of the temperatures at the wall of the bores, within 2 degC for $H=10^6$, but not necessarily of the temperature in the ground at the same depth. For H as large as 10^6, however, it is possible that turbulent flow might occur within the bore. In this event the temperature difference between the measuring instrument and the wall of the bore will be very small, and the pipe and the convecting fluid may be regarded as a very good conductor. The author, in an earlier paper (Donaldson 1959), determined the effect of a metal probe inserted into the ground on the temperatures in the ground. For a 3000 ft long probe, 10 in. in diameter, having a thermal conductivity of 8·4 joule cm^{-1} sec^{-1} $degC^{-1}$ (about twice that of copper) embedded in ground in which the temperature increases by 250 degC in 1000 ft, the maximum temperature discrepancy would be approximately 0·75 degC.

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