DEUTERON CAPTURE BY DEUTERONS AND THE STRUCTURE OF ^{4}He

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Summary

The four-body scattering matrix describing the various scattering states of two protons and two neutrons is analysed to see what information can be gained on the structure of the bound state, ^{4}He. It is found that an estimate can be given of the fraction of time that ^{4}He appears as two deuterons; this estimate is in turn used to calculate the cross sections for the process d(d, γ)^{4}He. The low energy n–^{3}He singlet elastic scattering parameters appear to be fairly well determined by the data studied.

I. INTRODUCTION

A large amount of information concerning the existence and structure of any bound states of a system is in principle contained in the positive energy scattering matrix for the system. This information can be extracted if the energy variation of the scattering matrix can be extrapolated back to the bound state energy; and such a procedure has been used with success to discuss the deuteron (see, e.g. Blatt and Weisskopf 1952) and the triton and ^{3}He (Delves 1960). However, in both cases the scattering matrix is, to a reasonable approximation, diagonal, and the small off-diagonal part of little importance, so that only one channel need be considered; it is of interest to show that the procedure can be carried through also when the off-diagonal terms are not intrinsically small and the coupled channels must be considered simultaneously. Of course extra information is obtained if this be done; in the language of the cluster-model, we obtain an estimate of the relative importance of various clusters in the expansion of the ground state. We carry through the procedure here for ^{4}He; the channels which enter are then the singlet S-state channels for n–^{3}He, p–^{3}H, and d–d scattering. The first two of these represent the principal clusters in ^{4}He, and we obtain an estimate of the importance of the d–d cluster in the ground state.

These results are used to calculate approximately the cross section for deuteron capture by deuterons, using the asymptotic form of the ^{4}He wave function. The accuracy of this procedure is discussed later.

The nucleus ^{4}He has three alternative two-particle photodisintegration modes

$$\left.\begin{aligned} {}^4\mathrm{He}+\gamma &\rightarrow \mathrm{p}+{}^3\mathrm{H}-19{\cdot}708\ \mathrm{MeV}, \quad (a)\\ &\rightarrow \mathrm{n}+{}^3\mathrm{He}-20{\cdot}472\ \mathrm{MeV}, \quad (b)\\ &\rightarrow \mathrm{d}+\mathrm{d}-23{\cdot}748\ \mathrm{MeV}, \quad (c) \end{aligned}\right\} \tag{1.1}$$

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of which the first two can proceed via an electric dipole interaction, while the third requires an electric quadrupole transition (since a singlet spin function representing two deuterons is symmetric under interchange of the deuterons, so that odd angular momentum for the space part of the wave functions is forbidden). Moreover, magnetic dipole transitions are forbidden for all three modes, if the space part of the ^{4}He wave function is totally symmetric in all four particles, and proceed for mode (*c*) only through any, presumably very small, part of the wave function which is space symmetric under simultaneous exchange of the two protons and two neutrons, but space antisymmetric if either the protons or neutrons alone are exchanged. The relative probability of decay through each of the modes presumably reflects in some way the relative importance in the ground state of ^{4}He of the clusters p–^{3}H, n–^{3}He, and d–d. Experimentally, mode (*c*) has never been observed, while the only cross-section measurements available are on mode (*a*) ; but in the absence of Coulomb effects, the proton and neutron channels are expected to enter identically, and there is plenty of evidence for this from the positive energy scattering experiments. We shall assume these two clusters are the most important in ^{4}He, and that the d+d grouping of the nucleons occurs only a small fraction of the time ; given this, we can analyse the positive energy scattering matrix to give an estimate of this fraction, and of the cross section for process (*c*).

II. The ^{4}He Scattering Matrix

The ground state of ^{4}He is a singlet *S*-state. We shall assume that **L** and **S** are good quantum numbers ; then the four-body binary reactions which can occur are the three elastic scattering processes

$$\left.\begin{array}{ll} \mathrm{p}+{}^3\mathrm{H} \rightarrow \mathrm{p}+{}^3\mathrm{H}, & (a) \\ \mathrm{n}+{}^3\mathrm{He} \rightarrow \mathrm{n}+{}^3\mathrm{He}, & (b) \\ \mathrm{d}+\mathrm{d} \rightarrow \mathrm{d}+\mathrm{d}, & (c) \end{array}\right\} \tag{2.1}$$

together with the reactions

$$\left.\begin{array}{ll} \mathrm{n}+{}^3\mathrm{He} \rightarrow \mathrm{p}+{}^3\mathrm{H}+0{\cdot}764\ \mathrm{MeV}, & (a) \\ \mathrm{d}+\mathrm{d} \rightarrow \mathrm{p}+{}^3\mathrm{H}+4{\cdot}040\ \mathrm{MeV}, & (b) \\ \rightarrow \mathrm{n}+{}^3\mathrm{He}+3{\cdot}276\ \mathrm{MeV}. & (c) \end{array}\right\} \tag{2.2}$$

The scattering matrix describing the processes is therefore 3×3 ; we label the channels (*a*), (*b*), (*c*) in (2.1) by suffixes p, n, d, respectively, so that

$$\mathbf{S}=\begin{pmatrix} S_{\mathrm{nn}} & S_{\mathrm{np}} & S_{\mathrm{nd}} \\ S_{\mathrm{pn}} & S_{\mathrm{pp}} & S_{\mathrm{pd}} \\ S_{\mathrm{dn}} & S_{\mathrm{dp}} & S_{\mathrm{dd}} \end{pmatrix} \tag{2.3}$$

and we diagonalize **S** with an orthogonal matrix **U** :

$$\mathbf{S}=\mathbf{U}^T\mathrm{e}^{2i\boldsymbol{\Delta}}\mathbf{U},$$
$$\boldsymbol{\Delta}=\begin{pmatrix} \delta_{\mathrm{n}} & & \\ & \delta_{\mathrm{p}} & \\ & & \delta_{\mathrm{d}} \end{pmatrix}, \tag{2.4}$$

$$\mathbf{U}=\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\gamma & \sin\gamma & 0\\ -\sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2.5}$$

The diagonal elements of $\mathbf{\Delta}$ are the eigenphaseshifts of $\mathbf{S}$; the ground state, ^{4}He, presumably corresponds (Delves 1960) to a singularity along the imaginary axis of one of these eigenphaseshifts, and the assumption that the d+d cluster is of minor importance in ^{4}He is equivalent to the assumption that the singularity is in either δ_p or δ_n. Further, it is then a good approximation to neglect the deuteron channels while discussing the neutron and proton channels, and we shall first do this, later, however, considering the interaction between deuteron and n, p channels explicitly. We can then write approximately for the 2×2 matrix involving n and p channels:

$$S'=\begin{pmatrix}S_{nn} & S_{np}\\ S_{pn} & S_{pp}\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}e^{2i\delta_n} & 0\\ 0 & e^{2i\delta_p}\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}, \tag{2.6}$$

$$\begin{aligned}S_{nn}&=\cos^2\alpha\ e^{2i\delta_n}+\sin^2\alpha\ e^{2i\delta_p},\\ S_{pp}&=\sin^2\alpha\ e^{2i\delta_n}+\cos^2\alpha\ e^{2i\delta_p},\\ S_{np}&=S_{pn}=\cos\alpha\ \sin\alpha\ (e^{2i\delta_n}-e^{2i\delta_p}).\end{aligned}$$

Now the neutron and proton channels are identical apart from Coulomb effects; that this is so is evidenced by the fact that the elastic scattering cross sections for the two channels are indistinguishable at energies and angles for which Coulomb effects should be unimportant. We neglect any difference (later we shall include the Coulomb damping of the wave functions) and write

$$\left.\begin{aligned}\delta_n&=\delta_p,\\ \alpha&=\tfrac{1}{4}\pi.\end{aligned}\right\} \tag{2.7}$$

The first of these assumptions is sufficient to diagonalize the scattering matrix, which becomes

$$S'=\begin{pmatrix}e^{2i\delta_n} & 0\\ 0 & e^{2i\delta_n}\end{pmatrix},$$

and it might be thought that the second is not necessary, α being undefined in this limit. However, this is not true; the ratio of the amplitudes A_p, A_n of the p and n channels in ^{4}He is given by

$$A_n/A_p=\tan\alpha$$

(the value of α being taken at the ^{4}He ground state energy) and the choice $\alpha=\frac{1}{4}\pi$ gives the two equal amplitude. Of course, with $\delta_n=\delta_p$, both channel eigenphaseshifts pass through $i\infty$ at the same energy; there are then two degenerate ground states with $A_n/A_p=\tan\alpha$, $\cot\alpha$ respectively, and any mixture of these eigenstates is also an eigenstate. Such a degeneracy does not occur physically.

With this value of α the charge exchange reaction (2.2 (a)) has zero cross section; and indeed this cross section is small except at energies for which Coulomb effects might be expected to be important. This is shown in Figure 1, which plots $(k_n^2\sigma_{tot}/\pi)(p+{}^3H\rightarrow n+{}^3He)$; the experimental points are taken from Jarvis *et al.* (1950), Bransden, Robertson, and Swan (1956), Jarvis (1957), Macklin and Gibbons (1958), and Seagrave, Cranberg, and Simmons (1960).

If only S-waves contribute to the cross section we have

$$\frac{k_n^2\sigma_{tot}}{\pi}=|\,S_{np}\,|^2\leqslant 1.$$

This is certainly not the case; the angular distributions are very far from isotropic even at quite low energies, and moreover the inequality is not satisfied at low energies. We have also plotted therefore an estimate of $|\,S_{np}\,|^2$, the

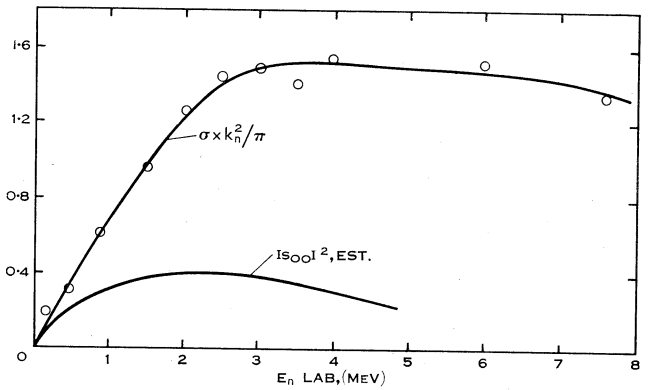

Fig. 1.—^{3}He(n,p)T cross sections. The upper curve gives the total cross sections for the reactions multiplied by the kinematic factor k_n^2/π; it is taken from Jarvis (1957). The lower curve is an estimate of the square of the S-wave matrix element obtained from this by optical model fits.

S-wave contribution to this cross section. This has been derived from optical model fits to the total absorption cross section for n–^{3}He scattering, which then give also the percentage absorption from each angular momentum state. I am grateful to Dr. B. Buck for running these calculations. Such an estimate is of course very crude; but in any case the total absorption matrix element $k_n^2\sigma_{tot}/\pi$ is decreasing at 4 MeV, so that we may safely conclude that S_{np} is negligible for energies greater than about 6 MeV. This makes (2.7) quite reasonable.

It is interesting to note that calculations of n–^{3}He and p–T elastic scattering (Bransden and Robertson 1958; Bransden, Hamilton, and Robertson 1960) also neglect S_{np} and assume the scattering matrix is diagonal. Since they take account of Coulomb effects so that $\delta_n\neq\delta_p$ this implies that they assume $\alpha=0$; in this case either n(^{3}He, γ)^{4}He or p(^{3}H, γ)^{4}He is forbidden.

Now if we return to the 3×3 scattering matrix (2.4), we retain the values (2.7) for δ_n, δ_p, and α; S is then independent of γ, and the matrix elements involving the deuteron channels are

$$\left.\begin{aligned} S_{nd}&=\sin\beta\cos\beta[e^{2i\delta_d}-e^{2i\delta_n}]/\sqrt{2}=-S_{pd},\\ S_{dd}&=\sin^2\beta\, e^{2i\delta_n}+\cos^2\beta\, e^{2i\delta_d}. \end{aligned}\right\} \tag{2.8}$$

Thus reactions (2.2 (*b*)) and (2.2 (*c*)) have the same cross sections; this is true experimentally to within 10% over the energy range with which we are concerned.

The matrix element S_{np} is no longer exactly zero, but is given by

$$S_{np}=\frac{\sin^2\beta}{2}[e^{2i\delta_n}-e^{2i\delta_d}],$$

which is still very small if β is small.

Thus in this approximation there are three parameters β, δ_n, and δ_d, which describe the four-body system at any energy.

III. The Structure of ^{4}He

A bound state of the four-body system occurs at an energy $-E_0$ for which either δ_n or δ_d passes through $i\infty$. We assume that there is only one such energy (so that ^{4}He has no bound excited states) and that, at the bound state energy, $\delta_n=i\infty$. The asymptotic form of the ^{4}He wave function is then, in the channel region for channel q (Delves 1960),

$$\psi_q(^4\mathrm{He})\to A_q[iF_q(\gamma_q r)+G_q(\gamma_q r)]\varphi_q\mathscr{Y}_0/r, \tag{3.1}$$

where $q=$n, p, d; F, G are the regular and irregular solutions of the force-free (or Coulomb) wave equation as defined in Delves (1958); $\mathscr{Y}_0$ is the normalized S-wave angular momentum eigenfunction; and φ_q is given by

$$\left.\begin{aligned} \varphi_n&=\psi(^3\mathrm{He}),\\ \varphi_p&=\psi(^3\mathrm{H}),\\ \varphi_d&=\psi_1(\mathrm{d})\psi_2(\mathrm{d}). \end{aligned}\right\} \tag{3.2}$$

The constant i is the value of cot δ_n at $-E_0$, and A_q is given by

$$A_q=A_0\left(\frac{m_q}{\hbar\gamma_q}\right)^{\frac{1}{2}}U_{nq} \tag{3.3}$$

where A_0 is an overall normalizing constant and

$$\left.\begin{aligned} \gamma_n&=\left[\frac{2m_n}{\hbar^2}(E_0-E(^3\mathrm{He}))\right]^{\frac{1}{2}}=0{\cdot}8608\ \mathrm{f}^{-1},\\ \gamma_p&=\left[\frac{2m_p}{\hbar^2}(E_0-E(^3\mathrm{H}))\right]^{\frac{1}{2}}=0{\cdot}8439\ \mathrm{f}^{-1},\\ \gamma_d&=\left[\frac{2m_d}{\hbar^2}(E_0-2E(\mathrm{d}))\right]^{\frac{1}{2}}=1{\cdot}065\ \mathrm{f}^{-1}. \end{aligned}\right\} \tag{3.4}$$

With the approximation to which we are working we have

$$u_{nn}=u_{np}=1/\sqrt{2}\ ;\quad u_{nd}=(1/\sqrt{2})(\sin\beta)_{E_0}. \tag{3.5}$$

The ratio $(A_d/A_n)^2$ is a measure of the proportion of the time that ^{4}He spends as two deuterons, and can be estimated if we can extrapolate the positive energy scattering matrix back to the energy $-E_0$ to give sin β at the energy $-E_0$. This can be done approximately as follows:

The cross section for reaction (2.2 (*b*)) is

$$\sigma(d+d\rightarrow n+{}^3He)=\frac{\pi\sin^2\beta\cos^2\beta}{2k^2_d}|\,e^{2i\delta_d}-e^{2i\delta_n}\,|^2. \tag{3.6}$$

For small energies near the threshold for the reaction, δ_d can be neglected compared with δ_n; moreover values for δ_n can be taken from the theoretical calculations of Bransden and Robertson (1958) and Bransden, Hamilton, and Robertson (1960).

Tan β has the following expansion about the d–d threshold (Delves 1958)

$$\tan\beta=e^{-\pi/k_dD_d}[\beta_0+\beta_1k^2_d+\ldots], \tag{3.7}$$

where $D_d=\hbar^2/me^2$, m the nucleon mass.

Brennan (1958) gives the following representation of the experimental cross section/energy curve near zero energy:

$$\sigma(d+d\rightarrow h+{}^3He)=\{0\cdot107\exp(-2\pi/k_dD_d)\}/E_d(\text{lab MeV}). \tag{3.8}$$

This is the cross section for both spin states; we assume the singlet cross section is one-quarter of this. We then find

$$\beta_0=0\cdot217. \tag{3.9}$$

It is not possible to estimate the higher terms in the expansion (3.7) without a phaseshift analysis of both d–d scattering (to give δ_d) and the d+d→n+^{3}He reaction; however, if we can use a single term in the expansion for tan β down to the ground state of ^{4}He we have

$$|\,A_n/A_d\,|^2=0\cdot075, \tag{3.10}$$

that is, ^{4}He is in the d+d state about 8% of the time. This is not an unreasonable figure, and supports the original assumption that the d+d cluster is relatively unimportant.

IV. Photodisintegration and Capture Cross Sections

If we use the asymptotic forms (3.1) over all of configuration space, we can calculate the photoelectric cross sections for reactions (2.1); however, this involves a knowledge not only of A_n/A_d, but of the overall normalizing constant A_0. As for the n–p and n–d case (Macklin and Gibbons 1958) this can be approximately expressed in terms of the "effective range" parameter r_0 in an expansion of k_n cot δ_n:

$$k_n\cot\delta_n=-\frac{1}{a_0}+\frac{r_0}{2}k^2_n+\ldots \tag{4.1}$$

Values of a_0, r_0, and A_0 are derived in the Appendix from the phaseshifts of Bransden and Robertson (1958) and Bransden, Hamilton, and Robertson (1960);

however, the value of A_0 thus derived is not expected to be very reliable. Another estimate can be made by comparing the cross section for reaction (3.1 (*a*) or (*b*)) with experimental values; and we shall use both of these estimates of A_0 to give an estimate of the cross section for reaction (3.1 (*c*)). We follow this procedure because the approximations made are inherently more accurate for the deuteron channel, since this proceeds via an electric quadrupole interaction; this enhances the role of the channel region compared with an electric dipole transition.

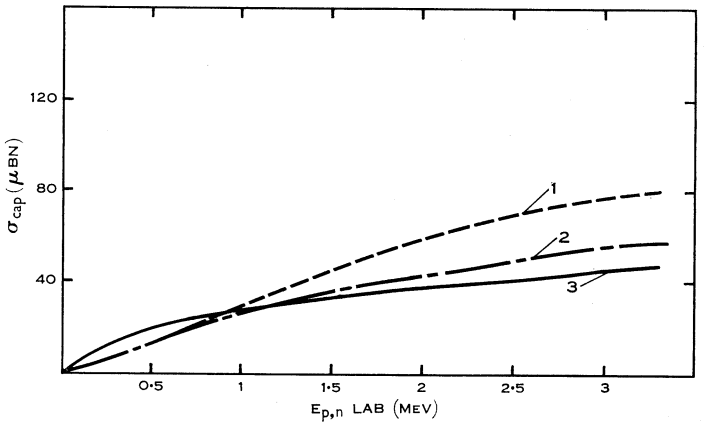

Fig. 2.—Capture cross sections for $\binom{n}{p,\gamma}{}^4$He. Curve (1), $\sigma[^3\text{H(p, }\gamma)^4\text{He}]$, experimental, from Bransden *et al.* (1956). Curve (2), a fit to the low energy end of curve (1) using equation (4.3); it corresponds to $A_n = 5{\cdot}35$ (fm)$^{-\frac{1}{2}}$. Curve (3), theoretical cross sections for $[^3\text{He(p, }\gamma)^4\text{He}]$ using this value of A_n.

We can follow the procedure of Delves (1960) to obtain in this approximation:

$$\sigma_{cap}(\text{n}{-}^3\text{He}) = \frac{8e^2k_n\hbar^2\pi \mid A_n \mid^2}{27m^2c^3(k_n^2+\gamma_n^2)}, \tag{4.2}$$

$$\sigma_{cap}(\text{p}{-}^3\text{H}) = \frac{32e^2\hbar^2\pi^2 \mid A_n \mid^2(1+k_p^2D_p^2)C_0^2(k_pD_p)}{27m^2c^3(k_p^2+\gamma_p^2)k_pD_p^3\gamma_p \sin(\pi/\gamma_pD_p)}, \tag{4.3}$$

where

$$C_0^2(x) = \frac{2\pi}{x}(e^{2\pi/x}-1)^{-1}. \tag{4.4}$$

These results use the wave functions (3.1), and neglect the final-state interaction; the Coulomb damping of the wave functions in (4.3) has been inserted as in Delves (1960). No experimental results exist for n–^{3}He capture; but we can fit (4.3) to the low energy p–^{3}H capture results given by Bransden, Robertson, and Swan (1956). The fit obtained is shown in Figure 2, for the value

$$A_n = 5{\cdot}35 \text{ (fm)}^{-\frac{1}{2}}, \tag{4.5a}$$

This compares with the value

$$A_q = 0{\cdot}44\text{–}0{\cdot}85 \tag{4.5b}$$

obtained in the appendix from extrapolation from the positive-energy values of δ_n. Such agreement is very poor, of course ; but the extent of the agreement between the experimental and theoretical cross sections of Figure 3, in the energy region where the final P-state interaction is expected to be unimportant, suggests that estimate (4.5a) is the more reliable of the two. The main sources of error in the estimate (4.5a) arise from the crudity of the Coulomb correction used and the inadequacy of the wave function (3.1) inside the interaction region. The first of these should be of comparable importance to the same approximation in the n–d case ; that is, the error introduced should be less than ~30% in A^2 (Delves 1960). The second is more difficult to estimate, but was expected to be relatively unimportant in ^{3}H (<10% error in A^2) ; while ^{4}He is more tightly bound than ^{3}H, we may hope that the estimate (4.3a) for A_n is reliable within

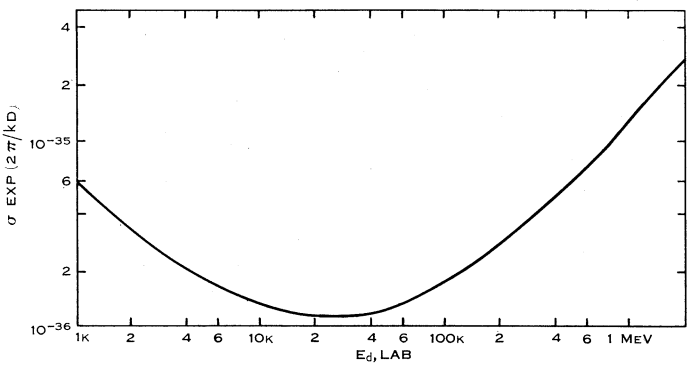

Fig. 3.—D–D capture cross section. The curve is that predicted by equation (4.6).

perhaps 50%, that is, a factor of 2 for A_n^2. We would thus conclude that the phaseshifts of Bransden and Robertson (1958) and Bransden, Hamilton, and Robertson (1960) are not sufficiently accurate to extrapolate back over the 20 MeV required—a not particularly surprising conclusion.

We can now discuss the branch (1.1 (c)) and its inverse ; that is, deuteron capture by deuterons. The electric quadrupole cross section for this reaction is, in the same approximation as before,

$$\sigma_{\mathrm{cap}}(\mathrm{d}+\mathrm{d})=\frac{\hbar^3 e^2\pi^2 \mid A_d^2 \mid C_0^2(k_d D_d)(1+k_d^2 D_d^2)(1+4k_d^2 D_d^2)}{1080 m^4 c^5(\gamma_p^2+k_p^2)k_d D_d^5 \gamma_d}, \tag{4.6}$$

where A_d is given by estimates (3.10) and (4.5a). This cross section is plotted against energy in Figure 3 ; it has a maximum of about 2×10^{-33} cm^2 at a deuteron laboratory energy of about 40 MeV.

V. Discussion

We have shown that the four-body scattering matrix can be analysed to give information on the cluster structure of ^{4}He, and thus on the photodisintegration cross sections of ^{4}He. The method used is a natural extension of the usual effective-range discussion of the deuteron photodisintegration which takes into

account the intrinsic non-diagonal nature of the four-body scattering matrix. Helium is not an ideal nucleus for calculations of this type which involve extrapolation from positive-energy to the ground state, due to its tight binding; nonetheless, it would appear that we can reproduce the low-energy photodisintegration results reasonably. The cross section we predict for the reaction $d+d\rightarrow\gamma+{}^4He$ is an order of magnitude less than that calculated by Flowers and Mandl (1951); this would appear to be due to their neglect of the off-diagonal nature of the d+d channel in ^{4}He. A measure of this effect is our estimate (3.10) of the asymptotic probability of this channel. This estimate involves the extrapolation of $\tan\beta$, one variable characterizing the mixture matrix $\mathbf{U}$; and we saw in the previous section that the corresponding extrapolation of $\cot\delta_n$ does not give reliable results. We cannot estimate the accuracy of our value for A_d; however, mixture parameters in general seem easier to predict than phaseshifts (they vary less rapidly), and further we have used to predict A the experimental values of the cross sections for $(d+d\rightarrow n+{}^3He)$ rather than theoretical phaseshifts which are themselves of doubtful accuracy. Thus we may hope that the predicted d+d capture cross section and reduced width of the d+d cluster in ^{4}He have at least order of magnitude reliability.

The method used is applicable also to other light nuclei for which the cluster model is expected to be reasonable, and for which sufficient experimental information is available. Tombrello and Phillips (1961) have recently carried through a somewhat similar analysis of the mass seven nuclei, in which they use wave functions of type (3.1) to calculate photodisintegration cross sections into the ^{3}He+^{4}He and n+^{6}Li channels; comparison with experimental results then gives the asymptotic amplitude for the channels and therefore their relative importance in a cluster expansion of the ground state. Their analysis can then be extended to give an estimate of the cross section for the reaction $^3H+{}^4He\rightarrow n+{}^6Li$. Other systems which might repay study are those involving bound hyperons, the hyperfragments; here again the scattering properties are ill known, but the binding energies certainly small, so that observation of the capture rate can give information on the scattering processes.

Finally, we note that, as shown in the Appendix, the calculations of Bransden and Robertson (1958) and Bransden, Hamilton, and Robertson (1960) seem to define quite well the singlet n–^{3}He scattering length a_0 and effective range r_0 they give

$$a_0=1\cdot3\text{ f},$$
$$r_0=2\text{ f}.$$

VI. Acknowledgments

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APPENDIX

Effective Range Expansion of $k_n \cot \delta_n$

It was shown in Delves (1960) that, as for the simple single-channel n–p case, the asymptotic amplitudes A of the wave functions (3.1) can be related to the coefficients of a power series expansion of the eigenphaseshift matrix; in our case, A_n is related to the expansion of the quantity $kn \cot \delta_n$. However, for such relations to be valid, the form of the expansion of $k \cot \delta_n$ about zero k must be valid also at the ground state energy $-E_0$ (strictly expansions about zero k and about $-E_0$ must have a common region of convergence). For the n–d scattering case treated in Delves (1960) it was seen to be rather likely that $k \cot \delta_n$ had a pole between $k=0$ and the bound state; in this case the form of the expansion included the pole explicitly. There is no information at all on whether a similar pole exists in ^{4}He; however, we shall see that the results we obtain do not depend strongly on the existence or position of such a pole. If ^{4}He has no bound excited states there cannot be more than one such pole.

If there be such a pole, $k_n \cot \delta_n$ can be expanded about $k_n=0$ in the form

$$k_n \cot \delta_n = \frac{\delta_0+\delta_1 k^2+\delta_2 k^4+\delta_3 k^6+\ldots}{1+\Gamma^2 k^2}, \tag{A1}$$

where the pole occurs at $k_n^2=-\Gamma^{-2}$. We find, as in Delves (1960),

$$A^2=\frac{2\gamma_n U_{nn}^2(1-\Gamma^2\gamma_n^2)}{6\delta_0/\gamma_n-4\delta_1\gamma_n+5+2\delta_2\gamma_n^3-3\Gamma^2\gamma_n^2}, \tag{A2}$$

where U_{nn} is the n–n element of the mixture matrix $\mathbf{U}$; in our approximation

$$U_{nn}=2^{-\frac{1}{2}}.$$

The coefficients δ_i, Γ^2 of (A1) can be found from the theoretical phaseshifts of Bransden and Robertson (1958) and Bransden, Hamilton, and Robertson (1960), which fit the experimental data reasonably well for $E_{n\,lab}>5$ MeV, and from the observed binding energy $-E_0$ of ^{4}He. In view of the uncertainty as to the presence or absence of the pole at $-\Gamma^2$, and as to the accuracy of the

phaseshifts of Bransden and Robertson (1958) and Bransden, Hamilton, and Robertson (1960) the fitting has been done in several ways.

(1) Fitting δ_0, δ_1, and δ_2 putting $\delta_3=\Gamma^2=0$; that is, taking three terms of (5.10) assuming there is no pole between $k_n=0$, $k_n=i\gamma$.
(2) Fitting δ_0, δ_1, δ_2, δ_3.
(3) Fitting δ_0, δ_1, δ_2, Γ^2.
(4) Fitting δ_0, δ_1, δ_2, δ_3, Γ^2.

In each case the binding energy of ^{4}He was fitted. The results, together with the resulting values of A_n/U_{nn} and the scattering length $a_0=-1/\delta_0$, and effective range $r_0=2\delta_1-2\Gamma^2\delta_0$ are given in Table 1.

TABLE 1

EFFECTIVE RANGE PARAMETERS FOR n–^{3}He ELASTIC SCATTERING PHASESHIFT δ_n

Fitting Procedure	δ_0	δ_1	δ_2	δ_3	Γ^2	a_0	r_0	$\frac{A_n}{U_{nn}}$
1	−0·723	0·671	0·655			1·38	1·34	1·07
2	−0·761	0·998	0·431	−0·990		1·31	2·00	0·735
3	−0·768	−0·243	1·60		1·804	1·30	2·29	0·625
4	−0·593	16·252	−14·08	−11·72	−23·11	1·69	5·09	1·20

Several points may be noted from Table 1 ; the first is that the equations leading to set 4 appear to have been ill-conditioned, so that the results of this set are meaningless. Secondly, the inclusion or otherwise of a term corresponding to a pole in k_n cot δ_n does not significantly affect the value of A_n/U_{nn} calculated ; it is interesting that the value for Γ^2 given by set 3 does put the pole between $k_n=0$ and $i\gamma$, but the reliability of the result is not such as to warrant any weight being put upon this. Finally, the scattering length and effective range appear to be quite well determined by the phaseshifts ; probable best sets to take would be

$$a_0=1\cdot3\text{ f}\,;\quad r_0=2\text{ f}. \tag{A3}$$