THE RAY PATHS OF WHISTLING ATMOSPHERICS: DIFFERENTIAL GEOMETRY*

By R. F. MULLALY†

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Summary

Expressions are developed on the basis of geometrical optics for the curvature of a whistler ray path in terms of the gradients of electron density, magnetic field strength, and field direction, the analysis being restricted for simplicity to paths which are plane curves. It is shown that there is in general no tendency for the rays to follow the lines of force closely unless the wave-normal is very nearly at right angles to the direction of the ray.

Expressions are also given for the curvature of low frequency disturbances when the magnetic field is weak (" nose " whistlers). It is shown that the ray may be inclined at a very considerable angle to the direction of the field.

I. INTRODUCTION

The study of whistling atmospherics offers interesting possibilities for investigating the distribution of ionization and the form of the geomagnetic field at very great heights above the Earth's surface. In order to carry through this programme in detail, however, extensive knowledge is needed of the effect on the ray paths followed by the whistlers, of different hypothetical forms of the field and distributions of electron density. At first only rather crude attempts at this were made. Thus Storey (1953) relied on the fact that for the whistler mode of propagation the greatest angle possible between the ray direction and the direction of the geomagnetic field is $\cot^{-1}\sqrt{8}=19°\ 28'$, so that, *very approximately*, the whistlers must be propagated along the lines of magnetic force. More recently Maeda and Kimura (1956) have given a theoretical study of whistler ray propagation but, despite quite heavy analysis, their suggested method of ray plotting amounts to no more than dividing the medium into a number of slabs in each of which the magnetic field is assumed constant, and applying Snell's laws to find the refraction in this hypothetical medium. It has also been suggested (see e.g. Northover 1959) that the energy is essentially guided in " ducts " formed by streamers of high electron density extended along lines of magnetic force. Longitudinal propagation in this form is often assumed in current whistler studies. But, whether or not high-density ducts play a major role in guiding the energy, it seems desirable to possess a good knowledge of the basic geometrical optics for the whistler mode of propagation in a slowly-varying medium. In this and succeeding papers we shall attempt such a study.

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† Division of Radiophysics, C.S.I.R.O., University Grounds, Chippendale, N.S.W.

Here we propose to give an expression for the *ray curvature* at a given point in the medium, in terms of the gradients of the electron density and field strength and of the curvature of the lines of magnetic force. This should provide a simple and reasonably powerful means of ray plotting by means of a series of circular arcs. There is also the intrinsic interest of exhibiting the effects of the different elements of inhomogeneity in the medium (inhomogeneity of electron density, of field strength, and of field direction) on the ray paths of the whistlers. Finally, it will be possible to draw some general conclusions about ray paths when the expression for the curvature has been obtained. The analysis given here is restricted to rays whose trajectories are plane curves; methods similar to those used here could be used to plot such ray trajectories in three dimensions, but at the price of added complexity.

II. The Normal Whistler Mode

Storey (1953) developed the theory of the propagation of whistling atmospherics under the assumption that the frequency of the disturbances is so low that the strong inequalities

$$X \gg Y \gg 1 \tag{1}$$

hold throughout the medium. Here X and Y are the well-known Appleton parameters of magneto-ionic theory,

$$X = Ne^2/\varepsilon_0 mp^2,$$
$$Y = eB_0/mp,$$

where e and m are the charge and mass of an electron, N is the electron number-density, p is the angular frequency of the wave, B_0 is the Earth's magnetic induction, and ε_0 is the permittivity of free space. Under these conditions propagation is in the extraordinary mode and the quasi-longitudinal (Q.L.) approximation holds for the phase refractive-index μ. Assuming collision damping to be negligible we have

$$\mu^2 = X/Y \cos\theta, \tag{2}$$

where θ is the angle between the wave-normal and the Earth's magnetic field. Propagation in this mode is possible for angles $\theta < \theta_c$ where

$$\theta_c = \cos^{-1}[(X+Y^2-1)/XY^2] \sim \tfrac{1}{2}\pi - 1/Y, \tag{3}$$

that is, for all directions of the wave-normal except those almost exactly transverse to the magnetic field.

A note of warning must be struck here. The condition for the validity of the Q.L. approximation (2), and so for the other approximations of this paper is

$$| Y \sin^2\theta/2(1-X)\cos\theta | \ll 1,$$

so that (2) holds for all directions for which $|\theta| < \theta'$ say, when θ' is some angle close to $\tfrac{1}{2}\pi$, for which the condition first fails. However, (1) is not *sufficient* to ensure that $\theta_c < \theta'$, the condition for which is $X \gg Y^2 \gg 1$. Thus, if this latter condition is violated though (1) holds, there may be a narrow range of angles between θ' and θ_c for which propagation is possible, but the approximation (2) is not valid. This situation will perhaps arise only rarely in practical ray tracing, but its possibility should be noted.

The direction of the ray (that is, the direction in which electromagnetic energy is propagated) lies in the plane determined by the Earth's magnetic field and the wave-normal, and is inclined to the latter at an angle α given by

$$\tan\alpha=-1/\mu-\partial\mu/\partial\theta, \tag{4}$$

the sign convention (in the first quadrant) being that α is positive if the wave-normal lies between the magnetic field and the ray. We shall also write $\varphi=\theta+\alpha$ for the angle between the geomagnetic field and the ray. Using (2), (4) becomes

$$\tan\alpha=-\tfrac{1}{2}\tan\theta, \tag{5}$$

so that

$$\tan\varphi=\sin\theta\cos\theta/(1+\cos^2\theta), \tag{6}$$

and in view of this it is easily shown that $\varphi\leqslant\cot^{-1}\sqrt{8}=19°\ 28'$, the property used by Storey. When $\varphi=19°\ 28'$, $\theta=54°\ 21'$. It will be noted that θ is in

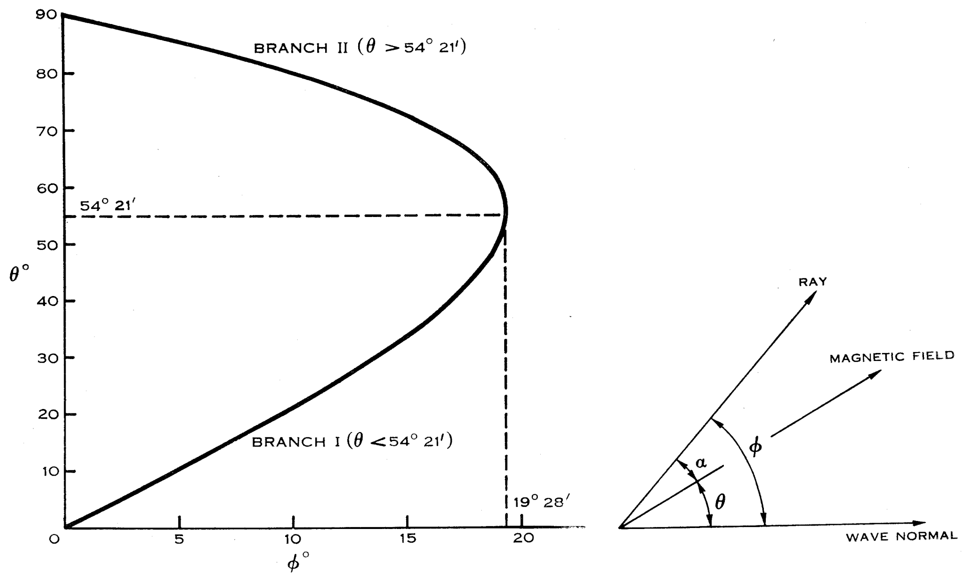

Fig. 1.—The angle θ between the wave-normal and the magnetic field as a function of the angle φ between the ray and the field, for the normal whistler mode.

general a two-valued* function of φ (Fig. 1). Thus, corresponding to a given ray direction (φ), there are two distinct types of propagation differing in the direction of phase propagation, the values of the refractive index and group velocity, and we shall see, in the effect produced on the ray path by inhomogeneity of the medium.

* If we go to a higher approximation than that used in this paper we find that, for very small values of $|\varphi|$, θ is a *three*-valued function of φ. The greatest error in the approximation (5) occurs when θ is close to $\frac{1}{2}\pi$, and α is then given with an error of the order of $2/Y$, which by (1) is small.

The (ray) group refractive index for whistlers is given by

$$\begin{aligned}\mu_G&=\tfrac{1}{2}\mu\cos\alpha\\&=\left(\frac{4\cos\theta}{1+3\cos^2\theta}\,\frac{X}{Y}\right)^{\frac{1}{2}},\end{aligned}\tag{7}$$

and the group delay is

$$(1/c)\int\mu_G\mathrm{d}s$$

taken along the ray. The quantity μ_G differs little from $(X/Y)^{\frac{1}{2}}$ for values of θ less than about 70° and then falls sharply to zero as θ tends to 90° (see Storey 1953).

III. Formulae for the Ray Curvature

It has been shown elsewhere (Mullaly 1962) that the curvature of a ray path in a slowly-varying inhomogeneous magneto-ionic medium may be expressed in terms of X, Y, θ, and the gradients of these three quantities. If these three gradients all lie in the plane determined by the wave-normal and the geomagnetic field, the ray will remain in this plane and the curvature of its path may be expressed as

$$\varkappa=\sum_{X,Y,\psi}\{Z_L^X\nabla_L X+Z_T^X\nabla_T X\},\tag{8}$$

where $\nabla_L X$, $\nabla_T X$, etc. denote the components of $\boldsymbol{\nabla}X$ etc. respectively along and transverse to the direction of the *ray*. Here $\boldsymbol{\nabla}\psi$ is the gradient of the angle between the direction of the geomagnetic field and a direction fixed in space so far as the partial differentiations involved are concerned. In the present applications the direction to which ψ is measured will be taken as the wave-normal, and we shall write $\boldsymbol{\nabla}\theta$ rather than $\boldsymbol{\nabla}\psi$. The components of $\boldsymbol{\nabla}\theta$ at any point along and transverse to the magnetic field, equal respectively the curvature of the line of force and of the magnetic equipotential, through that point.

In a region *free of currents* it may further be shown that $\boldsymbol{\nabla}Y$ is perpendicular to $\boldsymbol{\nabla}\theta$ and has Y times its length, so that

$$\left.\begin{aligned}\nabla_L\theta&=-(1/Y)\nabla_T Y,\\ \nabla_T\theta&=(1/Y)\nabla_L Y.\end{aligned}\right\}\tag{9}$$

In the absence of currents it is thus possible to eliminate $\boldsymbol{\nabla}\theta$ and express the curvature in terms of the gradients of two quantities, X and Y.

The coefficients in (8) are given by

$$\left.\begin{aligned}Z_L^X&=\frac{\zeta\sin 2\alpha}{2\mu}\left(\frac{\partial\mu}{\partial X}\right)_\theta+\left(\frac{\partial\alpha}{\partial X}\right)_\theta,\\ Z_T^X&=\frac{\zeta\cos^2\alpha}{\mu}\left(\frac{\partial\mu}{\partial X}\right)_\theta,\end{aligned}\right\}\tag{10}$$

with similar expressions for Z_L^Y, Z_T^Y, and

$$\left.\begin{aligned}Z_L^\theta&=\zeta\cos^2\alpha-1,\\ Z_T^\theta&=-\tfrac{1}{2}\zeta\sin 2\alpha,\end{aligned}\right\}\tag{11}$$

where

$$\zeta=1+\partial\alpha/\partial\theta.$$

In a general magneto-ionic medium for which μ is related to X, Y, and θ by the Appleton-Hartree expression (Appleton 1932) these coefficients are of great algebraic complexity, but for the whistler mode of propagation the approximations (2) and (5) lead to a notable simplification. We find after some reduction

$$\left.\begin{aligned} Z_Y^X&=f(\varphi)/X, & Z_T^X&=g(\varphi)/X,\\ Z_L^Y&=-f(\varphi)/Y, & Z_T^Y&=-g(\varphi)/Y,\\ Z_L^\theta&=2g(\varphi)-1, & Z_T^\theta&=-2f(\varphi), \end{aligned}\right\} \tag{12}$$

where

$$\left.\begin{aligned} f(\varphi)&=\frac{(3\cos^2\theta-1)\sin 2\alpha}{4(3\cos^2\theta+1)},\\ g(\varphi)&=f(\varphi)\cot\alpha, \end{aligned}\right\} \tag{13}$$

in which α and θ are determined as functions of φ by (5) and (6).

The curvature may thus also be written

$$\varkappa=\left\{f(\varphi)\frac{\partial}{\partial x_L}+g(\varphi)\frac{\partial}{\partial x_T}\right\}\log X/Y+\{2g(\varphi)-1\}\frac{\partial\theta}{\partial x_L}-2f(\varphi)\frac{\partial\theta}{\partial x_T}, \tag{14}$$

where x_L,x_T are coordinates respectively along and transverse to the ray. In a region free of currents this becomes in virtue of the relation (9) between $\nabla\theta$ and ∇Y,

$$\varkappa=\left\{f(\varphi)\frac{\partial}{\partial x_L}+g(\varphi)\frac{\partial}{\partial x_T}\right\}\log X/Y+\frac{\partial}{\partial x_T}\log Y. \tag{15}$$

It is also of interest to calculate the rate at which the angle θ between the wave-normal and the field changes along the ray. If s represents arc length along the ray we have

$$\begin{aligned} \theta_s\equiv\frac{d\theta}{ds}&=\frac{d\varphi}{ds}-\frac{d\alpha}{ds}\\ &=\varkappa+\nabla_L\theta-\frac{d\alpha}{ds}\\ &=\varkappa+\nabla_L\theta-\frac{\partial\alpha}{\partial\theta}\theta_s, \end{aligned}$$

using (5). Hence

$$\begin{aligned} \theta_s&=\frac{3\cos^2\theta+1}{3\cos^2\theta-1}\{\varkappa+\nabla_L\theta\}\\ &=\frac{3\cos^2\theta+1}{3\cos^2\theta-1}\left\{\left(f(\varphi)\frac{\partial}{\partial x_L}+g(\varphi)\frac{\partial}{\partial x_T}\right)\log X/Y+2\left(g(\varphi)\frac{\partial}{\partial x_L}-f(\varphi)\frac{\partial}{\partial x_T}\right)\theta\right\}. \end{aligned} \tag{16}$$

The function $f(\varphi)$ and $g(\varphi)$ have been computed and are shown in Figure 2 for positive values of φ. Both are two-valued functions of φ, and in the figure the two branches of the curves have been marked I and II corresponding respectively to propagation with $|\theta|<54°\,21'$ and with $|\theta|>54°\,21'$, this being the value of θ for which φ attains its maximum value of $19°\,28'$.

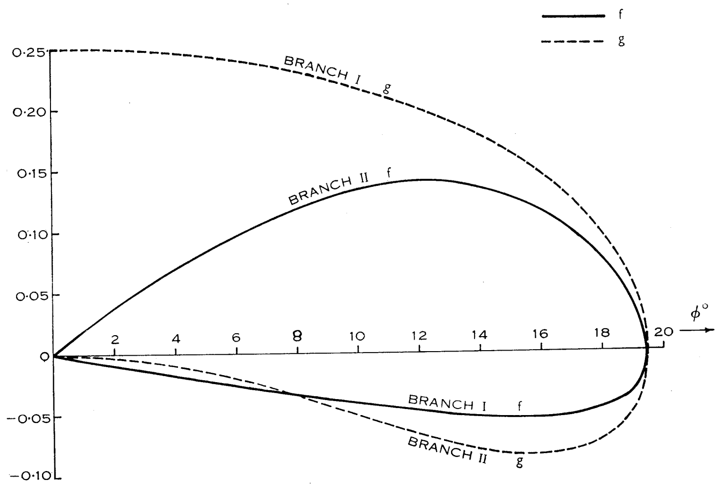

Fig. 2.—The functions $f(\varphi)$ and $g(\varphi)$. For branch I, $\theta < 54° 21'$ and for branch II $\theta > 54° 21'$.

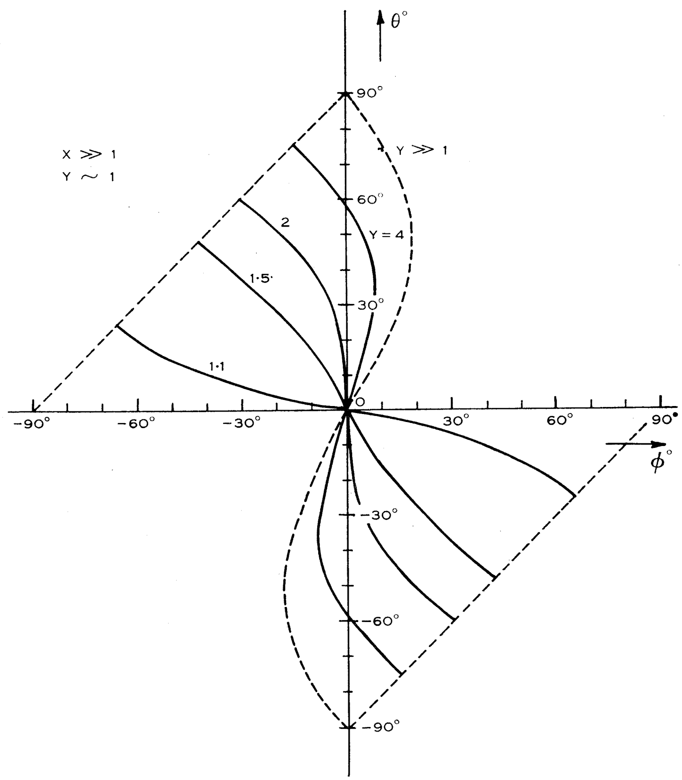

Fig. 3.—The relation between θ and φ for propagation conditions in which $X \gg 1$, but Y is close to unity (e.g. for " nose " whistlers).

IV. The Curvature of "Nose" Whistlers

At increasingly great heights above the Earth's surface the magnetic field becomes progressively weaker so that near the top of a ray trajectory the condition $Y \gg 1$ may no longer hold. So far as the dispersion curve is concerned, this gives rise to the so-called "nose" whistlers (Helliwell *et al.* 1956).

Here we shall consider the ray paths in such a medium for which Y is greater than, but of the order of, unity, assuming that the condition $X \gg 1$ still holds.

Propagation is possible for angles between the wave-normal and the field for which

$$|\theta| < \cos^{-1}(1/Y),$$

and the quasi-longitudinal approximation is valid in the form

$$\mu^2 = X/(Y_L - 1),$$

so that

$$\tan\alpha = -\frac{1}{\mu}\frac{\partial\mu}{\partial\theta} = \frac{Y_T}{2(1-Y_L)}, \tag{17}$$

where $Y_L = Y\cos\theta$, $Y_T = Y\sin\theta$. We find also

$$\tan\varphi = \frac{2\sin\theta - Y\sin\theta\cos\theta}{2\cos\theta - Y(1+\cos^2\theta)}. \tag{18}$$

It follows that the values of φ corresponding to the extreme values of θ, namely to $\pm\cos^{-1}(1/Y)$, are $\pm\{\frac{1}{2}\pi - \cos^{-1}(1/Y)\}$. Also if $Y>2$, φ is zero for $\theta = \pm\cos^{-1}(2/Y)$ and θ is a three-valued function of φ. For $Y<2$, θ is a single-valued function of φ. The relation of φ and θ is illustrated in Figure 3 for several values of Y. It will be seen that for values of Y not much above unity the direction of the ray may diverge very greatly from that of the magnetic field.

The coefficients of curvature are:

$$\left.\begin{aligned} Z_L^X &= \zeta Y_T(1-Y_L)\Gamma X^{-1}, \\ Z_T^X &= 2\zeta(1-Y_L)^2\Gamma X^{-1}, \\ Z_L^Y &= (2+\zeta Y_L)\Gamma\sin\theta, \\ Z_T^Y &= 2\zeta(1-Y_L)\Gamma\cos\theta, \\ Z_L^\theta &= 4\zeta(1-Y_L)^2\Gamma - 1, \\ Z_T^\theta &= -2\zeta Y_T(1-Y_L)\Gamma, \end{aligned}\right\} \tag{19}$$

where

$$\Gamma = \{4(1-Y_L)^2 + Y_T^2\}^{-1},$$
$$\zeta = 1 + 2(Y_L - Y^2)\Gamma.$$

It will be seen that these are not only more complex algebraically than the expression (12) for the usual whistler mode, but *give rise to frequency-dependent terms in the curvature.* Thus the ray paths in a region of "nose" propagation will be different for rays at different frequencies.

V. Conclusion

The main conclusion is that *there is in general no tendency for a true longitudinal ray (both ray and wave-normal parallel to the field: $\varphi=0$ and $\theta=0$) to follow the lines of force closely.*

Two possibilities thus remain: either the ray swings back and forth about the direction of the magnetic field, or $|\theta|$ increases up to its limiting value θ_c, when the ray is again parallel to the field. The expressions (12) indicate that, when this happens, the gradients of X and Y are without effect on the trajectory, and that *the ray will follow the lines of force.* Also by (7) the group delay is small.

Work on ray plotting, and work in which explicit analytic expressions for ray paths have been obtained, suggest that both types of trajectory are possible. For the type with $|\theta|$ increasing towards the limit θ_c, it may be necessary to go a long way along the ray before the limit is approached closely. Since the wave-normal will then be almost transverse to the field, the approximation (2) for μ will be at its worst, as has been pointed out, and some caution may be needed in applying our results to the further plotting of the ray, especially if the inequalities (1) do not hold very strongly.

Finally, when the field is weak ("nose" whistlers) the ray paths exhibit a more complex behaviour; the paths depend markedly on the frequency, and the directions of the rays may be inclined at large angles to the field.

VI. Acknowledgments

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VII. References

Appleton, E. V. (1932).—*J. Inst. Elect. Engrs.* **71**: 642.

Helliwell, R. A., Crary, J. H., Pope, J. H., and Smith, R. L. (1956).—*J. Geophys. Res.* **61**: 139.

Maeda, K. I., and Kimura, I. (1956).—*Rep. Ionosph. Res. Japan* **10**: 105.

Mullaly, R. F. (1962).—*Aust. J. Phys.* **15**: 96.

Northover, F. H. (1959).—*J. Atmos. Terr. Phys.* **17**: 170.

Storey, L. R. O. (1953).—*Phil. Trans.* A **246**: 113.