SCATTERING OF ALPHA-PARTICLES BY A VIBRATIONAL NUCLEUS*

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Summary

The elastic and inelastic scattering of α-particles by a vibrational nucleus is calculated using plane-wave Born approximation. The excitation of both single-phonon and two-phonon states is considered. The effect of the diffuseness of the nuclear surface is included. The result for elastic scattering and the excitation of the single-phonon 2^+ and 3^- states is in good agreement with experiment for ^{60}Ni. The approximations used are discussed, and it is suggested that excitation of 0^+, 1^-, and 5^- states should provide the best experimental test of the theory of two-phonon excitation of nuclei. The energies of the vibrational states are also considered.

I. Introduction

One of the simplest models of the nucleus is the vibrational model (Bohr 1952). In this model, the nucleus is assumed to behave like a liquid drop which in equilibrium is spherically symmetric, and which makes small vibrations about equilibrium. The nuclear energy levels described by this model are examined, and the excitation of these levels by inelastic scattering is calculated by plane-wave Born approximation. The scattering of α-particles by ^{60}Ni is calculated and compared with experimental results. The treatment of inelastic scattering is similar to that of Lemmer, de Shalit, and Wall (1961) and of Walecka (1961, personal communication), but the effect of the diffuseness of the nuclear surface is included and the excitation of some additional levels is treated. It is essential to include the diffuseness of the nuclear surface before comparing the theory with experiment.

II. The Vibrational Model

At first we assume the nucleus has a sharp surface. The surface of the nucleus is described by Bohr (1952)

$$R'=R\left[1+\sum_{l=2}\sum_{m=-1}^{l}\left(\alpha_{lm}Y_{lm}-\frac{1}{4\pi}(-1)^m\alpha_{lm}\alpha_{l-m}\right)\right]. \tag{1}$$

The collective Hamiltonian is taken as

$$H=\sum_{lm}(\tfrac{1}{2}B_l\mid\dot{\alpha}_{lm}\mid^2+\tfrac{1}{2}C_l\mid\alpha_{lm}\mid^2). \tag{2}$$

Following the usual treatment (e.g. Reiner 1958) for quadrupole vibrations, $l=2$, we write

$$\alpha_{lm}=(\hbar/2B_l\omega_l)^{\frac{1}{2}}[b_{lm}+(-1)^m b_{l-m}^{+}], \tag{3}$$

$$B_l\dot{\alpha}_{lm}^{*}=i(\tfrac{1}{2}\hbar B_l\omega_l)^{\frac{1}{2}}[b_{lm}^{+}-(-1)^m b_{l-m}], \tag{4}$$

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where $\omega_l = (C_l/B_l)^{\frac{1}{2}}$. The quantization condition is

$$[b_{lm}, b_{l'm'}^{+}] = \delta_{ll'}\delta_{mm'}. \tag{5}$$

Then

$$H = \sum_{lm} \hbar\omega_l [n_{lm} + \tfrac{1}{2}(2l+1)], \tag{6}$$

where

$$n_{lm} = b_{lm} b_{lm}^{+}. \tag{7}$$

b_{lm} and b_{lm}^{+} are the annihilation and creation operators of phonons with angular momentum l, z-component of angular momentum m, and energy $\hbar\omega_l$.

If $| 0 >$ is the ground state of an even-even nucleus ($J = 0^+$),

$$| n_{lm} = 1 ;\ l,m\rangle = b_{lm}^{+} | 0\rangle \tag{8}$$

is a one-phonon state of spin $J = l$, z-component of spin $M = m$, and parity $(-1)^l$.

$$| n_{lm} = 1, n_{l'm'} = 1 ;\ J,M\rangle = (1+\delta_{ll'})^{-\frac{1}{2}} \sum_{m} C(l,l',J;m,m',M) b_{lm}^{+} b_{l'm'}^{+} |0\rangle \tag{9}$$

is a two-phonon state of spin J and parity $(-1)^{l+l'}$.

Values for the parameters B_l and C_l can be obtained from the hydrodynamical model (Bohr 1952),

$$(B_l)_{\text{hyd}} = l^{-1}\, \frac{3}{4\pi} AMR^2, \tag{10}$$

$$(C_l)_{\text{hyd}} = (l-1)(l+2)R^2 S, \tag{11}$$

where S is the nuclear surface energy. However, for the nickel isotopes, the experiments of Crannell *et al.* (1961) on electron scattering indicate that $B_l/(B_l)_{\text{hyd}}$ is about 20 for $l = 2$ and 3. It is also known that C_2 is sensitive to shell effects, being larger for closed shell nuclei than for nuclei with half-filled shells (Marumori, Suekane, and Yamamoto 1956). Lane and Pendlebury (1960) have shown that C_3 is less sensitive to shell structure effects than C_2 by an order of magnitude.

It is reasonable to expect C_l to be more sensitive to shell effects for even l than for odd l. A single phonon state with odd l has odd parity, and can be built up only of shell model states not contained in the ground state configuration. A single phonon state with even l and thus even parity may have shell model components within the ground state configuration. Thus the energy of a state of even l will be higher near closed shells where there are only a few states available in the ground state configuration than in a half-filled shell where many such states are available.

Although hydrodynamical estimates are unreliable for B_l and C_l, one can still hope to estimate the dependence on l from this model. However, from the above discussion it would seem unwise to compare single-phonon levels of different parity in this way. From equations (11) and (12) the ratios of the energies of single-phonon states are

$$\frac{\hbar\omega_{l'}}{\hbar\omega_l} = \left[\frac{l'(l'-1)(l'+2)}{l(l-1)(l+2)}\right]^{\frac{1}{2}}. \tag{12}$$

In particular

$$\frac{\hbar\omega_4}{\hbar\omega_2}=3, \qquad \frac{\hbar\omega_5}{\hbar\omega_3}=2\cdot 16.$$

For calculating inelastic scattering, we also require

$$\frac{A_l'}{A_l}=\left[\frac{(l-1)(l+2)/l}{(l'-1)(l'+2)/l'}\right]^{\frac{1}{2}}, \tag{13}$$

where $A_l=\hbar/(2B_l\omega_l)$.

The vibrational levels expected for Ni^{60}, using the experimentally determined energies of the lowest 2^+ and 3^- states as $\hbar\omega_2$ and $\hbar\omega_3$ respectively, are given in Table 1.

TABLE 1

VIBRATIONAL STATES OF NICKEL 60

Energy (MeV)	No. of Phonons	Spin and Parity
0	0	0^+
1·33*	$n_2=1$	2^+
2·66	$n_2=2$	0^+, 2^+, 4^+
3·99	$n_2=3$	0^+, 2^+, 3^+, 4^+, 6^+
3·99	$n_4=1$	4^+
4·1*	$n_3=1$	3^-
5·32	$n_2=1$, $n_4=1$	2^+, 3^+, 4^+, 5^+, 6^+
5·4	$n_2=1$, $n_3=1$	1^-, 2^-, 3^-, 4^-, 5^-
5·5	$n_6=1$	6^+

* Experimental values.

III. α-PARTICLE SCATTERING

Using plane-wave Born approximation for the scattering of α-particles, with momentum transfer $\hbar q$, by a vibrational nucleus (with ground state spin $J=0^+$),* we require the following matrix element.*

$$\langle J \mid \mathscr{M} \mid 0\rangle_h=\langle J \mid \textstyle\int \exp(i\mathbf{q}.\mathbf{r})\rho(\mathbf{r})\mathrm{d}^3\mathbf{r} \mid 0\rangle$$

$$=\frac{3A}{4\pi R^3}\langle J \mid \int \mathrm{d}\Omega\int_0^{R'} \mathrm{d}r\, r^2 \exp(i\mathbf{q}.\mathbf{r}) \mid 0\rangle, \tag{14}$$

where R' is given by equation (1).

We have assumed a homogeneous nucleus with a sharp surface and have neglected the range of the α-nucleon interaction. These effects will be included later. Choosing the z-axis along the $\mathbf{q}$-direction, we obtain

$$\langle J \mid \mathscr{M} \mid 0\rangle_h=\frac{3A}{(4\pi)^{\frac{1}{2}}R^3}i^J(2J+1)^{\frac{1}{2}}\langle J \mid \int \mathrm{d}\Omega\int_0^{R'} \mathrm{d}r\, r^2 j_J(\mathbf{qr})Y_{J,0} \mid 0\rangle. \tag{15}$$

Only states with $M_J=0$ are excited.

* In equations which apply only to a homogeneous nucleus with a sharp surface, the matrix element is written with a suffix h.

In the following, each case is calculated only to the first non-vanishing order in the deformation of the nucleus. For elastic scattering we obtain the usual result

$$\langle 0 \mid \mathscr{M} \mid 0 \rangle_h = \frac{A3j_1(qR)}{qR}. \tag{16}$$

For the excitation of single-phonon states, using equations (3) and (8) we obtain

$$\langle n_J = 1\,;\ J \mid \mathscr{M} \mid 0 \rangle_h = \frac{3A}{(4\pi)^{\frac{1}{2}}} i^J (2J+1)^{\frac{1}{2}} \left(\frac{\hbar}{2B_J \omega_J} \right)^{\frac{1}{2}} j_J(qR). \tag{17}$$

For the excitation of two-phonon states, using equation (9), we obtain for $J \neq 0$

$$\begin{aligned} &\langle n_l = 1,\ n_{l'} = 1\,;\ J \mid \mathscr{M} \mid 0 \rangle_h \\ &= \frac{3A}{4\pi} i^J \left(\frac{\hbar}{2B_l \omega_l} \right)^{\frac{1}{2}} \left(\frac{\hbar}{2B_{l'} \omega_{l'}} \right)^{\frac{1}{2}} (1+\delta_{ll'})^{-\frac{1}{2}} [(2l+1)(2l'+1)]^{\frac{1}{2}} \\ &\quad \times C(ll'J;000) \times \{(J+2) j_J(qR) - qR j_{J+1}(qR)\}, \end{aligned} \tag{18}$$

and for $J=0$ $(l=l')$

$$\langle n_l = 2,\ 0 \mid \mathscr{M} \mid 0 \rangle_h = \frac{3A}{4\sqrt{(2)}\pi} (2l+1)^{\frac{1}{2}} \frac{\hbar}{2B_l \omega_l} qR j_1(qR). \tag{19}$$

For small qR,*

$$\langle n_l = 2\,;\ 0 \mid \mathscr{M} \mid 0 \rangle \propto l(qR)^2. \tag{20}$$

Equation (19) differs from the result of Lemmer, de Shalit, and Wall (1961), as their matrix element is not zero for $qR=0$. Because they omit the term $\alpha_{lm}\alpha_{l-m}$ in equation (1), nuclear matter is not conserved to second order in the α_{lm}, and their excited 0^+ state is not orthogonal to the ground state. It is easily seen that for $qR=0$,

$$\langle J \mid \mathscr{M} \mid 0 \rangle = 0, \tag{21}$$

for all inelastic scattering provided the interaction between the scattered particle and a nucleon is independent of the spin and isotopic spin of the nucleon. The matrix element can be written in nucleon coordinates,

$$\langle J \mid \mathscr{M} \mid 0 \rangle = \langle J \mid \sum_{k=1}^{A} \exp\,(i\mathbf{q}.\mathbf{r}_k) \mid 0 \rangle. \tag{22}$$

Then, for $q=0$,

$$\langle J \mid \mathscr{M} \mid 0 \rangle = A \langle J \mid 0 \rangle, \tag{23}$$

which vanishes because of the orthogonality of $\mid J \rangle$ and $\mid 0 \rangle$.

Except for $J=0$ and $J=1$,

$$\langle J \mid \mathscr{M} \mid 0 \rangle \propto l(qR)^J, \tag{24}$$

for small qR. The exceptional behaviour for $J=1$ can also be seen from equation (22). For small q,

$$\begin{aligned} \langle J \mid \mathscr{M} \mid 0 \rangle &= \langle J \mid [1 + i\mathbf{q}.\sum_{k=1}^{A} \mathbf{r}_k \ldots] \mid 0 \rangle \\ &= Ai\mathbf{q}.\langle J \mid \mathbf{R} \mid 0 \rangle \ldots, \end{aligned} \tag{25}$$

* We drop the suffix h in equations (20) to (26), as these equations also apply to an inhomogeneous nucleus.

where $\mathbf{R}$ is the centre of mass of the nucleus, and thus $\langle J \mid \mathbf{R} \mid 0\rangle=0$. We find that

$$\langle J=1 \mid \mathscr{M} \mid 0\rangle \propto l(qR)^3. \tag{26}$$

Although among two-phonon states for which $l \neq l'$, states with spins

$$J=|\,l-l'\,|,\ |\,l-l'\,|+1, \ldots, l+l'-1,\ l+l',$$

all occur, only those states for which $l+l'+J$ is even can be excited by inelastic scattering.

IV. Nucleus with Diffuse Surface

The results of the previous sections are valid only for a homogeneous nucleus with a sharp surface. The energies of the vibrational states are not very sensitive to the thickness of the nuclear surface (e.g. Lane and Pendlebury 1960), but the matrix elements for scattering must be corrected. The simplest way to include the effect of the finite thickness of the nuclear surface is to follow the procedure of Helm (1956). Helm assumes that ρ_t, the transition density involved in a nuclear transition, is given by

$$\rho_t(\mathbf{r})=\int \rho_{h,t}(\mathbf{r}')\rho_s(|\,\mathbf{r}-\mathbf{r}'\,|)\mathrm{d}^3\mathbf{r}', \tag{27}$$

where $\rho_{h,t}$ is the transition density for a homogeneous nucleus,

ρ_s describes the finite thickness of the nuclear surface,

Assuming ρ_s is a gaussian, we obtain

$$\langle J \mid \mathscr{M} \mid 0\rangle=\langle J \mid \mathscr{M} \mid 0\rangle_h \exp\left(-g(qR)^2\right), \tag{28}$$

where $\langle J \mid \mathscr{M} \mid 0\rangle_h$ is the matrix element for a homogeneous nucleus as given by equations (16) to (19).

Assuming the α-nucleon potential is a gaussian, the correction for the finite range of the α-nucleon interaction is also of the form of equation (28). Because of the short mean free path of α-particles in nuclear matter, α-particles may not completely penetrate the nuclear surface. Thus the apparent thickness of the nuclear surface may be less for the scattering of α-particles than for the scattering of electrons or protons. For these reasons, we cannot expect g to be simply related to the similar parameter in the nuclear form factor for electron scattering.

V. Scattering of α-Particles by Nickel 60

The scattering cross section is given by

$$\frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\omega}=D\,|\,\langle J \mid \mathscr{M} \mid 0\rangle\,|^2, \tag{29}$$

where D depends on the strength of the α-nucleon interaction. Figures 1, 2, and 3 show the calculated scattering of 43 MeV α-particles by ^{60}Ni using the following values of the parameters,

$$D=2\cdot18\times10^5 \text{ mb/ster},\quad R=7\cdot35\times10^{-13} \text{ cm};\quad g=0\cdot002;$$

$$\frac{\hbar}{2B_2\omega_2}=2\cdot19\times10^{-3};\qquad \frac{\hbar}{2B_3\omega_3}=4\cdot82\times10^{-4}.$$

These values were chosen to fit the experimental data (Broek *et al.* 1961, personal communication) on the elastic scattering and the inelastic scattering to the 1·33 and 4·1 MeV levels, shown in Figures 1 and 2 respectively. The excitation of the single-phonon 4^+ state is calculated using

$$\hbar/(2B_4\omega_4)=\tfrac{2}{3}\hbar/(2B_2\omega_2),$$

as given by equation (13).

Figures 1 and 2 show there is good agreement between theory and experiment for the elastic scattering and the inelastic scattering to the 1·33 and 4·1 MeV states. The complicated angular distributions obtained for the excitation of

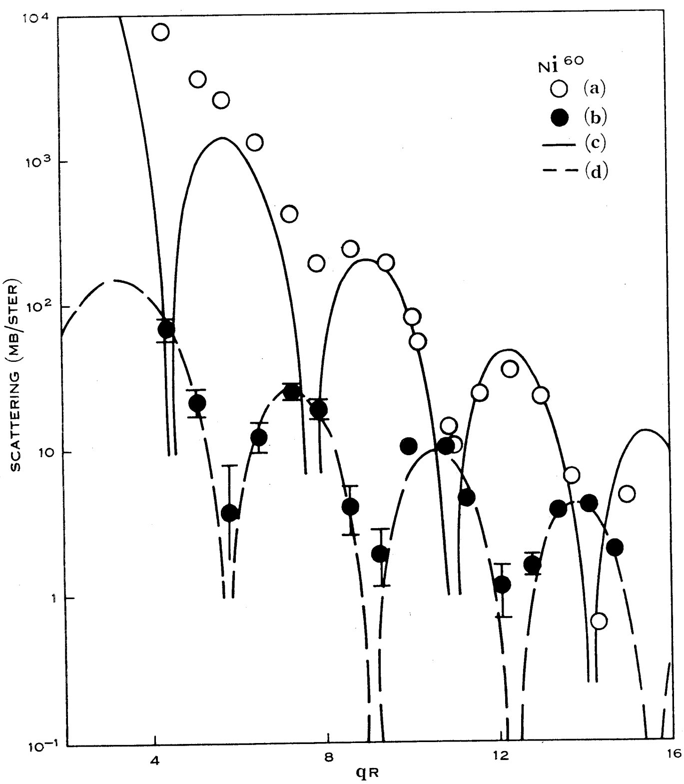

Fig. 1.—Scattering of 43 MeV α-particles by ^{60}Ni. The experimental points are those of Broek *et al.* for (*a*) elastic scattering, (*b*) inelastic scattering to the 1·33 MeV state. The curves are calculated for (*c*) elastic scattering, (*d*) inelastic scattering to the one-phonon 2^- state.

two-phonon states show that care must be taken in trying to determine spins and parities of nuclear energy levels from the positions of maxima in the experimental angular distributions for inelastic scattering. For instance, the maxima in the experimental angular distribution for the 4·1 MeV state occur at the same values of qR as the first three maxima of the angular distribution for excitation of the $J=4^+$ state formed by two 2^+ phonons. However, it is only for the

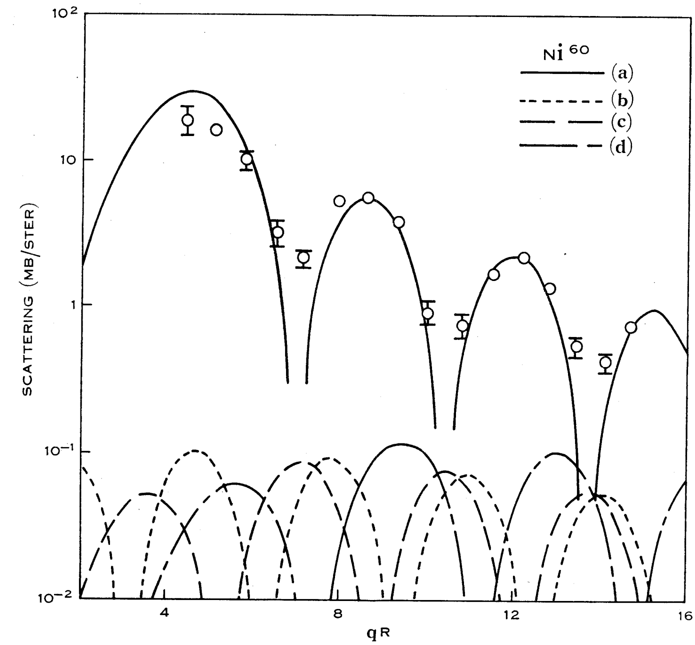

Fig. 2.—Scattering of 43 MeV α-particles by ^{60}Ni. The experimental points are those of Broek *et al.* for inelastic scattering to the 4·1 MeV state. The curves are calculated for inelastic scattering to (*a*) the one-phonon 3^- state, (*b*) the 1^- state, (*c*) the 3^- state, and (*d*) the 5^- state formed by coupling a 2^+ phonon and a 3^- phonon.

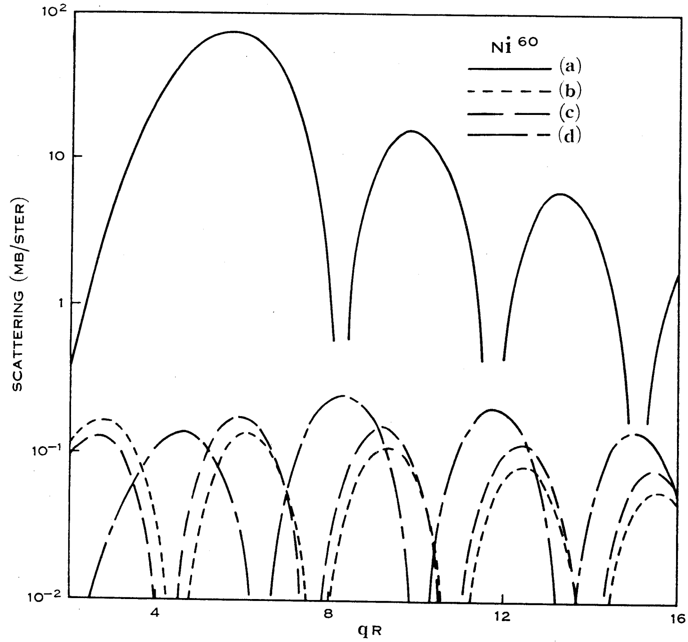

Fig. 3.—Inelastic scattering of 43 MeV α-particles, with excitation of (*a*) the single-phonon 4^+ state, (*b*) the 0^+ state, (*c*) the 2^+ state, and (*d*) the 4^+ state formed by coupling two 2^+ phonons.

spin assignment $J=3^-$, that the observed relative intensities of the maxima are reproduced by the theory.

VI. Discussion

We have used plane-wave Born approximation to calculate α-particle scattering. It is expected that a calculation using distorted-wave Born approximation should be more accurate. In such a calculation, plane waves would be replaced by the wavefunction of a particle moving in a complex potential well. However, some of the effects of a complex potential well on the α-particle wave function have been included in our calculation by regarding R and g as parameters which are determined by fitting experimental data. Part of the effect of the potential well is to increase the apparent wavenumber of the α-particle, thus increasing the value of q. As q only occurs as qR, this is equivalent to increasing R.

Part of the effect of the absorption of α-particles by nuclear matter, as described by the imaginary part of the complex potential well, is that the α-particles do not completely penetrate the nuclear surface. As discussed in Section IV, some of this effect is to alter the apparent value of g. It cannot be expected that g will have the same value for all transitions, but, from the comparison with experiment in Section V, it is seen that the scattering to three different states can be fitted with the same value of g. However, the effect of absorption may still be different for the excitation of two-phonon states.

Another weakness of our calculation is that mixing of two-phonon and single-phonon states is not included. As the scattering to two-phonon states is very much weaker than the scattering to single-phonon states, this phonon mixing will not have much effect on the analysis of single-phonon states. On the other hand, a small admixture of single-phonon amplitude into a two-phonon state will greatly alter the inelastic scattering. Fortunately, there can be no admixture of single-phonon amplitude into the 0^+ and 1^- states, and as the single-phonon 5^- state is expected near 9 MeV in Ni^{60}, the 5^- state expected near $5\cdot4$ MeV should be almost a pure two-phonon state. Thus, the 0^+, 1^-, and 5^- states should provide the best test of this theory. In the two-phonon 2^+ and 4^+ states, some admixture must be expected of single-phonon 2^+ and 4^+ states respectively, and the theory given here should be least reliable for these two states.

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VIII. References

Bohr, A. (1952).—*Math-fys. Medd.* **26**: 14.

Crannell, H., Helm, R., Kendall, H., Oeser, J., and Yearian, M. (1961).—*Phys. Rev.* **123**: 923.

Helm, R. H. (1956).—*Phys. Rev.* **104**: 1466.

Lane, A. M., and Pendlebury, E. D. (1960).—*Nucl. Phys.* **15**: 39.

Lemmer, R. H., de Shalit, A., and Wall, N. S. (1961).—*Phys. Rev.* **124**: 1155.

Marumori, T., Suekane, S., and Yamamoto, A. (1956).—*Progr. Theor. Phys. Osaka* **16**: 320.

Reiner, A. S. (1958).—Ph.D. thesis, Amsterdam.