RESONANT BEHAVIOUR OF AN ACOUSTICAL TRANSMISSION LINE

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Summary

In the passage of elastic waves through a cylindrical rod, phenomena are exhibited that closely resemble those occurring in an electromagnetic waveguide. Following a discussion of the classical Pochhammer-Chree theory for an infinite cylinder, elastic wave propagation in a finite rod is considered under conditions when the rod may be regarded as analogous to an electrical transmission line. In the case of a resonant system, normal mode analysis provides a simplified treatment and it is shown that the results are in agreement with transmission-line theory. A brief discussion is given regarding the identity of the pulse response of a system and its response to continuous waves.

I. INTRODUCTION

In the study of wave propagation in finite rods and plates, it is not possible to obtain exact solutions of the wave equation because of the difficulty in satisfying the boundary conditions. Green (1960) has discussed various approximation methods that have been suggested for waves in rods, particularly for the low frequency range.

In the case of elastic waves in an infinite isotropic elastic cylinder, exact solutions of the wave equation have been obtained independently by Pochhammer (1876) and Chree (1889). Three types of wave motion are possible in a cylinder, namely, longitudinal, torsional, and flexural. For each wave type a series of waveguide modes is obtained, as in the analogous case of an electromagnetic waveguide, for which the phase velocity is a function of the parameter a/λ, where a is the radius of the cylinder and λ is the wavelength. Provided interest can be confined to one particular mode, such as the first longitudinal or " Young's modulus " mode, then an acoustical system may be treated as analogous to an electrical transmission line. The theory so developed may be applied to both infinite and finite systems.

It is the aim of this paper to develop the theory of the acoustical transmission line (Section III). The classical theory for an infinite cylinder is summarized in Section II, while in Sections IV and V it is shown that, for a resonant system, when the main interest is in the response at resonance, transmission-line theory may be simplified and is then found to be in agreement with the analysis of the system in terms of its normal modes as developed by Skudrzyk (1958). As Skudrzyk has shown, each normal mode of vibration may be represented by a series-resonant circuit. It is finally shown that the pulse response of a finite rod gives the same results as continuous wave analysis provided the responses produced by the multiply reflecting pulses are additive.

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II. Waves in a Solid Cylinder : Pochhammer-Chree Theory

Wave motion in an infinite isotropic elastic solid must satisfy the wave equations

$$\nabla^2\varphi=\frac{1}{c_l^2}\frac{\partial^2\varphi}{\partial t^2}, \qquad \nabla^2\psi_i=\frac{1}{c_t^2}\frac{\partial^2\psi_i}{\partial t^2}, \tag{1}$$

where the displacement u_i is written in terms of a scalar potential φ and a vector potential ψ_i so that $u_i=\text{grad}\ \varphi+\text{curl}\ \psi_i$, the suffix i taking the values 1, 2, 3. c_l is then the velocity of longitudinal or dilatational waves, c_t the velocity of transverse or rotational waves. Solutions of the wave equations are sought using cylindrical coordinates r, θ, and z (the axis of the cylinder being taken in the z direction) together with the appropriate boundary conditions. When the particle displacements are independent of θ, the solutions correspond to longitudinal waves. If the displacements are independent of r and z, the solutions correspond to torsional waves. If the displacements are a function of r, θ, and z, the solutions correspond to flexural waves.

(a) *Solutions for Longitudinal Waves*

The wave equations in cylindrical coordinates are

$$\left.\begin{aligned}\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}&=\frac{1}{c_l^2}\frac{\partial^2\varphi}{\partial t^2},\\ \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}&=\frac{1}{c_t^2}\frac{\partial^2\psi}{\partial t^2}.\end{aligned}\right\} \tag{2}$$

The harmonic solutions of interest have the form

$$\left.\begin{aligned}\varphi&=A\mathrm{J}_0(k_lr)\exp \mathrm{i}(\omega t-k_zz),\\ \psi&=B\mathrm{J}_0(k_tr)\exp \mathrm{i}(\omega t-k_zz),\end{aligned}\right\} \tag{3}$$

where

$$k_l^2=(\omega/c_l)^2-k_z^2,$$
$$k_t^2=(\omega/c_t)^2-k_z^2,$$

and k_z is the component of the wave vector in the z direction, ω is the angular frequency.

For an infinitely long cylinder of radius a, the boundary conditions are that the normal and tangential stresses must be zero at the surface of the cylinder, that is, $\sigma_{rr}=0$ and $\sigma_{zr}=0$ at $r=a$. As shown in Love (1927), application of the boundary conditions leads to the frequency equation

$$k_z^2\frac{k_t\mathrm{J}_0(k_ta)}{\mathrm{J}_1(k_ta)}-\tfrac{1}{2}\left(\frac{\omega}{c_t}\right)^2\frac{1}{a}+\left[\tfrac{1}{2}\left(\frac{\omega}{c_t}\right)^2-k_z^2\right]^2\frac{\mathrm{J}_0(k_la)}{k_l\mathrm{J}_1(k_la)}=0. \tag{4}$$

Equation (4) may be solved at any frequency to give the phase velocity corresponding to a particular mode (Bancroft 1941 ; Davies 1948). Similarly, equations may be obtained for the particle displacements u_r and u_z.

The Bessel functions may be expanded to give approximate solutions. Taking only the first-order terms in a gives

$$\omega/k_z \equiv c_L = (E/\rho)^{\frac{1}{2}}, \tag{5}$$

where c_L is the low frequency velocity of plane longitudinal waves in a thin rod with Young's modulus E and density ρ. When second-order terms in a are included

$$\omega/k_z = c_L[1-\sigma^2\pi^2(a/\lambda)^2], \tag{6}$$

where σ is Poisson's ratio. Equation (6) was first derived by Rayleigh (1894) taking into account the lateral inertia of the rod.

Davies (1948) has published curves for the phase velocity of the first three longitudinal modes of a solid cylinder having a Poisson's ratio of 0·29. In Figure 1 is shown the form of the phase velocity distributions for these three

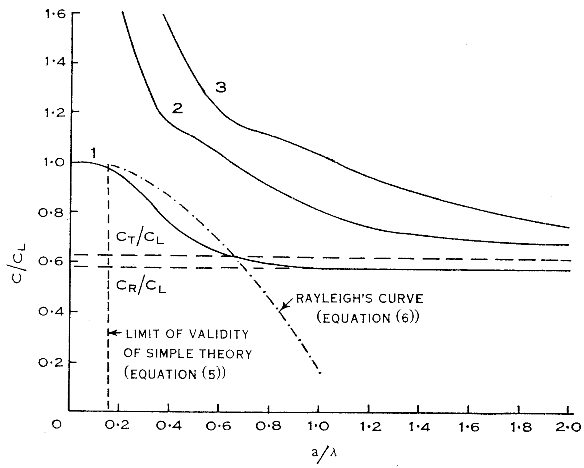

Fig. 1.—Phase velocity distributions for the first three longitudinal modes of a solid cylinder having Poisson's ratio 0·29 (after Davies 1948). C_L, C_T, and C_R represent the velocities of longitudinal, torsional, and Rayleigh waves respectively.

modes as a function of a/λ. Also shown is a curve corresponding to Rayleigh's formula (equation (6)). For the first longitudinal mode, also described as the (1,1) mode or Young's modulus mode, the phase velocity is constant at low frequencies and is given by equation (5). The upper frequency limit of validity of this simple relationship occurs when $a/\lambda \geqslant 1/6$. In the (1,1) mode the wave fronts are planes whose normals are in the direction of the axis of the cylinder. At high frequencies, that is, increased values of a/λ, the phase velocity of the (1,1) mode approaches the velocity of Rayleigh surface waves.

The higher order longitudinal modes exhibit a cut-off phenomenon, similar to the cut-off effect in an electromagnetic waveguide. For these higher modes the wave fronts are inclined at an angle θ to the axis of the cylinder with the phase velocity c given by $c = c_l/\sin\theta$. Cut-off occurs when $\theta \to 0$ and hence $c \to \infty$. With increasing frequency, the phase velocity of the higher modes approaches that for transverse waves.

(b) *Solutions for Torsional Waves*

If $u_r=0$, $u_z=0$ and u_θ is finite and independent of θ, then the appropriate solutions of the wave equations (2) in terms of displacements that satisfy the boundary conditions are

$$u_\theta = A\mathrm{J}_1(k_t r)\exp \mathrm{i}(\omega t - k_z z), \quad \text{when } k_t \neq 0, \tag{7a}$$
$$u_\theta = Br \exp \mathrm{i}(\omega t - \omega z/c_t), \quad \text{when } k_t = 0. \tag{7b}$$

Equation (7a) gives a series of modes with cut-off frequencies similar to the higher modes for longitudinal waves. Equation (7b) represents the fundamental torsional mode for which the phase velocity is independent of frequency. An arbitrary pulse will therefore be transmitted without distortion in this mode. Much use has been made of the fundamental torsional mode in the design of delay lines. In Figure 2 are shown the phase velocity distributions for the first three torsional modes in a cylinder having a Poisson's ratio of $0 \cdot 29$.

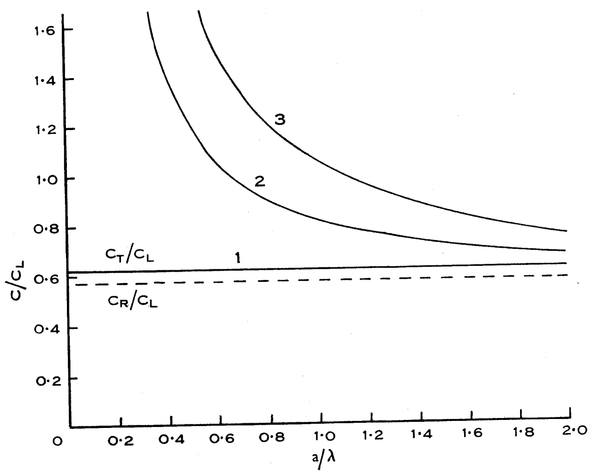

Fig. 2.—Phase velocity distributions for the first three torsional modes of a solid cylinder having Poisson's ratio $0 \cdot 29$ (after Davies 1956).

(c) *Solutions for Flexural Waves*

Although the solutions of the wave equations are more complicated in this case, since the displacements depend on r, θ, and z, they may be accomplished by a similar process as before. Sittig (1957) has derived the frequency equation in the form of the determinant

$$\begin{vmatrix} n^2-1-a^2k_z^2(x-1) & n^2-1-a^2k_z^2(2x-1) & 2(n^2-1)[\gamma_n(k_l a)-n]-a^2k_z^2(2x-1) \\ \gamma_n(k_l a)-n-1 & \gamma_n(k_t a-n-1) & 2n^2-2[\gamma_n(k_t a)-n]-a^2k_z^2(2x-1) \\ \gamma_n(k_t a)-n & -(x-1)[\gamma_n(k_t a)-n] & n^2 \end{vmatrix} = 0, \tag{8}$$

where

$$\gamma_n(ka) = \frac{ka\mathrm{J}_{n-1}(ka)}{\mathrm{J}_n(ka)},$$
$$x = \frac{c^2}{2c_t^2},$$
$$n = 0,\ 1,\ 2\ \ldots$$

The longitudinal and torsional modes are found as special cases of equation (8) by putting $n=0$. In Figure 3 are shown the phase velocity distributions for the first three flexural modes for a cylinder with Poisson's ratio 0·29. As a/λ increases, the phase velocity of the first flexural mode approaches the velocity of Rayleigh surface waves while, for the higher modes, the phase velocity approaches the velocity of transverse waves.

(*d*) *Pulse Response*

Although the Pochhammer-Chree theory applies strictly only for continuous waves it may be used to describe the propagation of pulses provided $a/\lambda<1$. At high frequencies when $a/\lambda>1$, " trailing pulses " appear owing to mode conversion at the boundaries. The modified theory proposed by Redwood (1959) may then be employed. For instance, considering an initial longitudinal

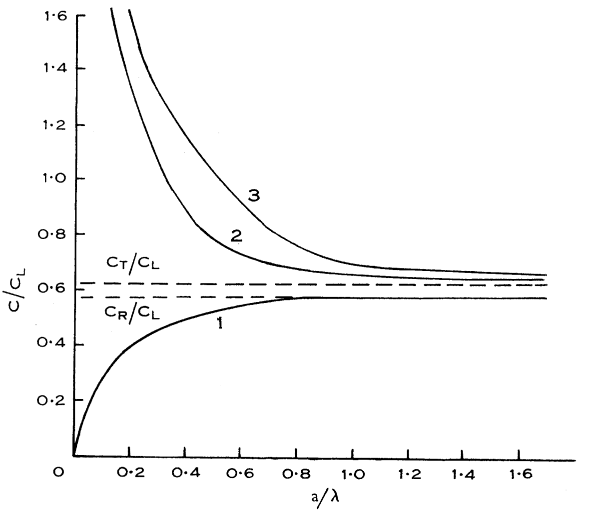

Fig. 3.—Phase velocity distributions for the first three flexural modes of a solid cylinder having Poisson's ratio 0·29 (after Abramson 1957).

pulse, the solutions must provide for the fact that transverse waves, generated by mode conversion at the boundaries, cannot appear until after the initial longitudinal waves have reached the boundary. The Bessel function solutions assume both types of waves present from the start. Redwood (1959) has overcome this problem by the use of a Hankel function in place of the Bessel function in the second of equations (3). The modified solutions are

$$\left.\begin{aligned}\varphi&=A\mathrm{J}_0(k_l r)\exp \mathrm{i}(\omega t-k_z z),\\ \psi&=B\mathrm{H}_0^{(1)}(k_t r)\exp \mathrm{i}(\omega t-k_z z).\end{aligned}\right\}\tag{9}$$

Applying the boundary conditions leads to the modified frequency equation

$$k_z^2\,\frac{k_t\mathrm{H}_0^{(1)}(k_t a)}{\mathrm{H}_1^{(1)}(k_t a)}-\tfrac{1}{2}\left(\frac{\omega}{c_t}\right)^2\frac{1}{a}+\left[\tfrac{1}{2}\left(\frac{\omega}{c_t}\right)^2-k_z^2\right]^2\frac{\mathrm{J}_0(k_l a)}{k_l\mathrm{J}_1(k_l a)}=0.\tag{10}$$

III. Acoustical Transmission-line Theory

(a) *Boundary Conditions*

When only one mode, such as the Young's modulus mode, is under consideration, it is convenient to make use of the analogies that may be drawn between the propagation of elastic waves in a rod and electrical transmission-line theory. It will first of all be necessary to establish the effect of a boundary on the propagation of a longitudinal wave in a rod. At any free surface, the boundary conditions for the stress are that there must be no resultant normal or tangential stresses. As shown by Kolsky (1953), if a stress pulse is incident on a free end of the rod, the shape of the pulse is unaltered on reflection from the free end but it changes sign, that is, a compressional pulse will be reflected as a tensile pulse. In addition, the value of both the particle displacement and particle velocity at a free end is twice the value when travelling along the rod. In the case of a clamped end, the resultant displacement and velocity will be zero whereas the resultant stress has twice the value when travelling along the rod.

For an electrical transmission line which is short-circuited at one end, the end voltage is zero and hence an incident voltage pulse will be reflected with phase reversal. A current pulse is reflected from a short-circuited end with phase unchanged and hence has double amplitude at the end. For an electrical transmission line which is open-circuited at one end, the end current is zero and hence a current pulse will be reflected with phase reversal. On the other hand, a voltage pulse is reflected with phase unchanged and hence has double amplitude at the end. If stress or force is taken as analogous to voltage and velocity analogous to current (Olson 1958), it may be concluded that the electrical analogue of a rod with free ends is a short-circuited transmission line. Similarly, the electrical analogue of a rod with clamped ends is an open-circuited transmission line. Although this form of analogy is convenient for the present purpose, it is also feasible to have force analogous to current and velocity analogous to voltage. The theory of the acoustical transmission line will now be developed with the aid of acoustical-electrical analogies.

(b) *Infinite Rod*

Consider plane longitudinal waves travelling along an isotropic cylindrical rod whose axis is in the x direction. Lateral motion due to " Poisson coupling ", as included in equation (6), may be ignored at low frequencies when $\lambda \gg a$. A wave equation similar to equation (1) may be written in terms of displacement. The general solution for plane waves is of the form

$$\mathbf{s}_x = \mathbf{A} \exp \mathrm{i}(\omega t - \mathbf{k}x) + \mathbf{B} \exp \mathrm{i}(\omega t + \mathbf{k}x), \tag{11}$$

where $\mathbf{s}_x$ is the complex displacement, $\mathbf{A}$ the complex amplitude of a wave proceeding in the $+x$ direction, $\mathbf{B}$ is the complex amplitude of a wave proceeding in the $-x$ direction, $\mathbf{k} = k - \mathrm{i}\alpha$ is the complex wave vector, $k = \omega/c$, α is the attenuation coefficient. Frequently, the propagation coefficient γ is introduced where $\gamma = \mathrm{i}\mathbf{k} = \alpha + \mathrm{i}k$.

The corresponding particle velocity, $\mathbf{u}_x$ is then given by

$$\mathbf{u}_x \equiv \frac{\partial \mathbf{s}_x}{\partial t} = \mathrm{i}\omega \mathbf{A} \exp \mathrm{i}(\omega t - \mathbf{k}x) + \mathrm{i}\omega \mathbf{B} \exp \mathrm{i}(\omega t + \mathbf{k}x). \tag{12}$$

Assuming that Hooke's Law is obeyed, the stress $\boldsymbol{\sigma}_{xx}$ is then found from $\sigma_{xx} = E\boldsymbol{\epsilon}_{xx} = E\partial \mathbf{s}_x/\partial x$, where E is Young's modulus and $\boldsymbol{\epsilon}_{xx}$ is the principal strain in the x direction. Thus,

$$\boldsymbol{\sigma}_{xx} = -\mathrm{i}\mathbf{k}E\mathbf{A} \exp \mathrm{i}(\omega t - \mathbf{k}x) + \mathrm{i}\mathbf{k}E\mathbf{B} \exp \mathrm{i}(\omega t + \mathbf{k}x). \tag{13}$$

As in electrical theory, we may introduce a mechanical impedance, $\mathbf{Z}_m$, defined as the ratio of force to velocity at any point x along the axis. If the area of cross section is taken as unity, $\mathbf{Z}$ is then a measure of the ratio of stress to velocity and is known as the specific acoustic impedance. For the case under consideration,

$$\mathbf{Z}_m = -\frac{\boldsymbol{\sigma}_{xx}}{\mathbf{u}_x} S = \mathbf{Z}_0 \frac{\mathbf{A} \exp(-\mathrm{i}\mathbf{k}x) - \mathbf{B} \exp(\mathrm{i}\mathbf{k}x)}{\mathbf{A} \exp(-\mathrm{i}\mathbf{k}x) + \mathbf{B} \exp(\mathrm{i}\mathbf{k}x)}, \tag{14}$$

where $\mathbf{Z}_0 = \mathbf{k}ES/\omega$ is the characteristic impedance of the medium and S is the area of cross section. If the attenuation is small, Z_0 is real and is given by $Z_0 \doteq ES/c = \rho c S$. The negative sign appears in equation (14) since a positive u_x would result from a compression applied to the rod for which σ_{xx} is customarily taken as negative. For a travelling wave in an infinite rod in the $+x$ direction, $\mathbf{B} = 0$ and hence $\mathbf{Z}_m = \mathbf{Z}_0$. That is, in the absence of any backward wave arising by virtue of reflection, the impedance experienced by the wave is the characteristic impedance of the medium.

(c) *Finite Rod*

Consider, now, a finite rod of length l with ends at $x=0$ and $x=l$. The end termination may be represented by an impedance $\mathbf{Z}_T$. In the extreme cases, if the rod is free, $\mathbf{Z}_T = 0$; if clamped, $\mathbf{Z}_T = \infty$. The terminating impedance at $x=l$ is found from equation (10)

$$\mathbf{Z}_T = \mathbf{Z}_0 \frac{\mathbf{A} \exp(-\mathrm{i}\mathbf{k}l) - \mathbf{B} \exp(\mathrm{i}\mathbf{k}l)}{\mathbf{A} \exp(-\mathrm{i}\mathbf{k}l) + \mathbf{B} \exp(\mathrm{i}\mathbf{k}l)}. \tag{15}$$

Hence,

$$\frac{\mathbf{B}}{\mathbf{A}} \exp(2\mathrm{i}\mathbf{k}l) = \frac{\mathbf{Z}_0 - \mathbf{Z}_T}{\mathbf{Z}_0 + \mathbf{Z}_T} \equiv \mathbf{R}, \tag{16}$$

where $\mathbf{R}$ is the complex reflection coefficient. From equations (13) and (16) it is observed that the reflection coefficient for stress is $\mathbf{R}(\sigma) = -\mathbf{R}$. From equations (12) and (16), the velocity reflection coefficient is $\mathbf{R}(u) = \mathbf{R}$. Thus, for a free end, $\mathbf{Z}_T = 0$, and equation (16) shows that $\mathbf{R}(\sigma) = -1$ and $\mathbf{R}(u) = 1$. It is thus confirmed that in terms of stress, an incident wave is reflected with phase reversed giving a resultant stress of zero at the free end, while in terms of velocity an incident wave is reflected without phase change giving rise to a doubling in velocity amplitude at the free end. When $\mathbf{Z}_T = \mathbf{Z}_0$, $\mathbf{R} = 0$, that is, no reflection takes place when a rod is terminated by its characteristic impedance.

The results developed above are identical to those applicable to electrical transmission lines (Slater 1959) when force and velocity are replaced by voltage and current. The comparison is further emphasized by calculating the input and transfer impedances for a finite rod.

The input impedance, $\mathbf{Z}_{11}$, at $x=0$ is defined as $\mathbf{Z}_{11}\equiv f_1/u_1$, where f_1 is the input force and u_1 is the input velocity.

From equation (15)

$$\mathbf{Z}_{11}=\mathbf{Z}_0\frac{1-\mathbf{B}/\mathbf{A}}{1+\mathbf{B}/\mathbf{A}}.$$

Combining with equation (16) yields

$$\mathbf{Z}_{11}=\mathbf{Z}_0\frac{\mathbf{Z}_0 \sinh \mathbf{i}\mathbf{k}l+\mathbf{Z}_T \cosh \mathbf{i}\mathbf{k}l}{\mathbf{Z}_0 \cosh \mathbf{i}\mathbf{k}l+\mathbf{Z}_T \sinh \mathbf{i}\mathbf{k}l}. \tag{17}$$

Equation (17) is identical to the equation giving the input impedance of an electrical transmission line of length l with terminating impedance $\mathbf{Z}_T$. Important special cases of equation (17) arise for the extremes of $\mathbf{Z}_T=0$ and $\mathbf{Z}_T=\infty$.

$$\text{For } \mathbf{Z}_T=0: \qquad \mathbf{Z}_{11}=\mathbf{Z}_0 \tanh \mathbf{i}\mathbf{k}l, \tag{18}$$

$$\text{for } \mathbf{Z}_T=\infty: \qquad \mathbf{Z}_{11}=\mathbf{Z}_0 \coth \mathbf{i}\mathbf{k}l. \tag{19}$$

Equation (18) gives the ratio of force to velocity at the input end of a rod with free ends or a short-circuited transmission line. Equation (19) applies to a rod with clamped ends or an open-circuited transmission line. Both equations represent a steady-state solution and describe the set of standing waves in a rod when the velocity of an end face is measured from the same end as the force is applied. It also follows that $\mathbf{Z}_{11}$ (free) . $\mathbf{Z}_{11}$ (clamped)$=\mathbf{Z}_0^2$.

If the force is applied at one end of a rod and the velocity measured at the opposite end, the transfer impedance $\mathbf{Z}_{12}\equiv f_1/u_2$ is more appropriate, where f_1 is the input force and u_2 is the output velocity. By analogy with the corresponding electrical equations (Cherry 1949), the force and velocity at any point along a rod may be written

$$\left.\begin{aligned} f_x&=f_1 \cosh \mathbf{i}\mathbf{k}x-u_1\mathbf{Z}_0 \sinh \mathbf{i}\mathbf{k}x,\\ u_x&=u_1 \cosh \mathbf{i}\mathbf{k}x-\frac{f_1}{\mathbf{Z}_0} \sinh \mathbf{i}\mathbf{k}x.\end{aligned}\right\} \tag{20}$$

From equations (18) and (20), the transfer impedance of a rod of length l with free ends then follows,

$$\mathbf{Z}_{12}\equiv f_1/u_2=\mathbf{Z}_0 \sinh \mathbf{i}\mathbf{k}l. \tag{21}$$

In terms of the input impedance,

$$\mathbf{Z}_{12}=\mathbf{Z}_{11} \cosh \mathbf{i}\mathbf{k}l. \tag{22}$$

Similarly, for a rod with clamped ends,

$$\mathbf{Z}_{12}=\mathbf{Z}_0 \operatorname{cosech} \mathbf{i}\mathbf{k}l=\mathbf{Z}_{11} \operatorname{sech} \mathbf{i}\mathbf{k}l. \tag{23}$$

Thus, according to the experimental conditions, either $\mathbf{Z}_{11}$ or $\mathbf{Z}_{12}$ may be used to compute the total response of the system.

IV. Normal Mode Analysis

While a computation of the total response of a finite system, as described in the previous section, may sometimes be necessary, it is often the response near resonant peaks that is of most interest. A considerable simplification may then be made by expressing the response of the system in terms of its normal modes of vibration. While this approach has often been described for simple systems (for instance, Slater and Frank 1947), the theory for multi-resonant systems has been developed in detail by Skudrzyk (1958, 1959), Skudrzyk, Kautz, and Greene (1961) and some of the consequences have been examined experimentally by Biesterfeldt, Lange, and Skudrzyk (1960).

Instead of attempting to describe the mechanical system in terms of a wave equation and appropriate boundary conditions, parameters are introduced that represent physically measurable properties of the system, such as resonant frequencies, bandwidths, and mechanical impedances. Once these parameters are known for each mode of vibration of the system, the response of the system may be computed for any given excitation function. The main features of Skudrzyk's theory will now be summarized.

The differential equation of motion for a mechanical system with continuously distributed mass and compliance, vibrating in simple harmonic motion, may be written in terms of the normal modes of vibration of the system in the form

$$m\left[-\omega^2\sum_n \mathbf{s}_n+\sum_n \omega_n^2\mathbf{s}_n(1+i\eta)\right]=\mathbf{f}, \tag{24}$$

where m is the mass per unit area or volume, $\mathbf{s}_n$ is the displacement of the nth normal mode, ω_n is the resonant angular frequency of the nth mode, $\eta=1/Q$ is the loss factor for the whole system, and $\mathbf{f}$ is the external force per unit area or volume at angular frequency ω. Quantities such as $\mathbf{s}$ and $\mathbf{f}$ are functions of the coordinates x, y, z. It is advantageous to introduce two mode constants that will be incorporated into the parameters of the system. A mode constant, $q_n(A)$, represents the mean square of the displacement at an arbitrary point A in the system and is defined by

$$q_n(A)=\langle s_n^2\rangle/s_n^2(A). \tag{25}$$

where $\langle s_n^2\rangle$ is the mean square of $|s_n|$ over the system. A second mode constant, $\varkappa_n(A)$, represents the fraction of the total force available for exciting the nth mode of the system and is defined by

$$\varkappa_n(A)=\int_\sigma \frac{\mathbf{f}\mathbf{s}_n}{\mathbf{f}_0\mathbf{s}_n(A)}\mathrm{d}\sigma, \tag{26}$$

where $\mathbf{f}_0$ is the total driving force and σ represents the area or volume. It may be observed that $q_n(A)$ is a function of the mathematical form of the normal modes only and is not a function of their amplitudes nor of the force distribution. On the other hand, $\varkappa_n(A)$ depends on both the force distribution and the form of the normal modes.

Skudrzyk (1958) shows that an equation of motion may now be written for the nth normal mode in the form

$$[-\omega^2 M_n(A)+i\omega R_n(A)+1/C_n(A)]\,\mathbf{s}_n(A)=\mathbf{f}_0, \tag{27}$$

where

$$\left.\begin{aligned} M_n(A)&=\{q_n(A)/\varkappa_n(A)\}M,\\ R_n(A)&=(\omega_n^2/\omega)\eta M_n(A),\\ C_n(A)&=1/\omega_n^2 M_n(A), \end{aligned}\right\} \tag{28}$$

$M=m\sigma$ is the total mass of the system,
$M_n(A)$ is the effective mass associated with the nth mode,
$R_n(A)$ is the effective mechanical resistance,
$C_n(A)$ is the effective compliance.
The above quantities refer to the motion of an arbitrary point A in the system.

The solution for the equation of motion of the complete system may then be written in the form

$$\frac{\mathbf{u}(A)}{\mathbf{f}_0}=\frac{1}{\mathbf{Z}(A)}=\sum_n \frac{1}{\mathbf{Z}_n(A)}, \tag{29}$$

where

$$\mathbf{u}(A)=i\omega\sum_n \mathbf{s}_n(A),$$

and

$$\mathbf{Z}_n(A)=R_n(A)+i\omega M_n(A)+1/i\omega C_n(A). \tag{30}$$

$\mathbf{u}(A)$ is the total particle velocity, $\mathbf{Z}(A)$ is the total mechanical impedance, and $\mathbf{Z}_n(A)$ is an impedance associated with the nth normal mode, the particular type of force, and the point of observation in the system relative to the point of application of the force. The mode impedance, given by equation (30), reduces to $Z_n=R_n$ at resonance and for points just off resonance may be written in the form

$$\mathbf{Z}_n=R_n(1+iQ_n\gamma_n), \tag{31}$$

where

$$Q_n=\omega_n M_n/R_n,$$

and

$$\gamma_n=\omega/\omega_n-\omega_n/\omega.$$

It is of interest to note that, from the definition of Q_n and equations (28), it follows that $Q_n=\omega/\omega_n\eta$ and is therefore independent of the reference point A.

The above analysis shows that, provided there is a Fourier component of the force at a particular frequency, that is, $\varkappa_n(A)$ is finite, the corresponding normal mode of the system will be excited. The resulting velocity is then the same as would be obtained from a simple mass-spring system having the same impedance. This conclusion may be applied to complex systems having any number of natural frequencies and degrees of freedom. The behaviour of a mechanical system with continuously distributed mass and compliance can therefore be represented by an electrical circuit consisting of an infinite number of parallel branches. Each branch consists of a series-resonant circuit which represents a normal mode of the system.

Skudrzyk (1958) has also computed the impedance values at the anti-resonances between resonant peaks. It is found that the geometric mean of the maximum and minimum impedance is equal to Z_0.

(a) *Mode Parameters for Longitudinal Vibrations of a Free Rod*

The normal modes have the form

$$s_n = |s_n| \cos (k_n x),$$

where $k_n = \omega_n/c$, $\omega_n = n\pi c/l$, l being the length of the rod.
Thus, the mode constant $q_n(A)$ for a given reference point A is

$$q_n(A) = \langle s_n^2 \rangle / s_n^2(A) = 1/2 \cos^2 k_n x_A.$$

If the point A coincides with a point of maximum displacement, such as a free end of a rod, $q_n(A) = \frac{1}{2}$.

If a point force is applied at the point $x = F$ and the motion is observed at $x = A$, then, application of equation (26) leads to $\varkappa_n(F,A) = s_n(F)/s_n(A)$. For normal modes as above, $\varkappa_n(F,A) = \cos (k_n x_F)/\cos (k_n x_A)$.

If F is at one end of the rod and A is at the other end of the rod, $\varkappa_n = \pm 1$ according as n is even or odd. If F and A coincide, $\varkappa_n = 1$. In Table 1 are summarized the mode parameters required for longitudinal wave propagation in a rod.

TABLE 1

MODE PARAMETERS FOR LONGITUDINAL WAVES IN A ROD*

Parameter	F and A at Same End of Rod	F and A at Opposite Ends of Rod
q_n	$\frac{1}{2}$	$\frac{1}{2}$
$\varkappa_n$	1	± 1
M_n	$\frac{1}{2}M$	$\pm\frac{1}{2}M$
R_n	$\omega_n M/2Q_n$	$\pm\omega_n M/2Q_n$
C_n	$2/\omega_n^2 M$	$\pm 2/\omega_n^2 M$

* The + sign applies if the mode number n is even, the — sign if n is odd. The sign governs the phase relations and may be omitted if magnitudes only are relevant.

As an example of the application of these parameters, the following data were obtained for the first longitudinal resonant mode of a brass rod (composition: Cu 61%, Zn 36%, Pb 3%): length 0·199 m, density $8 \cdot 43 \times 10^3$ kg m^{-3}, radius $3 \cdot 18 \times 10^{-3}$ m, mass 0·0534 kg, $\omega_1 = 2\pi \times 9080$ s^{-1}, $Q_1 = 1 \cdot 62 \times 10^4$. Then, the parameters for this mode (omitting minus signs) are: $q_n = \frac{1}{2}$, $\varkappa_n = 1$, $M_1 = 0 \cdot 0267$ kg, $R_1 = 0 \cdot 094 \Omega$, $C_1 = 1 \cdot 15 \times 10^{-8}$ m.N^{-1}. A series electrical circuit that would have a similar behaviour at a frequency of 9080 c/s would have the following parameters: L = 26·7 mH, $R = 0 \cdot 094 \Omega$, $C = 0 \cdot 0115$ μF.

V. Equivalence of Transmission Line and Normal Mode Analysis near Resonance

From equation (18), giving the input impedance of a short-circuited line, $\mathbf{Z}_{11}$ may be written in terms of a resistance R and reactance X, so that $\mathbf{Z}_{11}=R_{11}+\mathrm{i}X_{11}$. R_{11} and X_{11} may be found by expanding equation (18) and separating real and imaginary parts :

$$R_{11}=Z_0\frac{\sinh \alpha l \cosh \alpha l}{\cosh^2 \alpha l \cos^2 kl+\sinh^2 \alpha l \sin^2 kl}, \tag{32}$$

$$X_{11}=Z_0\frac{\sin kl \cos kl}{\cosh^2 \alpha l \cos^2 kl+\sinh^2 \alpha l \sin^2 kl}. \tag{33}$$

In order to simplify equations (32) and (33) two steps are possible, (i) when the damping is small, $\sinh \alpha l\to\alpha$, $\cosh \alpha l\to 1$ and Z_0 is real, (ii) l can be taken as a multiple of $\frac{1}{4}\lambda$, that is, $l=m.\frac{1}{4}\lambda$ or $kl=m.\frac{1}{2}\pi$, where m is an integer. If m is odd, $\cos kl=0$ and $\sin kl=1$. If m is even, $\cos kl=1$ and $\sin kl=0$. Thus, for $\frac{1}{4}\lambda$ line (m odd), $R_{11}=Z_0/\alpha l$ and $X_{11}=0$. For a $\frac{1}{2}\lambda$ line (m even), $R_{11}=Z_0\alpha l$ and $X_{11}=0$. As shown by Slater (1959), Q_n for a resonant transmission line may be defined in terms of the bandwidth, $\Delta\nu$, of the resonant curve,

$$Q_n\equiv\nu_0/\Delta\nu=k_n/2\alpha_n, \tag{34}$$

where ν_0 is the resonant frequency. It follows from equation (34) that, for a line whose length is a multiple of $\frac{1}{4}\lambda$,

$$\alpha_n l=m\pi/4Q_n. \tag{35}$$

Thus, the resistance values computed above may be written

$$R_{11}=Z_0/\alpha_n l=4Q_nZ_0/m\pi, \qquad \text{if } m \text{ is odd}, \tag{36}$$

$$R_{11}=Z_0\alpha_n l=m\pi Z_0/4Q_n, \qquad \text{if } m \text{ is even}. \tag{37}$$

The normal acoustical line is a half-wavelength resonator for which m is even and R_{11} varies as $1/Q_n$. For a highly resonant specimen, R_{11} will therefore be very low. So-called antiresonances occur at quarter wavelength intervals when m is odd. R_{11} now varies directly as Q_n and will generally be high.

A similar treatment to the above may be carried out in terms of the transfer impedance of a short-circuited line. Expanding equation (21) and applying the same approximations as before leads to the following results :

$$R_{12}=0\,; \qquad X_{12}=Z_0, \qquad \text{for } m \text{ odd}, \tag{38}$$

$$R_{12}=Z_0\alpha_n l=m\pi Z_0/4Q_n\,; \quad X_{12}=0, \qquad \text{for } m \text{ even}. \tag{39}$$

In Table 2 the above conclusions are summarized.

It may be noted that for the $\frac{1}{2}\lambda$ line at resonance, the reactance is zero while the resistance has the same value both when driver and detector are at the same end of the specimen and when driver and detector are at opposite ends of the specimen. The data shown in Table 2 are in agreement with the normal mode theory. It is easy to show that the value of R_{11} in equation (37) is identical with the value of R_n in Table 1, that is, $R_{11}=m\pi Z_0/4Q_n=\omega_n M/2Q_n$ since $Z_0=\rho cS$,

$\rho = M/lA$, $c = \omega_n l/n\pi$, where the mode number $n = \frac{1}{2}m$, M is the total mass of the rod, l is the length, S is the area of cross section, and ω_n is the angular frequency at resonance. In addition, Mason (1948) has shown that the equivalent inductance, L_n, and capacitance, C_n, of a half-wavelength electrical line are given by

$$L_n = \tfrac{1}{2}Ll\ ; \qquad C_n = (2/\pi^2)Cl, \tag{40}$$

where L and C are the inductance and capacitance per unit length of line respectively and l is the length of the line. When the resonant frequency $\omega_n = (L_n C_n)^{-\frac{1}{2}}$ is introduced, equations (40) become identical with the first and third of equations (28). In addition, the geometric mean of the two values of R_{11} given by equations (36) and (37) is Z_0, in agreement with the corresponding result from normal mode theory.

TABLE 2

RESISTANCE AND REACTANCE OF RESONANT LINES

	m Odd ($\frac{1}{4}\lambda$ line)	m Even ($\frac{1}{2}\lambda$ line)
A. Driver and receiver at same end :		
R_{11}	$4Q_n Z_0/m\pi$	$m\pi Z_0/4Q_n$
X_{11}	0	0
B. Driver and receiver at opposite ends :		
R_{12}	0	$m\pi Z_0/4Q_n$
X_{12}	Z_0	0

It has been demonstrated that normal mode theory and transmission line theory produce the same results for systems whose lengths are multiples of $\frac{1}{4}\lambda$ or $\frac{1}{2}\lambda$. The transmission line theory is the more general, since it may be applied to find the response at any point along a line, whether resonant or not. In the acoustical case, where the lengths involved are usually multiples of $\frac{1}{2}\lambda$, a discussion in terms of the mode parameters is then sufficient.

VI. Pulse Response of a Finite Rod

When a stress pulse is applied to one end of a finite rod a series of pulses is developed by multiple reflection from the ends. If the duration of the initial pulse is less than the transit time in the specimen the series of pulses will be clearly separated. On the other hand, the separate pulses may overlap so that the response of the end faces of the rod may be represented by a series.

If the ends of the rod are taken at $x=0$ and $x=l$ and, for simplicity, the pulse, $f(x,t)$, is assumed to be in the form of a gated continuous wave so that

$$\begin{aligned} f(x,t) &= u_0 \exp i(\omega t - \mathbf{k}x), && \text{for } 0<t<\tau, \\ &= 0, && \text{elsewhere,} \end{aligned}$$

where u_0 represents the input particle velocity amplitude, then Pollard (1962) has shown that the total response at the input end of the rod is

$$\begin{aligned}\mathbf{u}_1 &= u_0 + 2u_0 \exp(-2\mathbf{ik}l) + 2u_0 \exp(-4\mathbf{ik}l) + \ldots \\ &= u_0 \coth \mathbf{ik}l,\end{aligned}$$

assuming that the separate pulse responses are in phase (corresponding to a resonant condition). If $\mathbf{f}_1$ is the force applied at $x=0$, then $\mathbf{Z}_{11}=\mathbf{f}_1/\mathbf{u}_1=\mathbf{Z}_0 \tanh \mathbf{ik}l$, which is identical to equation (18).

Similarly, the total response at the end face at $x=l$ may be found to be $\mathbf{Z}_{12}=\mathbf{Z}_0 \sinh \mathbf{ik}l$ which is identical to equation (21).

The above discussion indicates that, when the series of reflected pulses are additive, the pulse response becomes identical with the response under continuous wave excitation. A similar argument could be developed for an arbitrary pulse shape when it would be necessary to consider a superposition process involving all the Fourier components of the initial pulse.

VII. References

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