THE CALIBRATION OF LINEAR SCALES BY THE METHOD OF HANSEN-PERARD

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Summary

The exact least-squares solutions for X_j^0, X_i^0 and $\lambda_{(j-i)}$ of the set of equations

$$X_j^0 - X_i^0 - \lambda_{(j-i)} = b_{ji} \quad (i = 0, 1, \ldots, n-1;\; j = 1, \ldots, n;\; 0<(j-i)<n),$$

and for X_l^j, X_k^i and $\lambda_{(j-i)}$ of the set of equations

$$X_l^j - X_k^i - \lambda_{(j-i)} = b_{ji} \quad (j, i = 0, 1, \ldots, m;\; |i-j|<m),$$

are obtained by matrix inversion. These sets of equations correspond to the "simple calibration" and the "cross calibration", of linear scales by the method of Hansen-Pérard. Tables are given of the inverses of the coefficient matrices of the normal equations for most practical cases and expressions are derived for the variance to be associated with any interval between calibrated scale graduations.

I. Introduction

The classical methods of scale calibration originally formulated by Hansen (1873) and developed by Pérard (1917) and referred to here, after Pérard, as the methods of "simple calibration" and "cross calibration", have an extensive literature (Judson 1927; Platanov 1941). The rigorous least-squares solution in the linear case is complicated, however, and the simplified method of Thiesen (1879), known as "graphical squares" or the "method of means" (Johnson 1923) has generally been used.

Unfortunately, unless special measures are taken to ensure that the reduced equations are statistically independent, as has been pointed out by Cook (1954), the Thiesen estimates do not in general correspond to the most probable values. To ensure independence between the entries in the squares in the Thiesen solution it is necessary to repeat sufficient of the observations for the entries in the square to be obtained from distinct observations. This is in fact what Gilet and Watson (1953) have done in their treatment of the subject, and it is contended, in disagreement with Cook, that the Gilet and Watson treatment is exact for the observational procedure adopted. When there is independence between the observations the reduced equations are independent, the Thiesen estimates are the most probable values, and the derivation of the variances given by Gilet and Watson follow in the case of both simple and cross calibration. When circular scales are treated in the manner of Gilet and Watson, symmetry is lost and a distinction is made between the graduations in the scheme of observation. It is then no longer true

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that the variance associated with an angle is independent of its position relative to some zero graduation, i.e. the zero of the scale is not arbitrary as it is when the scheme described by Cook is followed. The Thiesen procedure is, of course, always valid when distinct entities are being compared.

There is an advantage in using the exact least-squares solution in the linear case as set out by Pérard (1917), if the complexity of the arithmetic can be avoided, in that the number of observations necessary is of the order of half that required for the exact Thiesen solution. A large part of the computation can be avoided if the inverse of the coefficients matrix of the normal equations is available.

Experimentally the Hansen method is inferior in one respect to the rigorous Thiesen treatment. In the Thiesen treatment it is possible to arrange that the reduced equations are derived from the differences between successive pairs of microscope readings. These microscope readings are taken at very short time intervals and the danger of systematic errors due to relative microscope displacement is reduced. In the Hansen procedure the stability of the microscope separation must be reliable for a much longer period. The comparators used in subdivisional work of this nature, however, have been designed with this in view and extraordinary stability has been achieved by the methods of beam construction and support. Further, the observational procedure is usually such that all the observations of a given set can be reduced to refer to some specific time during the set, and in this way compensation is made for any linear changes which occur while the observations are being made.

While it is possible to estimate the microscope separations from the least-squares solution it is not possible to estimate the variance attributable to instability separately from the observational variance. The latter variance, however, has a component due to instability, and any marked change from its usual magnitude should be examined closely.

With the advent of the automatic digital computing machine, the inversion of the coefficients matrix of the normal equations is a relatively simple matter and once the inverse has been obtained it can be used whenever required. Tables of the inverses for a number of calibrations of various sizes for both simple and cross calibrations are given in the Appendices. The rigorous least-squares solutions, together with the corresponding variances and covariances, can then be obtained using these tables and a desk calculator.

In the following sections a method of obtaining the least-squares solutions of the equations resulting from the simple and cross calibration of a scale by the Hansen-Pérard method is given and the variance of the estimate of the interval between any pair of scale graduations is derived.

II. Notation

As the use of symbols in the literature is not uniform, a nomenclature is introduced which, without being unduly complex, allows the various cases to be shown clearly. In later sections use is made of matrix notation as being the best means of expressing results which would otherwise be extremely cumbersome.

The matrix treatment of least squares in the general linear case will be found in a recent work by Plackett (1960).

General formulae have been given to cover most particular cases. While the analysis is given in terms of linear scales it will be found to be appropriate in a number of other fields of measurement.

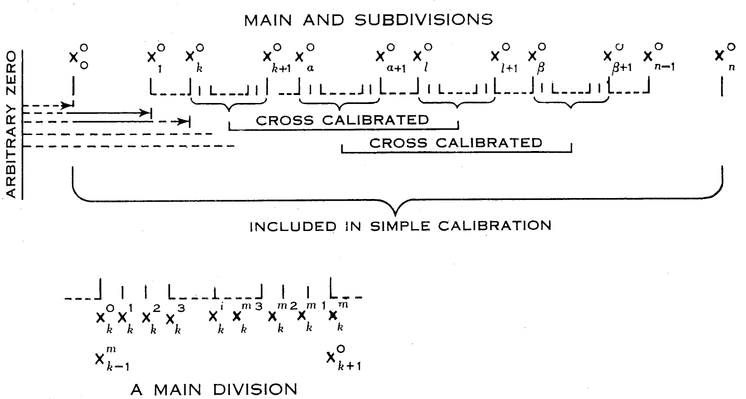

Fig. 1.—Illustration of notation.

The hierarchy of calibrations treated here is illustrated in Figure 1. Intervals are given in terms of differences from nominal length and positions are defined with respect to an arbitrary zero to the left of X_0^0. It is assumed that there are n main divisions with leading ends at X_0^0, X_1^0, X_2^0, . . . , X_k^0, . . ., X_{n-1}^0 respectively. The comparisons of the scale against a reference standard to determine the deviations of X_0^0 and X_n^0 from their nominal positions is the sole reference to an outside standard; this determination is assumed to introduce uncertainties measured by their variances, $\sigma^2_{X_0^0}$, $\sigma^2_{X_n^0}$, and their covariance $\sigma_{X_0^0X_n^0}$. Most frequently X_0^0 equals zero identically, i.e. the zero is taken at X_0^0; then $\sigma^2_{X_0^0}$ and $\sigma_{X_0^0X_n^0}$ are zero. The notation X_k^0 will be used to mean either the graduation or its deviation from nominal position; no confusion should result from this double use of symbols.

A simple calibration now serves to determine the deviations of the main graduations X_1^0, X_2^0, . . ., X_{n-1}^0 from their nominal positions.

Two cross calibrations are shown: the m subdivisions contained in $X_k^0\ X_{k+1}^0$ have been involved in a cross calibration with those of $X_l^0\ X_{l+1}$; the m subdivisions of $X_a^0\ X_{a+1}^0$, have been involved in a cross calibration with those of $X_\beta^0\ X_{\beta+1}^0$. By extension of the nomenclature, X_k^i is taken to mean the ith graduation of the main division $X_k^0\ X_{k+1}^0$. This leads to a small amount of redundancy in that the main graduations have two equivalent symbols; thus X_k^m, meaning the trailing end of the mth subdivision of $X_k^0\ X_{k+1}^0$, is also the leading end of a main division, in which context it is written equivalently as X_{k+1}^0.

It will be seen later that in allocating variances to intervals between graduations of the scale, cases fall into distinct groups which are now defined according to the graduations being used.

1. End-points falling upon main graduations:
   (*a*) End-points falling on main graduations other than X_0^0 and X_n^0.
   (*b*) End-points falling on either X_0^0 or X_n^0 and on one of the other main graduations.
2. End-points falling within main divisions involved in the same cross calibration:
   (*a*) Both end-points within the left-hand division of the cross calibration, typified by $X_k^i\ X_k^j$.
   (*b*) Both end-points within the right-hand division of the cross calibration, typified by $X_l^i\ X_l^j$.
   (*c*) One end-point within the left-hand division, and the other within the right-hand division, typified by $X_k^i\ X_l^j$.
3. End-points falling within main divisions involved in different cross calibrations:
   (*a*) Typical end-points: $X_k^i\ X_\alpha^j$.
   (*b*) Typical end-points: $X_l^i\ X_\beta^j$.
   (*c*) Typical end-points: $X_l^i\ X_\alpha^j$.
   (*d*) Typical end-points: $X_k^i\ X_\beta^j$.
4. One end-point falling upon a main graduation, the other end-point within a main division, typified by

$$\left.\begin{array}{l}(a)\ X_t^0\ X_k^i\\ (b)\ X_t^0\ X_l^i\end{array}\right\}\text{including the cases } t = k,\ l,\ 0, \text{ or } n.$$

In using a scale, a knowledge is required not only of the positions of the graduations with respect to some zero but also of the errors in the intervals from any one calibrated graduation to any other calibrated graduation. For example, the error in the interval X_p^0 to X_q^0 is obtained by subtracting the estimate of X_p^0 from that of X_q^0, and the variance to be associated with the length of this interval, as a measure of the uncertainty of the result, is given by

$$\text{Var}\ (X_p^0-X_q^0) = \text{Var}\ (X_p^0)+\text{Var}\ (X_q^0)-2\ \text{Cov}\ (X_p^0,\ X_q^0).$$

One of the advantages of the matrix method of solution of the normal equations is that it yields not only the variances of each of the estimated lengths but also the covariance required for determining the uncertainty of any interval not commencing at the zero. In this respect the following notation is introduced: $\text{Cov}\ (X_k^0,\ X_l^0) = \sigma_{X_k^0X_l^0}$ is the covariance of the quantities X_k^0 and X_l^0; it measures the correlation between these quantities arising jointly from the simple calibration and from the errors made in determining X_0^0 and X_n^0.

In particular, when $k = l$, $\text{Cov}\ (X_k^0,\ X_k^0)$ is $\text{Var}\ (X_k^0)$, the variance of X_k^0, which is also written $\sigma^2_{X_k^0}$.

$[\text{Cov}\ (X_k^0,\ X_l^0)]$ is the set of variances and covariances written as an $(n+1)\times(n+1)$ array as k, l separately run through the values $0, 1, \ldots, n$, and is referred to as the variance-covariance matrix of the X^0's.

When superscripts other than 0 occur, as in X_k^i, X_k^j (referring to the ith and the jth graduations of the m subdivisions of a main division) the array will be found by allowing i, j to take separately the values 1, 2, . . ., m in succession, as in

$$[\mathrm{Cov}\,(X_k^i,\, X_k^j)].$$

Simple and cross calibrations are considered in the next two sections. It will be shown later how estimates of the variance-covariance matrices are obtained.

III. Simple Calibration

The set of observational equations is

$$\begin{aligned} X_j^0-X_i^0-\lambda_{(j-i)} = b_{ji}, \quad & i = 0, 1, \ldots, n-1; \\ & j = 1, \ldots, n; \\ & 0 < (j-i) < n, \end{aligned}$$

where $\lambda_{(j-i)}$ is the deviation of the microscope separation from the nominal separation and b_{ji} is the observed quantity. There are then $\frac{1}{2}(n-1)(n+2)$ equations involving $2(n-1)$ unknowns of which $(n-1)$ are λ's; of the $(n+1)$ X's, X_0^0 and X_n^0 are known, leaving $(n-1)$ unknown X's.

The observational equations are written in vector form as

$$\mathbf{AX} = \mathbf{a},$$

where

$$\mathbf{A} = \left[\begin{array}{cccccccccccccccc}
-1 & 0 & 0 & 0 & \ldots & 0 & \quad & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & & -1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & & 0 & -1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots} & & \multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
-1 & 0 & 0 & 0 & \ldots & 0 & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 1 \\
-1 & 0 & 0 & 0 & \ldots & 0 & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & \ldots & 0 & & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & \ldots & 0 & & -1 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & \ldots & 0 & & 0 & -1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots} & & \multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & -1 & 0 & 0 & \ldots & 0 & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 1 \\
0 & -1 & 0 & 0 & \ldots & 0 & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & & 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & & -1 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & & 0 & -1 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots} & & \multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & 0 & -1 & 0 & \ldots & 0 & & 0 & 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & \ldots & 0 & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & \ldots & 0 & & 0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \ldots & 0 & & -1 & 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots} & & \multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots} & & \multicolumn{10}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & 0 & 0 & 0 & \ldots & -1 & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -1 & & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{array}\right],$$

$$\mathbf{X} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \lambda_{n-1} \\ X_1^0 \\ \cdot \\ \cdot \\ X_{n-1}^0 \end{bmatrix}, \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} b_{10}+X_0^0 \\ b_{21} \\ \cdot \\ \cdot \\ b_{n(n-1)}-X_n^0 \\ b_{20}+X_0^0 \\ b_{31} \\ \cdot \\ \cdot \\ b_{n(n-2)}-X_n^0 \\ \cdot \\ \cdot \\ b_{(n-1)0}+X_0^0 \\ b_{n1}-X_n^0 \end{bmatrix}.$$

The general element of the vector $\mathbf{a}$ is

$$b_{ji}+\delta_{i0}X_0^0-\delta_{jn}X_n^0,$$

where

$$\begin{aligned} \delta_{ab} &= 0, \quad a \neq b, \\ &= 1, \quad a = b. \end{aligned}$$

It will be seen that the elements of $\mathbf{a}$ are the observations b_{ji} corrected for the value of X_0^0 and X_n^0 whenever they occur in the observational equations, i.e. these two values are brought to the right-hand side of the equations.

Least-squares procedure results in the normal equations

$$\mathbf{A}^T\mathbf{A}\hat{\mathbf{X}}-\mathbf{A}^T\mathbf{a} = 0;$$

the solution is written $\hat{\mathbf{X}} = (\mathbf{A}^T\mathbf{A})^{-1}\,\mathbf{A}^T\mathbf{a}$.

The residual sum of squares is $\mathbf{a}^T\mathbf{a}-\hat{\mathbf{X}}^T\mathbf{A}^T\mathbf{a}$, from which, on division by $\frac{1}{2}(n-1)(n-2)$, one obtains the estimate s^2 of the observational variance. The estimated variance-covariance matrix for the main divisions, not including X_0^0 or X_n^0 and ignoring any contribution from the uncertainties of these two quantities, is

$$s^2(\mathbf{A}^T\mathbf{A})^{-1}.$$

The quantities are all readily computed. The inverse matrix $(\mathbf{A}^T\mathbf{A})^{-1}$ has been tabulated for $n = 4, 5, 6, 8, 10, 12$ in Appendix I. The vector $\mathbf{A}^T\mathbf{a}$ is obtained directly as the product of the transpose of the observational matrix $\mathbf{A}$ and the column vector of the observed deviations b_{ji} corrected for X_0^0 and X_n^0. The vector $\hat{\mathbf{X}}$ is then the product of $(\mathbf{A}^T\mathbf{A})^{-1}$ with the column vector $\mathbf{A}^T\mathbf{a}$. The scalar product $\mathbf{a}^T\mathbf{a}$ is the sum of squares of the $\frac{1}{2}(n-1)(n+2)$ observed quantities corrected for X_0^0, X_n^0, as described; if $\hat{\mathbf{X}}^T\mathbf{A}^T\mathbf{a}$ is subtracted from it the residual sum of squares is the remainder.

Example: For illustration, the procedure will be applied to the data of Table 1 from a simple calibration involving four main divisions. The quantities given under the heading "Comparison" are the subscripts of main divisions wherein, for example, 0/1 indicates the interval X_0^0 to X_1^0; the values listed under b are the observed quantities with signs in accordance with the observational equations already given.

TABLE 1

SIMPLE CALIBRATION: $n = 4$

Unit: 10^{-6} in.

Comparison	b	Comparison	b	Comparison	b
0/1	−37	0/2	0	0/3	+24
1/2	−14	1/3	0	1/4	−16
2/3	−33	3/4	−75		
3/4	−91				

$X_0^0 = 0$; $X_n^0 = 221$: $s^2_{X_0^0} = 0$, $s^2_{X_n^0} = 117 \times 10^{-12}$ in^2, $s_{X_0^0 X_n^0} = 0$

For simplicity $\mathbf{A}$, not $\mathbf{A}^T$, is written followed by the column vector $\mathbf{a}$, whence $\mathbf{A}^T\mathbf{a}$ is computed by taking cross products of the columns of $\mathbf{A}$ in succession with the vector $\mathbf{a}$. Thus $\mathbf{A}$, $\mathbf{a}$ are written

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -37 & \\ -14 & \\ -33 & \\ -91 & -221 \\ 0 & \\ 0 & \\ -75 & -221 \\ 24 & \\ -16 & -221 \end{bmatrix},$$

so that

$$\mathbf{A}^T\mathbf{a} = \begin{bmatrix} 37+\ 14+\ 33+312 \\ 0+\ 0+296 \\ -24+237 \\ -37+\ 14+237 \\ -14+\ 33+296 \\ -33+312+\ 24 \end{bmatrix} = \begin{bmatrix} 396 \\ 296 \\ 213 \\ 214 \\ 315 \\ 303 \end{bmatrix},$$

$$\mathbf{a}^T\mathbf{a} = 37^2+14^2+33^2+312^2+ \ldots +237^2 = 244359.$$

The inverse matrix $(\mathbf{A}^T\mathbf{A})^{-1}$ is symmetrical, so $\hat{\mathbf{X}} = (\mathbf{A}^T\mathbf{A})^{-1}\,\mathbf{A}^T\mathbf{a}$ may be interpreted as row × column or column × column multiplication, and for ease of computation the latter is chosen, i.e. multiply $\mathbf{A}^T\mathbf{a}$ in succession by the columns of $(\mathbf{A}^T\mathbf{A})^{-1}$, taken from Table 1·1, Appendix I. Thus

$$\hat{\mathbf{X}} = (1/20)\underbrace{\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 2 & -2 & 0 & 2 \\ 0 & 2 & 13 & -3 & 0 & 3 \\ 0 & -2 & -3 & 7 & 2 & 1 \\ 0 & 0 & 0 & 2 & 6 & 2 \\ 0 & 2 & 3 & 1 & 2 & 7 \end{bmatrix}}_{(\mathbf{A}^T\mathbf{A})^{-1}} \underbrace{\begin{bmatrix} 396 \\ 296 \\ 213 \\ 214 \\ 315 \\ 303 \end{bmatrix}}_{\mathbf{A}^T\mathbf{a}} = (1/20)\begin{bmatrix} 1980 \\ 2972 \\ 3628 \\ 1200 \\ 2924 \\ 4196 \end{bmatrix} = \begin{bmatrix} 99\cdot0 \\ 148\cdot6 \\ 181\cdot4 \\ 60\cdot0 \\ 146\cdot2 \\ 209\cdot8 \end{bmatrix}.$$

The vector $\hat{\mathbf{X}}^T\mathbf{A}^T\mathbf{a}$ is conveniently computed by writing $\hat{\mathbf{X}}$ and $\mathbf{A}^T\mathbf{a}$ side by side and summing the products of the pairs of numbers on the one line:

$$(1/20)\begin{bmatrix} 1980 \\ 2972 \\ 3628 \\ 1200 \\ 2924 \\ 4196 \end{bmatrix}\begin{bmatrix} 396 \\ 296 \\ 213 \\ 214 \\ 315 \\ 303 \end{bmatrix} = (1/20)\{(1980\times396)+(2972\times296)+\ldots+(4196\times303)\}$$

$$= 244290.$$

Hence $s^2 = (244359-244290)/3 = 23$.

The only portion of the variance-covariance matrix of interest is the 3×3 submatrix in the lower right-hand corner of $s^2(\mathbf{A}^T\mathbf{A})^{-1}$, which is

$$(23/20)\begin{bmatrix} 7 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 8 \end{bmatrix}.$$

We now have to take account of the variances and covariances of the determinations of X_0^0 and X_n^0 listed at the bottom of Table 1, and of the covariances between these two quantities and X_1^0, X_2^0, X_3^0; this corresponds to 1(*b*) of the groupings mentioned in Section II.

The solution for X_k^0 may be written* alternatively as

$$\hat{X}_k^0 = \hat{x}_k^0+(k/n)(X_n^0-X_0^0),$$

where $\hat{\mathbf{x}}^0 = (\mathbf{A}^T\mathbf{A})^{-1}\,\mathbf{A}^T\mathbf{b}$.

* To avoid complexity in the notation, the circumflex, ^, indicating "estimate", has been omitted where confusion is unlikely to arise.

Hence for k or l having the values $1, 2, \ldots, n-1$,

$$\begin{aligned}[\mathrm{Cov}\,(X_k^0, X_l^0)] &= [\mathrm{Cov}\,(x_k^0, x_l^0)]+[(kl/n^2)\sigma^2_{(X_n^0-X_0^0)}] \\ &= \sigma^2(\mathbf{A}^T\mathbf{A})^{-1}+[(kl/n^2)\sigma^2_{(X_n^0-X_0^0)}] \\ &= \sigma^2(\mathbf{A}^T\mathbf{A})^{-1}+[(kl/n^2)\sigma^2_{X_n^0}],\end{aligned}$$

when X_0^0 is identically zero.

By similar reasoning, for one of the components equal to X_0^0 or X_n^0

$$\mathrm{Cov}\,(X_0^0, X_k^0) = (k/n)\,\mathrm{Cov}\,(X_0^0, X_n^0).$$

When X_0^0 is identically zero,

$$\mathrm{Cov}\,(X_0^0, X_k^0) = 0,$$

and

$$\mathrm{Cov}\,(X_k^0, X_n^0) = (k/n)\sigma^2_{X_n^0}.$$

In the example, then, the elements of the 3×3 variance-covariance matrix are augmented by allocation of $s^2_{X_n^0} = 117$ in the proportion $(kl/16)$ and the addition of a row and column corresponding to $(k/n)s^2_{X_n^0}$, giving

$$117\begin{bmatrix} 1/16 & 2/16 & 3/16 & 1/4 \\ 2/16 & 4/16 & 6/16 & 2/4 \\ 3/16 & 6/16 & 9/16 & 3/4 \\ 1/4 & 2/4 & 3/4 & 1 \end{bmatrix},$$

leading finally to

$$[\mathrm{Cov}\,(X_k^0, X_l^0)] = \begin{bmatrix} 15 & 17 & 23 & 29 \\ 17 & 36 & 46 & 58 \\ 23 & 46 & 74 & 88 \\ 29 & 58 & 88 & 117 \end{bmatrix},$$

where the unit is 10^{-12} in^2.

The variance of any specified interval may now be obtained immediately using the propagation of variance formula given in Section II.

IV. Cross Calibration

Apart from mechanical limitations, a cross calibration may be made between the subdivisions of any pair of main divisions of a scale. Several such cross calibrations could be made and the analysis would follow the same pattern in each case.

In a single cross calibration involving m subdivisions, (m^2+2m-1) observations are taken and a total of $(4m-3)$ constants have to be fitted; $(2m-1)$ of these refer to the microscope separations and $2(m-1)$ to the subdivisional graduations.

From a cross calibration between the intervals $X_k^0\,X_k^m$ and $X_l^0\,X_l^m$ the following set of (m^2+2m-1) observational equations with $(4m-3)$ unknowns is obtained:

$$X_l^j - X_k^i - \lambda_{(j-i)} = b_{ji},\quad j = 0, 1, \ldots, m;\quad i = 0, 1, \ldots, m;\ |i-j|<m.$$

Following the notation used before, we write

$$\mathbf{AX} = \mathbf{a},$$

where $\mathbf{A}$ is the coefficients matrix of the observational equations,

$$\mathbf{X} = \begin{bmatrix} \lambda_{-(m-1)} \\ \lambda_{-(m-2)} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_0 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{(m-1)} \\ X_k^1 \\ \cdot \\ \cdot \\ \cdot \\ X_k^{m-1} \\ X_l^1 \\ \cdot \\ \cdot \\ \cdot \\ X_l^{m-1} \end{bmatrix} \text{ and } \mathbf{a} = \begin{bmatrix} b_{0(m-1)} - X_l^0 \\ b_{1m} + X_k^m \\ b_{0(m-2)} - X_l^0 \\ b_{1(m-1)} \\ b_{2m} + X_k^m \\ b_{0(m-3)} - X_l^0 \\ \cdot \\ \cdot \\ \cdot \\ b_{00} + X_k^0 - X_l^0 \\ \cdot \\ \cdot \\ \cdot \\ b_{mm} + X_k^m - X_l^m \\ b_{10} + X_k^0 \\ \cdot \\ \cdot \\ \cdot \\ b_{m1} - X_l^m \end{bmatrix},$$

where the vector $\mathbf{b}$ has been modified to $\mathbf{a}$ by transferring X_k^0, X_k^m, X_l^0, X_l^m, whenever they occur, to the right-hand side of the observational equations.

The general element of the vector $\mathbf{a}$ is

$$b_{ji} - \delta_{0j} X_l^0 + \delta_{mi} X_k^m + \delta_{0i} X_k^0 - \delta_{mj} X_l^m,$$

where

$$\delta_{ab} = 0, \quad a \neq b,$$
$$= 1, \quad a = b.$$

The solution for $\mathbf{X}$ is then

$$\hat{\mathbf{X}} = (\mathbf{A}^T\mathbf{A})^{-1}\, \mathbf{A}^T\mathbf{a}.$$

Example: This example is a continuation of the example treated in the case of simple calibration. The main divisions involved are $X_0^0\ X_1^0$ and $X_2^0\ X_3^0$ of that example. In Table 2 the observed quantities b_{ji} are given.

The observational matrix $\mathbf{A}$ and the vector $\mathbf{a}$ are tabulated in Table 3.

The calculation proceeds as for the simple case. The vector $\mathbf{A}^T\mathbf{a}$ is obtained by forming the product of $\mathbf{A}$, column by column, with the column vector $\mathbf{a}$; for convenience, this vector is written transposed at the foot of $\mathbf{A}$ in Table 3. It is worth while to run a check on $\mathbf{A}^T\mathbf{a}$ by adding all its elements and checking this sum against the product of $\mathbf{a}$ and the vector which is the row sum of $\mathbf{A}$. The $(4m-3)\times(4m-3)$ inverse matrix $(\mathbf{A}^T\mathbf{A})^{-1}$ is given in Table 2.2, Appendix II; its product with $\mathbf{A}^T\mathbf{a}$ is now obtained and yields the solution $\hat{\mathbf{X}} = (\mathbf{A}^T\mathbf{A})^{-1}\,\mathbf{A}^T\mathbf{a}$ of the normal equations. Unfortunately, all the $(4m-3)$ elements of $\hat{\mathbf{X}}$ are required in computing the residual error, although the first $(2m-1)$, which refer to the arbitrary microscope separations, are not otherwise needed. The quantities of direct interest are the last $2(m-1)$ elements. The first $(m-1)$ of these correspond to the deviations from nominal length of the subdivisions of $X_k^0\,X_{k+1}^0$ and the second $(m-1)$ to the deviations from nominal length of the subdivisions of $X_l^0\,X_{l+1}^0$.

TABLE 2

CROSS CALIBRATION: $m = 5$

b_{ji}; $i = 0, \ldots, 5$; $j = 0, \ldots, 5$

Unit: 10^{-6} in.

j \ i	0	1	2	3	4	5
0	−12	+55	0	0	0	—
1	0	−59	−31	−8	−39	−31
2	−16	39	−79	8	16	−16
3	−24	8	−39	−31	24	24
4	0	31	−16	43	−12	47
5	—	8	−39	12	43	−20

$X_0^0 = 0$; $X_1^0 = 60$; $X_2^0 = 146$; $X_3^0 = 210$

The sum of squares, $\mathbf{a}^T\mathbf{a}$, of the elements of $\mathbf{a}$, is now computed and then the product of the vectors $\hat{\mathbf{X}}$, $\mathbf{A}^T\mathbf{a}$; as before, $\mathbf{a}^T\mathbf{a}-\hat{\mathbf{X}}^T\mathbf{A}^T\mathbf{a}$ is the residual sum of squares, based on (m^2-2m+2) degrees of freedom, the divisor for the calculation of the residual variance.

We have then in the present example

$$\hat{\mathbf{X}} = (\mathbf{A}^T\mathbf{A})^{-1}\,(117,\ 141,\ 54,\ \ldots,\ 153)^T,$$

where $(\mathbf{A}^T\mathbf{A})^{-1}$ is given in Table 2.2, Appendix II; we find

$$\begin{aligned}\hat{\mathbf{X}}^T = [&92\cdot1,\ 105\cdot4,\ 81\cdot7,\ 92\cdot8,\ 165\cdot0,\ 121\cdot0,\\ &158\cdot9,\ 182\cdot5,\ 205\cdot0,\ -3\cdot1,\ 65\cdot1,\ 40\cdot0,\\ &49\cdot8,\ 117\cdot0,\ 148\cdot1,\ 162\cdot6,\ 205\cdot0],\end{aligned}$$

and

$$\hat{\mathbf{X}}^T\mathbf{A}^T\mathbf{a} = 337416.$$

Thus

$$\mathbf{a}^T\mathbf{a}-\hat{\mathbf{X}}^T\mathbf{A}^T\mathbf{a} = 338546-337416 = 1130,$$

and

$$s^2 = 1130/17 = 66\cdot5.$$

TABLE 3

OBSERVATIONAL MATRIX FOR $m = 5$

A: $\lambda_{(j-i)}$, $(j-i)$									A: X_k^i, i				A: X_l^j, j				**a** Unit: 10^{-6} in.	Check
−4	−3	−2	−1	0	1	2	3	4	1	2	3	4	1	2	3	4		
−1												−1					−146 = (0−146)	−2
−1													1				29 = (−31+60)	0
	−1										−1						−146 = (0−146)	−2
	−1											−1	1				−39	−1
	−1													1			44 = (−16+60)	0
		−1								−1							−146 = (0−146)	−2
		−1									−1		1				−8	−1
		−1										−1		1			16	−1
		−1													1		84 = (24+60)	0
			−1						−1								−91 = (55−146)	−2
			−1							−1			1				−31	−1
			−1								−1			1			8	−1
			−1									−1			1		24	−1
			−1													1	107 = (47+60)	0
				−1													−158 = (−12−146+0)	−1
				−1					−1				1				−59	−1
				−1						−1				1			−79	−1
				−1							−1				1		−31	−1
				−1								−1				1	−12	−1
				−1													−170 = (−20−210+60)	−1
					−1								1				0 = (0+0)	0
					−1				−1					1			39	−1
					−1					−1					1		−39	−1
					−1						−1					1	43	−1
					−1							−1					−167 = (43−210)	−2
						−1								1			−16 = (−16+0)	0
						−1			−1						1		8	−1
						−1				−1						1	−16	−1
						−1					−1						−198 = (12−210)	−2
							−1								1		−24 = (−24+0)	0
							−1		−1							1	31	−1
							−1			−1							−249 = (−39−210)	−2
								−1								1	0 = (0+0)	0
								−1	−1								−202 = (8−210)	−2

$\mathbf{a}^T\mathbf{A} = [+117, +141, +54, -17, +509, +124, +222, +242, +202, +274, +560, +332, +324, -108, +12, +22, +153]$. Check sum $+3163$

$\mathbf{a}^T\mathbf{a} = 146^2+29^2+146^2+ \dots +202^2 = 338546$

As in the case of simple calibration, the solutions for the X's may be written in an alternative form.

Thus

$$\hat{X}_k^i = \hat{x}_k^i + \mathbf{X}_{kl}^T \mathbf{K}_i,$$
$$\hat{X}_l^j = \hat{x}_i^j + \mathbf{X}_{kl}^T \mathbf{L}_j,$$

where

$$\mathbf{K}_i = [3/4 - i/2m,\ 1/4 + i/2m,\ 1/4 - i/2m,\ -1/4 + i/2m]^T,$$
$$\mathbf{L}_j = [1/4 - j/2m,\ -1/4 + j/2m,\ 3/4 - j/2m,\ 1/4 + j/2m]^T,$$
$$\mathbf{X}_{kl}^T = [X_k^0,\ X_{k+1}^0,\ X_l^0,\ X_{l+1}^0],$$

and

$$\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\ \mathbf{A}^T\mathbf{b}.$$

We note that $\mathbf{X}_{kl}\mathbf{X}_{kl}^T$ is a 4×4 matrix and that the expectation of $(\mathbf{X}_{kl} - \mathbf{X}_{kl})(\mathbf{X}_{kl} - \overline{\mathbf{X}}_{kl})^T$, where

$$\overline{\mathbf{X}}_{kl}^T = [\overline{X}_k^0,\ \overline{X}_{k+1}^0,\ \overline{X}_l^0,\ \overline{X}_{l+1}^0]$$

is the variance-covariance matrix* of the vector $\mathbf{X}_{kl}$. For brevity the following notation is adopted

$$\mathbf{\Sigma}_{kl} = \xi\{(\mathbf{X}_{kl} - \overline{\mathbf{X}}_{kl})(\mathbf{X}_{kl} - \overline{\mathbf{X}}_{kl})^T\}$$
$$= \begin{bmatrix} \sigma^2_{X_k^0} & \sigma_{X_k^0 X_{k+1}^0} & \sigma_{X_k^0 X_l^0} & \sigma_{X_k^0 X_{l+1}^0} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sigma^2_{X_{l+1}^0} \end{bmatrix},$$

$$\mathbf{K} = [\mathbf{K}_1,\ \mathbf{K}_2,\ \ldots,\ \mathbf{K}_{m-1}],$$

the matrix whose column vectors are $\mathbf{K}_i$,

$$\mathbf{L} = [\mathbf{L}_1,\ \mathbf{L}_2,\ \ldots,\ \mathbf{L}_{m-1}],$$

the matrix whose column vectors are $\mathbf{L}_j$.

For a second cross calibration involving $X_\alpha^0\ X_{\alpha+1}^0,\ X_\beta^0\ X_{\beta+1}^0$ the matrices $\mathbf{K}$ and $\mathbf{L}$ will appear again but the vector $\mathbf{X}_{\alpha\beta}$ will replace $\mathbf{X}_{kl}$ and $\mathbf{\Sigma}_{\alpha\beta}$ will replace $\mathbf{\Sigma}_{kl}$.

To establish the variances and covariances a number of separate cases must be distinguished corresponding to those enumerated under 2, 3, and 4 of Section II. The derivations of the relevant formulae in each of these cases follow essentially the same reasoning and so those corresponding to case 2(a) will be derived, the remainder merely being listed. Case 2(a) then corresponds to end-points falling within main divisions involved in the same cross calibrations and also falling within the left-hand main division.

* See footnote, p. 89.

We then have,

$$\begin{aligned}\operatorname{Cov}(X_k^i, X_k^j) &= \xi\{(X_k^i-\overline{X}_k^i)(X_k^j-\overline{X}_k^j)\}\\ &= \xi\{(\{x_k^i+\mathbf{X}_{kl}^T\mathbf{K}_i\}-\{\bar{x}_k^i+\overline{\mathbf{X}}_{kl}^T\mathbf{K}_i\})(\{x_k^j+\mathbf{X}_{kl}^T\mathbf{K}_j\}-\{\bar{x}_k^j+\overline{\mathbf{X}}_{kl}^T\mathbf{K}_j\})\},\end{aligned}$$

leading to $\xi(\{x_k^i-\bar{x}_k^i\}\{x_k^j-\bar{x}_k^j\})+\xi(\mathbf{K}_i^T(\mathbf{X}_{kl}-\overline{\mathbf{X}}_{kl})(\mathbf{X}_{kl}-\overline{\mathbf{X}}_{kl})^T\mathbf{K}_j)$
since the x_k's and the $\mathbf{X}_{kl}$'s are independent.
Hence finally

$$\operatorname{Cov}(X_k^i, X_k^j) = \operatorname{Cov}(x_k^i, x_k^j)+\mathbf{K}_i^T\mathbf{\Sigma}_{kl}\mathbf{K}_j, \quad i, j = 1, 2, \ldots, m-1;$$

more concisely

$$[\operatorname{Cov}(X_k^i, X_k^j)] = [\operatorname{Cov}(x_k^i, x_k^j)]+\mathbf{K}^T\mathbf{\Sigma}_{kl}\mathbf{K},$$

where $[\operatorname{Cov}(x_k^i, x_k^j)]$ is obtained from the inverse $(\mathbf{A}^T\mathbf{A})^{-1}$ on multiplication by the scalar σ^2. (In practice the estimate of σ^2, s^2, is used.)

A complete list is now given of covariances associated with a cross calibration.* In this list the only restriction on the superscripts i and j is that they should not equal 0 or m; j may equal i.

1. Elements within the same cross calibrations:
   (*a*) A length falling entirely within $X_k^0\, X_{k+1}^0$,
   $$[\operatorname{Cov}(X_k^i, X_k^j)] = [\operatorname{Cov}(x_k^i, x_k^j)]+\mathbf{K}^T\,\mathbf{\Sigma}_{kl}\,\mathbf{K}.$$
   (*b*) A length falling entirely within $X_l^0\, X_{l+1}^0$,
   $$[\operatorname{Cov}(X_l^i, X_l^j)] = [\operatorname{Cov}(x_l^i, x_l^j)]+\mathbf{L}^T\,\mathbf{\Sigma}_{kl}\,\mathbf{L}.$$
   (*c*) A length extending from a graduation in $X_k^0\, X_{k+1}^0$ to a graduation in $X_l^0\, X_{l+1}^0$,
   $$[\operatorname{Cov}(X_k^i, X_l^j)] = [\operatorname{Cov}(x_k^i, x_l^j)]+\mathbf{K}^T\,\mathbf{\Sigma}_{kl}\,\mathbf{L}.$$
2. Elements within two different cross calibrations:
   (*a*) A length extending from a graduation in the left-hand main division of a first cross calibration to a graduation in the left-hand main division of a second cross calibration. For a graduation in $X_k^0\, X_{k+1}^0$ to a graduation in $X_\alpha^0\, X_{\alpha+1}^0$
   $$[\operatorname{Cov}(X_k^i, X_\alpha^j)] = \mathbf{K}^T\,[\operatorname{Cov}(X_{kl}, X_{\alpha\beta})]\mathbf{K},$$
   where the expression on the right-hand side is a matrix of bilinear forms each involving the 4×4 matrix

$$[\operatorname{Cov}(X_{kl}, X_{\alpha\beta})] = \begin{bmatrix} \sigma_{X_k^0X_\alpha^0} & \sigma_{X_k^0X_{\alpha+1}^0} & \sigma_{X_k^0X_\beta^0} & \sigma_{X_k^0X_{\beta+1}^0}\\ \sigma_{X_{k+1}^0X_\alpha^0} & \sigma_{X_{k+1}^0X_{\alpha+1}^0} & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \sigma_{X_{l+1}^0X_{\beta+1}^0}\end{bmatrix}.$$

* The cases here numbered 1–3 are cases 2–4 of Section II. (Case 1 of Section II involved only simple calibrations.)

(*b*) A length extending from a graduation in the right-hand main division of a first cross calibration to a graduation in the right-hand main division of a second cross calibration. For a graduation in $X_l^0\, X_{l+1}^0$ to a graduation in $X_\beta^0\, X_{\beta+1}^0$,

$$[\mathrm{Cov}\ (X_l^i,\ X_\beta^j)] = \mathbf{L}^T\ [\mathrm{Cov}\ X_{kl},\ X_{\alpha\beta})]\mathbf{L}.$$

(*c*) A length extending from a graduation in the right-hand main division of a first cross calibration to a graduation in the left-hand main division of a second cross calibration. For a graduation in $X_l^0\, X_{l+1}^0$ to a graduation in $X_\alpha^0\, X_{\alpha+1}^0$,

$$[\mathrm{Cov}\ (X_l^i,\ X_\alpha^j)] = \mathbf{L}^T\ [\mathrm{Cov}\ (X_{kl},\ X_{\alpha\beta})]\mathbf{K}.$$

(*d*) A length extending from a graduation in the left-hand main division of a first cross calibration to a graduation in the right-hand main division of a second cross calibration. For a graduation in $X_k^0\, X_{k+1}^0$ to a graduation in $X_\beta^0\, X_{\beta+1}^0$,

$$[\mathrm{Cov}\ (X_k^i,\ X_\beta^j)] = \mathbf{K}^T\ [\mathrm{Cov}\ (X_{kl},\ X_{\alpha\beta})]\mathbf{L}.$$

3. Main graduation to subdivisional graduation:

(*a*) A length extending from a main graduation to a left-hand subdivisional graduation of a cross calibration,

$$\mathrm{Cov}\ (X_t^0,\ X_k^i) = [\sigma_{X_t^0X_k^0},\ \sigma_{X_t^0X_{k+1}^0},\ \sigma_{X_t^0X_l^0},\ \sigma_{X_t^0X_{l+1}^0}]\mathbf{K}_i,$$

where t may assume any of the integral values 0 to n.

(*b*) A length extending from a main graduation to a right-hand subdivisional graduation of a cross calibration,

$$\mathrm{Cov}\ (X_t^0,\ X_l^j) = [\sigma_{X_t^0X_k^0},\ \sigma_{X_t^0X_{k+1}^0},\ \sigma_{X_t^0X_l^0},\ \sigma_{X_t^0X_{l+1}^0}]\mathbf{L}_j,$$

where again t may assume any of the integral values 0 to n.

It is not suggested that it would be profitable to calculate the entire variance-covariance matrix in every case but rather that the values be calculated as they are required. As an example of these calculations the values corresponding to $[\mathrm{Cov}\ (X_0^i,\ X_2^j)]$ in the previous example will be calculated (case 1(*c*) in the above list).

The elements corresponding to $[\mathrm{Cov}\ (x_0^i,\ x_2^j)]$ are obtained from the appropriate elements of the inverse $(\mathbf{A}^T\mathbf{A})^{-1}$ for the cross calibration on multiplication by $s^2 = 66\cdot 5$, i.e.

$$(66\cdot 5/2160)\begin{bmatrix} 4 & -37 & -73 & -104 \\ -37 & -74 & -101 & -73 \\ -73 & -101 & -74 & -37 \\ -104 & -73 & -37 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ -1 & -2 & -3 & -2 \\ -2 & -3 & -2 & -1 \\ -3 & -2 & -1 & 0 \end{bmatrix}$$

on rounding.

To this 4×4 matrix is added the matrix

$$\mathbf{K}^T \mathbf{\Sigma}_{kl} \mathbf{L}.$$

Now

$$\mathbf{K}^T = \begin{bmatrix} 0\cdot65 & 0\cdot35 & 0\cdot15 & -0\cdot15 \\ 0\cdot55 & 0\cdot45 & 0\cdot05 & -0\cdot05 \\ 0\cdot45 & 0\cdot55 & -0\cdot05 & 0\cdot05 \\ 0\cdot35 & 0\cdot65 & -0\cdot15 & 0\cdot15 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} 0\cdot15 & 0\cdot05 & -0\cdot05 & -0\cdot15 \\ -0\cdot15 & -0\cdot05 & 0\cdot05 & 0\cdot15 \\ 0\cdot65 & 0\cdot55 & 0\cdot45 & 0\cdot35 \\ 0\cdot35 & 0\cdot45 & 0\cdot55 & 0\cdot65 \end{bmatrix},$$

and from the simple calibration

$$\mathbf{\Sigma}_{kl} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 15 & 17 & 23 \\ 0 & 17 & 36 & 46 \\ 0 & 23 & 46 & 74 \end{bmatrix}.$$

Multiplication then gives on rounding,

$$\mathbf{K}^T \mathbf{\Sigma}_{kl} \mathbf{L} = \begin{bmatrix} 3 & 4 & 4 & 5 \\ 7 & 8 & 8 & 9 \\ 10 & 11 & 13 & 14 \\ 13 & 15 & 17 & 19 \end{bmatrix},$$

leading finally to

$$[\mathrm{Cov}\,(X_0^i, X_2^j)] = \begin{array}{cc} & (j) \\ (i) & \begin{bmatrix} 3 & 3 & 2 & 2 \\ 6 & 6 & 5 & 7 \\ 8 & 8 & 11 & 13 \\ 10 & 13 & 16 & 19 \end{bmatrix} \end{array},$$

where the unit is 10^{-12} in^2.

V. Conclusion

The analysis given here is applicable not only to the treatment of the subdivision of linear scales but also to any series of observations which conforms to the pattern and has the same dependencies. For the simple calibration scheme of circular scales or polygons the analysis of Cook (1954) is appropriate. When, however, it is necessary to use a cross calibration in the circular case the analysis given here should be applied.

VI. Acknowledgments

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Appendices

$(\mathbf{A}^T\mathbf{A})^{-1}$

Inverse of the Matrix of the Normal Equations

Elements of the inverses are given as integral multiples of their actual values; the appropriate divisor d is given with each table. The row "column sums" has been added to facilitate a check on the arithmetic.

For the purpose of reproduction it has been necessary to partition some of the larger matrices. Where the partitioning has been made into two submatrices the division has been made so that one matrix, $\mathbf{\Lambda}$, is associated with the λ's and the other, $\mathbf{X}$, with the X's. In the case of the 37×37 matrix it has been necessary to partition into four submatrices. In this case, $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are associated with the λ's and $\mathbf{X}_1$ and $\mathbf{X}_2$ with the X's.

Appendix I

Simple Calibration, Tables 1.1 *to* 1.6

The number of main divisions is n; the number of λ's is $(n-1)$ and the number of X's is $(n-1)$.

Table 1.1

SIMPLE CALIBRATION: ORDER 6×6

$n = 4$

$d = 20$

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 2 & -2 & 0 & 2 \\ 0 & 2 & 13 & -3 & 0 & 3 \\ 0 & -2 & -3 & 7 & 2 & 1 \\ 0 & 0 & 0 & 2 & 6 & 2 \\ 0 & 2 & 3 & 1 & 2 & 7 \end{bmatrix}$$

Column Sums

5	10	15	5	10	15

TABLE 1.2

SIMPLE CALIBRATION: ORDER 8×8

$n = 5$
$d = 240$

48	0	0	0	0	0	0	0
0	68	12	16	−16	−2	2	16
0	12	108	24	−24	−18	18	24
0	16	24	152	−32	−4	4	32
0	−16	−24	−32	72	24	16	8
0	−2	−18	−4	24	63	17	16
0	2	18	4	16	17	63	24
0	16	24	32	8	16	24	72

Column Sums

48	96	144	192	48	96	144	192

TABLE 1.3

SIMPLE CALIBRATION: ORDER 10×10

$n = 6$
$d = 210$

35	0	0	0	0	0	0	0	0	0
0	46	6	8	10	−10	−2	0	2	10
0	6	66	18	15	−15	−12	0	12	15
0	8	18	94	20	−20	−16	0	16	20
0	10	15	20	130	−25	−5	0	5	25
0	−10	−15	−20	−25	55	20	15	10	5
0	−2	−12	−16	−5	20	49	15	11	10
0	0	0	0	0	15	15	45	15	15
0	2	12	16	5	10	11	15	49	20
0	10	15	20	25	5	10	15	20	55

Column Sums

35	70	105	140	175	35	70	105	140	175

TABLE 1.4

SIMPLE CALIBRATION: ORDER 14×14

$n = 8$
$d = 4536$

567	0	0	0	0	0	0	0	0	0	0	0	0	0
0	684	54	72	90	108	126	−126	−36	−18	0	18	36	126
0	54	873	156	195	234	189	−189	−162	−39	0	39	162	189
0	72	156	1160	316	312	252	−252	−216	−164	0	164	216	252
0	90	195	316	1529	390	315	−315	−270	−205	0	205	270	315
0	108	234	312	390	1980	378	−378	−324	−78	0	78	324	378
0	126	189	252	315	378	2709	−441	−126	−63	0	63	126	441
0	−126	−189	−252	−315	−378	−441	945	378	315	252	189	126	63
0	−36	−162	−216	−270	−324	−126	378	864	306	252	198	144	126
0	−18	−39	−164	−205	−78	−63	315	306	797	252	211	198	189
0	0	0	0	0	0	0	252	252	252	756	252	252	252
0	18	39	164	205	78	63	189	198	211	252	797	306	315
0	36	162	216	270	324	126	126	144	198	252	306	864	378
0	126	189	252	315	378	441	63	126	189	252	315	378	945

Column Sums

567	1134	1701	2268	2835	3402	3969	567	1134	1701	2268	2835	3402	3969

TABLE 1.5

SIMPLE CALIBRATION: ORDER 18×18

$$(\mathbf{A}^T\mathbf{A})^{-1} = [\mathbf{\Lambda} \mid \mathbf{X}]$$

$n = 10$
$d = 166320$

$\mathbf{\Lambda} =$

16632	0	0	0	0	0	0	0	0
0	19152	1008	1344	1680	2016	2352	2688	3024
0	1008	22932	2856	3570	4284	4998	5712	4536
0	1344	2856	28288	5660	6792	7924	7616	6048
0	1680	3570	5660	35695	9570	9905	9520	7560
0	2016	4284	6792	9570	44748	11886	11424	9072
0	2352	4998	7924	9905	11886	55447	13328	10584
0	2688	5712	7616	9520	11424	13328	70672	12096
0	3024	4536	6048	7560	9072	10584	12096	96768
0	−3024	−4536	−6048	−7560	−9072	−10584	−12096	−13608
0	−1008	−4032	−5376	−6720	−8064	−9408	−10752	−4536
0	−672	−1428	−4424	−5530	−6636	−7742	−3808	−3024
0	−336	−714	−1132	−4115	−4938	−1981	−1904	−1512
0	0	0	0	0	0	0	0	0
0	336	714	1132	4115	4938	1981	1904	1512
0	672	1428	4424	5530	6636	7742	3808	3024
0	1008	4032	5376	6720	8064	9408	10752	4536
0	3024	4536	6048	7560	9072	10584	12096	13608

Column Sums

16632	33264	49896	66528	83160	99792	116424	133056	149688

$\mathbf{X} =$

0	0	0	0	0	0	0	0	0
−3024	−1008	−672	−336	0	336	672	1008	3024
−4536	−4032	−1428	−714	0	714	1428	4032	4536
−6048	−5376	−4424	−1132	0	1132	4424	5376	6048
−7560	−6720	−5530	−4115	0	4115	5530	6720	7560
−9072	−8064	−6636	−4938	0	4938	6636	8064	9072
−10584	−9408	−7742	−1981	0	1981	7742	9408	10584
−12096	−10752	−3808	−1904	0	1904	3808	10752	12096
−13608	−4536	−3024	−1512	0	1512	3024	4536	13608
28728	12096	10584	9072	7560	6048	4536	3024	1512
12096	26712	10248	8904	7560	6216	4872	3528	3024
10584	10248	24892	8666	7560	6454	5348	4872	4536
9072	8904	8666	23503	7560	6737	6454	6216	6048
7560	7560	7560	7560	22680	7560	7560	7560	7560
6048	6216	6454	6737	7560	23503	8666	8904	9072
4536	4872	5348	6454	7560	8666	24892	10248	10584
3024	3528	4872	6216	7560	8904	10248	26712	12096
1512	3024	4536	6048	7560	9072	10584	12096	28728

Column Sums

16632	33264	49896	66528	83160	99792	116424	133056	149688

Table 1.6

SIMPLE CALIBRATION: ORDER 22×22

$$(\mathbf{A}^T\mathbf{A})^{-1} = [\Lambda \mid \mathbf{X}]$$

$n = 12$
$d = 308880$

$\Lambda =$

25740	0	0	0	0	0	0	0	0	0	0
0	28800	1080	1440	1800	2160	2520	2880	3240	3600	3960
0	1080	33156	3024	3780	4536	5292	6048	6804	7560	5940
0	1440	3024	39056	5920	7104	8288	9472	10656	10080	7920
0	1800	3780	5920	46835	9870	11515	13160	13320	12600	9900
0	2160	4536	7104	9870	56988	15006	15792	15984	15120	11880
0	2520	5292	8288	11515	15006	68987	18424	18648	17640	13860
0	2880	6048	9472	13160	15792	18424	82832	21312	20160	15840
0	3240	6804	10656	13320	15984	18648	21312	101196	22680	17820
0	3600	7560	10080	12600	15120	17640	20160	22680	128160	19800
0	3960	5940	7920	9900	11880	13860	15840	17820	19800	176220
0	−3960	−5940	−7920	−9900	−11880	−13860	−15840	−17820	−19800	−21780
0	−1440	−5400	−7200	−9000	−10800	−12600	−14400	−16200	−18000	−7920
0	−1080	−2268	−6192	−7740	−9288	−10836	−12384	−13932	−7560	−5940
0	−720	−1512	−2368	−6260	−7512	−8764	−10016	−5328	−5040	−3960
0	−360	−756	−1184	−1645	−5538	−6461	−2632	−2664	−2520	−1980
0	0	0	0	0	0	0	0	0	0	0
0	360	756	1184	1645	5538	6461	2632	2664	2520	1980
0	720	1512	2368	6260	7512	8764	10016	5328	5040	3960
0	1080	2268	6192	7740	9288	10836	12384	13932	7560	5940
0	1440	5400	7200	9000	10800	12600	14400	16200	18000	7920
0	3960	5940	7920	9900	11880	13860	15840	17820	19800	21780

Column Sums

25740	51480	77220	102960	128700	154440	180180	205920	231660	257400	283140

$\mathbf{X} =$

0	0	0	0	0	0	0	0	0	0	0
−3960	−1440	−1080	−720	−360	0	360	720	1080	1440	3960
−5940	−5400	−2268	−1512	−756	0	756	1512	2268	5400	5940
−7920	−7200	−6192	−2368	−1184	0	1184	2368	6192	7200	7920
−9900	−9000	−7740	−6260	−1645	0	1645	6260	7740	9000	9900
−11880	−10800	−9288	−7512	−5538	0	5538	7512	9288	10800	11880
−13860	−12600	−10836	−8764	−6461	0	6461	8764	10836	12600	13860
−15840	−14400	−12384	−10016	−2632	0	2632	10016	12384	14400	15840
−17820	−16200	−13932	−5328	−2664	0	2664	5328	13932	16200	17820
−19800	−18000	−7560	−5040	−2520	0	2520	5040	7560	18000	19800
−21780	−7920	−5940	−3960	−1980	0	1980	3960	5940	7920	21780
45540	19800	17820	15840	13860	11880	9900	7920	5940	3960	1980
19800	42840	17280	15480	13680	11880	10080	8280	6480	4680	3960
17820	17280	40284	14976	13428	11880	10332	8784	7236	6480	5940
15840	15480	14976	38144	13132	11880	10628	9376	8784	8280	7920
13860	13680	13428	13132	36563	11880	10957	10628	10332	10080	9900
11880	11880	11880	11880	11880	35640	11880	11880	11880	11880	11880
9900	10080	10332	10628	10957	11880	36563	13132	13428	13680	13860
7920	8280	8784	9376	10628	11880	13132	38144	14976	15480	15840
5940	6480	7236	8784	10332	11880	13428	14976	40284	17280	17820
3960	4680	6480	8280	10080	11880	13680	15480	17280	42840	19800
1980	3960	5940	7920	9900	11880	13860	15840	17820	19800	45540

Column Sums

25740	51480	77220	102960	128700	154440	180180	205920	231660	257400	283140

Appendix II

Cross Calibration, Tables 2.1 *to* 2.4

The number of subdivisions in a main division is m; the number of λ's is $(2m-1)$ and the number of X's is $2(m-1)$.

Table 2.1

CROSS CALIBRATION: ORDER 13×13

$m = 4$
$d = 360$

256	72	65	52	65	36	22	−22	−32	−76	76	32	22
72	216	90	72	90	72	36	−36	−72	−72	72	72	36
65	90	190	80	100	90	65	−65	−70	−65	65	70	65
52	72	80	136	80	72	52	−52	−56	−52	52	56	52
65	90	100	80	190	90	65	−65	−70	−65	65	70	65
36	72	90	72	90	216	72	−72	−72	−36	36	72	72
22	36	65	52	65	72	256	−76	−32	−22	22	32	76
−22	−36	−65	−52	−65	−72	−76	130	50	40	−4	−14	−22
−32	−72	−70	−56	−70	−72	−32	50	130	50	−14	−22	−14
−76	−72	−65	−52	−65	−36	−22	40	50	130	−22	−14	−4
76	72	65	52	65	36	22	−4	−14	−22	130	50	40
32	72	70	56	70	72	32	−14	−22	−14	50	130	50
22	36	65	52	65	72	76	−22	−14	−4	40	50	130

Column Sums

568	648	680	544	680	648	568	−208	−224	−208	568	584	568

Table 2.2

CROSS CALIBRATION: ORDER 17×17

$m = 5$
$d = 2160$

1454	358	332	300	250	300	188	142	86	−86	−127	−163	−374	374	163	127	86
358	1196	454	420	350	420	346	224	142	−142	−194	−356	−358	358	356	194	142
332	454	1046	480	400	480	434	346	188	−188	−331	−349	−332	332	349	331	188
300	420	480	936	420	504	480	420	300	−300	−330	−330	−300	300	330	330	300
250	350	400	420	710	420	400	350	250	−250	−275	−275	−250	250	275	275	250
300	420	480	504	420	936	480	420	300	−300	−330	−330	−300	300	330	330	300
188	346	434	480	400	480	1046	454	332	−332	−349	−331	−188	188	331	349	332
142	224	346	420	350	420	454	1196	358	−358	−356	−194	−142	142	194	356	358
86	142	188	300	250	300	332	358	1454	−374	−163	−127	−86	86	127	163	374
−86	−142	−188	−300	−250	−300	−332	−358	−374	644	253	217	176	4	−37	−73	−104
−127	−194	−331	−330	−275	−330	−349	−356	−163	253	641	254	217	−37	−74	−101	−73
−163	−356	−349	−330	−275	−330	−331	−194	−127	217	254	641	253	−73	−101	−74	−37
−374	−358	−332	−300	−250	−300	−188	−142	−86	176	217	253	644	−104	−73	−37	4
374	358	332	300	250	300	188	142	86	4	−37	−73	−104	644	253	217	176
163	356	349	330	275	330	331	194	127	−37	−74	−101	−73	253	641	254	217
127	194	331	330	275	330	349	356	163	−73	−101	−74	−37	217	254	641	253
86	142	188	300	250	300	332	358	374	−104	−73	−37	4	176	217	253	644

Column Sums

3410	3910	4160	4260	3550	4260	4160	3910	3410	−1250	−1375	−1375	−1250	3410	3535	3535	3410

Table 2.3

CROSS CALIBRATION: ORDER 21×21

$$(\mathbf{A}^T\mathbf{A})^{-1} = [\boldsymbol{\Lambda} \,\vdots\, \mathbf{X}]$$

$m = 6$
$d = 10080$

$\boldsymbol{\Lambda} =$

6521	1426	1341	1240	1127	966	1127	760	621	466	281
1426	5252	1818	1712	1582	1356	1582	1328	954	740	466
1341	1818	4545	1944	1827	1566	1827	1656	1377	954	621
1240	1712	1944	4064	1960	1680	1960	1856	1656	1328	760
1127	1582	1827	1960	3689	1722	2009	1960	1827	1582	1127
966	1356	1566	1680	1722	2916	1722	1680	1566	1356	966
1127	1582	1827	1960	2009	1722	3689	1960	1827	1582	1127
760	1328	1656	1856	1960	1680	1960	4064	1944	1712	1240
621	954	1377	1656	1827	1566	1827	1944	4545	1818	1341
466	740	954	1328	1582	1356	1582	1712	1818	5252	1426
281	466	621	760	1127	966	1127	1240	1341	1426	6521
−281	−466	−621	−760	−1127	−966	−1127	−1240	−1341	−1426	−1481
−418	−644	−810	−1232	−1246	−1068	−1246	−1328	−1386	−1412	−658
−543	−798	−1323	−1320	−1281	−1098	−1281	−1320	−1323	−798	−543
−658	−1412	−1386	−1328	−1246	−1068	−1246	−1232	−810	−644	−418
−1481	−1426	−1341	−1240	−1127	−966	−1127	−760	−621	−466	−281
1481	1426	1341	1240	1127	966	1127	760	621	466	281
658	1412	1386	1328	1246	1068	1246	1232	810	644	418
543	798	1323	1320	1281	1098	1281	1320	1323	798	543
418	644	810	1232	1246	1068	1246	1328	1386	1412	658
281	466	621	760	1127	966	1127	1240	1341	1426	1481

Column Sums

15876	18216	19476	20160	20412	17496	20412	20160	19476	18216	15876

$\mathbf{X} =$

−281	−418	−543	−658	−1481	1481	658	543	418	281
−466	−644	−798	−1412	−1426	1426	1412	798	644	466
−621	−810	−1323	−1386	−1341	1341	1386	1323	810	621
−760	−1232	−1320	−1328	−1240	1240	1328	1320	1232	760
−1127	−1246	−1281	−1246	−1127	1127	1246	1281	1246	1127
−966	−1068	−1098	−1068	−966	966	1068	1098	1068	966
−1127	−1246	−1281	−1246	−1127	1127	1246	1281	1246	1127
−1240	−1328	−1320	−1232	−760	760	1232	1320	1328	1240
−1341	−1386	−1323	−810	−621	621	810	1323	1386	1341
−1426	−1412	−798	−644	−466	466	644	798	1412	1426
−1481	−658	−543	−418	−281	281	418	543	658	1481
2561	1018	903	778	641	79	−58	−183	−298	−401
1018	2540	1014	908	778	−58	−188	−294	−380	−298
903	1014	2529	1014	903	−183	−294	−369	−294	−183
778	908	1014	2540	1018	−298	−380	−294	−188	−58
641	778	903	1018	2561	−401	−298	−183	−58	79
79	−58	−183	−298	−401	2561	1018	903	778	641
−58	−188	−294	−380	−298	1018	2540	1014	908	778
−183	−294	−369	−294	−183	903	1014	2529	1014	903
−298	−380	−294	−188	−58	778	908	1014	2540	1018
−401	−298	−183	−58	79	641	778	903	1018	2561

Column Sums

−5796	−6408	−6588	−6408	−5796	15876	16488	16668	16488	15876

TABLE 2.4

CROSS CALIBRATION: ORDER 37×37

$$(\mathbf{A}^T\mathbf{A})^{-1} = [\Lambda_1 \mid \Lambda_2 \mid \mathbf{X}_1 \mid \mathbf{X}_2]$$

$m = 10$
$d = 1663200$

$\Lambda_1 =$

984479	148958	143462	137246	130610	123686	116537	109188	101629	92390
148958	751516	191724	184892	177220	168972	160274	151176	141658	128780
143462	191724	629011	207338	200205	192208	183536	174264	164362	149420
137246	184892	207338	550904	212390	205364	197438	188712	179146	162860
130610	177220	200205	212390	495475	212640	205880	198120	189310	172100
123686	168972	192208	205364	212640	453484	210578	204072	196306	178460
116537	160274	183536	197438	205880	210578	420251	207324	200827	182570
109188	151176	174264	188712	198120	204072	207324	392976	203148	184680
101629	141658	164362	179146	189310	196306	200827	203148	369479	184690
92390	128780	149420	162860	172100	178460	182570	184680	184690	319100
101629	141658	164362	179146	189310	196306	200827	203148	203159	184690
78948	124296	150744	168552	181320	190632	197244	201456	203148	184680
71177	103154	133556	154598	170180	182018	190931	197244	200827	182570
63206	92812	112968	137444	156040	170604	182018	190632	196306	178460
55010	82020	101155	116690	138525	156040	170180	181320	189310	172100
46526	70652	88478	103424	116690	137444	154598	168552	179146	162860
37622	58444	74541	88478	101155	112968	133556	150744	164362	149420
27998	44796	58444	70652	82020	92812	103154	124296	141658	128780
16799	27998	37622	46526	55010	63206	71177	78948	101629	92390
−16799	−27998	−37622	−46526	−55010	−63206	−71177	−78948	−101629	−92390
−25198	−39196	−50044	−59452	−68020	−76012	−83554	−107496	−110858	−100780
−33247	−49694	−61416	−70978	−79280	−86718	−112381	−115044	−116237	−105670
−41071	−59742	−72113	−81604	−89415	−117674	−119383	−119892	−119141	−108310
−48715	−69430	−82270	−91510	−123850	−124510	−124045	−122580	−120065	−109150
−56191	−78782	−91923	−130984	−130565	−128994	−126523	−123252	−119141	−108310
−63487	−87774	−138836	−137338	−134580	−130958	−126661	−121764	−116237	−105670
−70558	−146716	−144124	−140092	−135220	−129772	−123874	−117576	−110858	−100780
−152879	−148958	−143462	−137246	−130610	−123686	−116537	−109188	−101629	−92390
152879	148958	143462	137246	130610	123686	116537	109188	101629	92390
70558	146716	144124	140092	135220	129772	123874	117576	110858	100780
63487	87774	138836	137338	134580	130958	126661	121764	116237	105670
56191	78782	91923	130984	130565	128994	126523	123252	119141	108310
48715	69430	82270	91510	123850	124510	124045	122580	120065	109150
41071	59742	72113	81604	89415	117674	119383	119892	119141	108310
33247	49694	61416	70978	79280	86718	112381	115044	116237	105670
25198	39196	50044	59452	68020	76012	83554	107496	110858	100780
16799	27998	37622	46526	55010	63206	71177	78948	101629	92390

Column Sums

2587100	2951000	3157400	3291800	3384200	3447800	3488900	3510000	3510100	3191000

TABLE 2.4 (*Continued*)

$m = 10$
$d = 1663200$

$\Lambda_2 =$

101629	78948	71177	63206	55010	46526	37622	27998	16799
141658	124296	103154	92812	82020	70652	58444	44796	27998
164362	150744	133556	112968	101155	88478	74541	58444	37622
179146	168552	154598	137444	116690	103424	88478	70652	46526
189310	181320	170180	156040	138525	116690	101155	82020	55010
196306	190632	182018	170604	156040	137444	112968	92812	63206
200827	197244	190931	182018	170180	154598	133556	103154	71177
203148	201456	197244	190632	181320	168552	150744	124296	78948
203159	203148	200827	196306	189310	179146	164362	141658	101629
184690	184680	182570	178460	172100	162860	149420	128780	92390
369479	203148	200827	196306	189310	179146	164362	141658	101629
203148	392976	207324	204072	198120	188712	174264	151176	109188
200827	207324	420251	210578	205880	197438	183536	160274	116537
196306	204072	210578	453484	212640	205364	192208	168972	123686
189310	198120	205880	212640	495475	212390	200205	177220	130610
179146	188712	197438	205364	212390	550904	207338	184892	137246
164362	174264	183536	192208	200205	207338	629011	191724	143462
141658	151176	160274	168972	177220	184892	191724	751516	148958
101629	109188	116537	123686	130610	137246	143462	148958	984479
−101629	−109188	−116537	−123686	−130610	−137246	−143462	−148958	−152879
−110858	−117576	−123874	−129772	−135220	−140092	−144124	−146716	−70558
−116237	−121764	−126661	−130958	−134580	−137338	−138836	−87774	−63487
−119141	−123252	−126523	−128994	−130565	−130984	−91923	−78782	−56191
−120065	−122580	−124045	−124510	−123850	−91510	−82270	−69430	−48715
−119141	−119892	−119383	−117674	−89415	−81604	−72113	−59742	−41071
−116237	−115044	−112381	−86718	−79280	−70978	−61416	−49694	−33247
−110858	−107496	−83554	−76012	−68020	−59452	−50044	−39196	−25198
−101629	−78948	−71177	−63206	−55010	−46526	−37622	−27998	−16799
101629	78948	71177	63206	55010	46526	37622	27998	16799
110858	107496	83554	76012	68020	59452	50044	39196	25198
116237	115044	112381	86718	79280	70978	61416	49694	33247
119141	119892	119383	117674	89415	81604	72113	59742	41071
120065	122580	124045	124510	123850	91510	82270	69430	48715
119141	123252	126523	128994	130565	130984	91923	78782	56191
116237	121764	126661	130958	134580	137338	138836	87774	63487
110858	117576	123874	129772	135220	140092	144124	146716	70558
101629	109188	116537	123686	130610	137246	143462	148958	152879

Column Sums

3510100	3510000	3488900	3447800	3384200	3291800	3157400	2951000	2587100

TABLE 2.4 (*Continued*)

$m = 10$
$d = 1663200$

$\mathbf{X}_1 =$

−16799	−25198	−33247	−41071	−48715	−56191	−63487	−70558	−152879
−27998	−39196	−49694	−59742	−69430	−78782	−87774	−146716	−148958
−37622	−50044	−61416	−72113	−82270	−91923	−138836	−144124	−143462
−46526	−59452	−70978	−81604	−91510	−130984	−137338	−140092	−137246
−55010	−68020	−79280	−89415	−123850	−130565	−134580	−135220	−130610
−63206	−76012	−86718	−117674	−124510	−128994	−130958	−129772	−123686
−71177	−83554	−112381	−119383	−124045	−126523	−126661	−123874	−116537
−78948	−107496	−115044	−119892	−122580	−123252	−121764	−117576	−109188
−101629	−110858	−116237	−119141	−120065	−119141	−116237	−110858	−101629
−92390	−100780	−105670	−108310	−109150	−108310	−105670	−100780	−92390
−101629	−110858	−116237	−119141	−120065	−119141	−116237	−110858	−101629
−109188	−117576	−121764	−123252	−122580	−119892	−115044	−107496	−78948
−116537	−123874	−126661	−126523	−124045	−119383	−112381	−83554	−71177
−123686	−129772	−130958	−128994	−124510	−117674	−86718	−76012	−63206
−130610	−135220	−134580	−130565	−123850	−89415	−79280	−68020	−55010
−137246	−140092	−137338	−130984	−91510	−81604	−70978	−59452	−46526
−143462	−144124	−138836	−91923	−82270	−72113	−61416	−50044	−37622
−148958	−146716	−87774	−78782	−69430	−59742	−49694	−39196	−27998
−152879	−70558	−63487	−56191	−48715	−41071	−33247	−25198	−16799
266279	108358	101287	93991	86515	78871	71047	62998	54599
108358	262916	105974	99782	93230	86342	79094	71396	62998
101287	105974	259411	103473	98195	92413	86091	79094	71047
93991	99782	103473	257014	102035	97584	92413	86342	78871
86515	93230	98195	102035	256175	102035	98195	93230	86515
78871	86342	92413	97584	102035	257014	103473	99782	93991
71047	79094	86091	92413	98195	103473	259411	105974	101287
62998	71396	79094	86342	93230	99782	105974	262916	108358
54599	62998	71047	78871	86515	93991	101287	108358	266279
21001	12602	4553	−3271	−10915	−18391	−25687	−32758	−39479
12602	4204	−3494	−10742	−17630	−24182	−30374	−36116	−32758
4553	−3494	−10491	−16813	−22595	−27873	−32611	−30374	−25687
−3271	−10742	−16813	−21984	−26435	−30214	−27873	−24182	−18391
−10915	−17630	−22595	−26435	−29375	−26435	−22595	−17630	−10915
−18391	−24182	−27873	−30214	−26435	−21984	−16813	−10742	−3271
−25687	−30374	−32611	−27873	−22595	−16813	−10491	−3494	4553
−32758	−36116	−30374	−24182	−17630	−10742	−3494	4204	12602
−39479	−32758	−25687	−18391	−10915	−3271	4553	12602	21001

Column Sums

−923900	−1007800	−1056700	−1083100	−1091500	−1083100	−1056700	−1007800	−923900

TABLE 2.4 (*Continued*)

$m = 10$
$d = 1663200$

$\mathbf{X}_2 =$

152879	70558	63487	56191	48715	41071	33247	25198	16799
148958	146716	87774	78782	69430	59742	49694	39196	27998
143462	144124	138836	91923	82270	72113	61416	50044	37622
137246	140092	137338	130984	91510	81604	70978	59452	46526
130610	135220	134580	130565	123850	89415	79280	68020	55010
123686	129772	130958	128994	124510	117674	86718	76012	63206
116537	123874	126661	126523	124045	119383	112381	83554	71177
109188	117576	121764	123252	122580	119892	115044	107496	78948
101629	110858	116237	119141	120065	119141	116237	110858	101629
92390	100780	105670	108310	109150	108310	105670	100780	92390
101629	110858	116237	119141	120065	119141	116237	110858	101629
78948	107496	115044	119892	122580	123252	121764	117576	109188
71177	83554	112381	119383	124045	126523	126661	123874	116537
63206	76012	86718	117674	124510	128994	130958	129772	123686
55010	68020	79280	89415	123850	130565	134580	135220	130610
46526	59452	70978	81604	91510	130984	137338	140092	137246
37622	50044	61416	72113	82270	91923	138836	144124	143462
27998	39196	49694	59742	69430	78782	87774	146716	148958
16799	25198	33247	41071	48715	56191	63487	70558	152879
21001	12602	4553	−3271	−10915	−18391	−25687	−32758	−39479
12602	4204	−3494	−10742	−17630	−24182	−30374	−36116	−32758
4553	−3494	−10491	−16813	−22595	−27873	−32611	−30374	−25687
−3271	−10742	−16813	−21984	−26435	−30214	−27873	−24182	−18391
−10915	−17630	−22595	−26435	−29375	−26435	−22595	−17630	−10915
−18391	−24182	−27873	−30214	−26435	−21984	−16813	−10742	−3271
−25687	−30374	−32611	−27873	−22595	−16813	−10491	−3494	4553
−32758	−36116	−30374	−24182	−17630	−10742	−3494	4204	12602
−39479	−32758	−25687	−18391	−10915	−3271	4553	12602	21001
266279	108358	101287	93991	86515	78871	71047	62998	54599
108358	262916	105974	99782	93230	86342	79094	71396	62998
101287	105974	259411	103473	98195	92413	86091	79094	71047
93991	99782	103473	257014	102035	97584	92413	86342	78871
86515	93230	98195	102035	256175	102035	98195	93230	86515
78871	86342	92413	97584	102035	257014	103473	99782	93991
71047	79094	86091	92413	98195	103473	259411	105974	101287
62998	71396	79094	86342	93230	99782	105974	262916	108358
54599	62998	71047	78871	86515	93991	101287	108358	266279

Column Sums

2587100	2671000	2719900	2746300	2754700	2746300	2719900	2671000	2587100