AN EXACTLY SOLUBLE TWO-BODY PROBLEM WITH NON-CENTRAL FORCES*

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Summary

A class of local potentials is given which includes hard cores and a finite-range central, tensor, and $L.S$ part, and for which the lowest two neutron-proton states ($J = 0$, $\pi = +1$ and $J = 1$, $\pi = +1$) are exactly soluble. A numerical example is given in which the form factor for the potentials is a hard core plus square well.

I. A CLASS OF EXACTLY SOLUBLE POTENTIALS

There has been some interest recently (Regge 1959; Barut and Calogero 1962; Bethe and Kinoshito 1962; Bhattacharjie and Sudarshan 1962; Nicholson 1962) in the analytic properties of the scattering amplitude for the non-relativistic Schrodinger equation, in the hope that these properties may have some relevance to quantum field theory; and in this connexion there has been continuing interest in soluble two-body systems. We give here a class of potentials for which the $J = 0$, $\pi = +1$ and $J = 1$, $\pi = +1$ states of the neutron-proton system are exactly soluble. These potentials may include hard cores; they contain in addition finite range central, tensor, and $L.S$ parts.

The $J = 0$, $\pi = +1$ state is an uncoupled S state, and is immediately soluble for the potentials we consider. The radial equations for the $J = 1$, $\pi = +1$ state are, in the usual notation (Blatt and Weisskopf 1952),

$$\left.\begin{aligned} \left[\frac{\hbar^2}{M}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+E-V_C(r)\right]u(r) &= \sqrt{8}\cdot V_T(r)w(r), \\ \left[\frac{\hbar^2}{M}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+E-\frac{6\hbar^2}{Mr^2}-V_C(r)+2V_T(r)-3V_{LS}(r)\right]w(r) &= \sqrt{8}\cdot V_T(r)u(r). \end{aligned}\right\} \tag{1}$$

We consider a potential with the following form in the triplet state

$$\left.\begin{aligned} V_C(r) &= -V_C v(r), \\ V_T(r) &= -V_T v(r), \\ V_{LS}(r) &= +V_{LS}v(r)-2\hbar^2/Mr^2, \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} \phi(r) &= u(r)+\beta w(r), \\ \beta^2+\frac{3V_{LS}+2V_T}{2\sqrt{2}V_T}\beta+1 &= 0, \\ V_E(r) &= 2\sqrt{2}\beta V_T(r)+V_C(r). \end{aligned}\right\} \tag{3}$$

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In terms of these quantities, equation (1) becomes

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{M}{\hbar^2}\left[E-V_E(r)\right]\right\}\phi = 0, \tag{4}$$

the solution of which is known if the Schrodinger equation for S states can be solved for the central potential $v(r)$. There are two solutions, corresponding to the two roots β_1 and β_2 of (3); these define $u(r)$ and $w(r)$.

TABLE 1

PARAMETERS OF THE TWO-BODY SYSTEM FOR THE POTENTIAL OF EQUATION (5)

r_0 (fm)	r_1 (fm)	V_C (MeV)	V_T (MeV)	V_L (MeV)	P_D (%)	Q (10^{-27}W^2)	r_{0S} (fm)	a_{0T} (fm)	r_{0T} (fm)
0·4	2·4	24·0	16·5	36·2	3·6	2·1	2·94	5·67	2·12
0·4	2·6	19·7	14·8	31·8	4·0	2·4	3·16	5·77	2·49
0·4	2·8	16·4	13·4	28·2	4·4	2·74	3·36	5·86	2·25
0·5	2·2	33·5	18·7	38·2	3·4	1·8	2·83	5·61	1·92
0·5	2·4	26·6	16·5	33·3	3·8	2·2	3·05	5·71	2·17
0·5	2·6	21·7	14·8	29·3	4·2	2·5	3·27	5·87	2·54
0·5	2·72	19·3	13·9	27·3	4·5	2·74	3·40	5·86	2·31
0·5	2·8	17·9	13·4	26·1	4·7	2·9	3·49	5·90	2·36
0·6	2·2	37·9	18·9	35·3	3·6	2·0	2·95	5·65	1·98
0·6	2·4	29·8	16·7	30·9	4·1	2·3	3·16	5·75	2·23
0·6	2·6	23·9	15·0	27·2	4·5	2·6	3·39	5·85	2·59
0·6	2·8	19·7	13·4	24·3	4·9	3·0	3·61	5·94	2·36

II. A NUMERICAL EXAMPLE

The D-state component of the wave function, $w(r)$, will not in general have the usual asymptotic form for this potential due to the term $-2\hbar^2/Mr^2$ in $V_{LS}(r)$. This is of no concern for the deuteron bound state, but does affect the scattering states; for instance, the usual effective range expansion is inapplicable. This feature can be removed by truncating the potentials, without affecting the solubility of the equations. We give here a numerical example for such a truncated potential, with a hard core of radius r_0 and an attractive square well of radius r_1. That is, we write

$$\left.\begin{aligned} &u(r) = w(r) = 0, && r < r_0, && \text{(i)}\\ &v(r) = 1, && r_0 \leqslant r \leqslant r_1, && \text{(ii)}\\ &V_C(r) = V_T(r) = V_{LS}(r) = 0, && r > r_1, && \text{(iii)} \end{aligned}\right\} \tag{5}$$

the solutions being obtained by matching the solution of equation (4) for region (ii) with those of equation (1) for region (iii), on the boundary $r = r_1$.

The results are given in Table 1 for a spread of hard core radius r_0 and potential width r_1. For a given choice of r_0 and r_1, V_C and V_T have been somewhat arbitrarily fitted to the deuteron binding energy and singlet scattering length. The parameter V_{LS} has been fitted by demanding that the expectation value of $V_{LS}(r)$ be zero over the deuteron ground state. Table 1 then gives for each potential the deuteron state probability P_D%, and quadruple moment Q, and the singlet effective range r_{0S} and triplet scattering length a_{0T} and effective range r_{0T}. The details of the computation can be found in Davies (1962). The numerical work was carried out on an English Electric DEUCE computer.

TABLE 2

REACTANCE MATRIX ELEMENTS FOR THE POTENTIAL OF EQUATION (5) WITH THE PARAMETERS $r_0 = 0\cdot5$ fm, $r_1 = 2\cdot72$ fm

E (MeV)	X_{00}	X_{02}	X_{22}
10^{-6}	$-9\cdot11\times10^{-4}$	$-7\cdot66\times10^{-12}$	$-1\cdot62\times10^{-19}$
10^{-4}	$-9\cdot11\times10^{-3}$	$-7\cdot67\times10^{-9}$	$-1\cdot62\times10^{-14}$
10^{-2}	$-9\cdot12\times10^{-2}$	$-7\cdot68\times10^{-6}$	$-1\cdot62\times10^{-9}$
1	$-1\cdot101$	$-9\cdot47\times10^{-3}$	$-1\cdot78\times10^{-4}$
2	$-1\cdot95$	$-3\cdot30\times10^{-2}$	$-1\cdot08\times10^{-3}$
3	$-3\cdot26$	$-8\cdot28\times10^{-2}$	$-3\cdot49\times10^{-3}$
4	$-6\cdot01$	$-2\cdot04\times10^{-1}$	$-9\cdot70\times10^{-3}$
5	$-17\cdot4$	$-7\cdot40\times10^{-1}$	$-3\cdot62\times10^{-2}$
6	$+29\cdot9$	$+1\cdot53$	$+7\cdot11\times10^{-2}$
7	$+8\cdot87$	$+5\cdot30\times10^{-1}$	$+2\cdot14\times10^{-2}$
8	$+5\cdot42$	$+3\cdot71\times10^{-1}$	$+1\cdot15\times10^{-2}$
9	$+3\cdot99$	$+3\cdot08\times10^{-1}$	$+5\cdot72\times10^{-3}$
10	$+3\cdot19$	$+2\cdot74\times10^{-1}$	$+8\cdot36\times10^{-4}$
12	$+2\cdot33$	$+2\cdot41\times10^{-1}$	$+8\cdot75\times10^{-3}$
14	$+1\cdot84$	$+2\cdot25\times10^{-1}$	$+1\cdot93\times10^{-2}$
16	$+1\cdot54$	$+2\cdot16\times10^{-1}$	$+3\cdot11\times10^{-2}$
18	$+1\cdot31$	$+2\cdot11\times10^{-1}$	$+4\cdot42\times10^{-2}$
20	$+1\cdot15$	$+2\cdot07\times10^{-1}$	$+5\cdot84\times10^{-2}$
25	$+8\cdot49\times10^{-1}$	$+2\cdot02\times10^{-1}$	$+9\cdot84\times10^{-2}$

For comparison, the measured values of the quantities calculated are (Blatt and Weisskopf 1952):

$$\left.\begin{array}{lll} B.E. = 2\cdot226\text{ MeV}, & P_D\% \sim 5\%, & Q = 2\cdot74\times10^{-27}\text{ cm}^2, \\ a_{0S} = -23\cdot7\text{ fm}, & r_{0S} \sim 2\cdot7\text{ fm}, & \\ a_{0T} = 5\cdot38\text{ fm}, & r_{0T} = 1\cdot7\text{ fm}. & \end{array}\right\} \tag{6}$$

For the potential ($r_0 = 0\cdot5$ fm, $r_1 = 2\cdot72$ fm), Table 2 lists the elements of the reactance matrix **X** (Delves 1960), for various centre-of-mass energies up to 25 MeV. It is seen that there is an S-wave resonance at between 5 and 6 MeV.

III. REFERENCES

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