MASSIVE STARS WITH UNIFORM COMPOSITION

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Summary

It is the purpose of the present paper to derive a set of tables which greatly facilitate the construction of models of massive stars with uniform composition, to show how these tables can be used and finally to apply the method to derive models of stars composed of pure hydrogen in which a small amount of ^{12}C is present.

I. Introduction

Three circumstances facilitate the construction of models of massive stars (Schwarzschild and Härm 1958). (1) Convective envelopes must not be considered since their surface temperatures are so high. (2) Their convective cores are very extensive and contain almost all the nuclear energy production. (3) Bound-free and free-free transitions do not contribute appreciably to the opacity, and electron scattering constitutes the main source of opacity.

On the other hand, the effects of radiation pressure must be taken fully into account.

Instead of solving the differential equations, giving the four basic equilibrium conditions, directly in terms of the physical quantities, it is usual to transform them.

First, since for massive stars we can assume that almost all the energy generation takes place in the convective core, it is possible to detach the luminosity equation and to evaluate it as a separate quadrature. In this way we have to solve a system of three differential equations in both core and envelope.

Secondly, it is convenient to eliminate the unknown radius by using as variable RT instead of T.

A third useful simplification is possible when the composition is uniform and the opacity is due to electron scattering. In this case it is possible to eliminate explicit reference to μ in the equations, resulting in one set of solutions which cover all models of massive stars with uniform composition.

The transformations used here are:

$$\begin{aligned} m &= \mu^2 M_r, \\ l &= \kappa\mu^2 L, \\ x &= r/R, \\ t &= \mu RT, \end{aligned}$$

where the opacity κ is given by

$$\kappa = 0\cdot 2004(1+X),$$

X being the abundance by weight of hydrogen.

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These, together with the equation of state

$$\rho = \frac{a\mu}{3\mathscr{R}} \cdot T^3 \cdot \frac{\beta}{1-\beta},$$

lead to the following differential equations.

(1) *In the core*

$$\frac{\mathrm{d}m}{\mathrm{d}x} = \frac{4\pi a}{3\mathscr{R}} x^2 t^3 \frac{\beta}{1-\beta}, \tag{I$_c$}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-2(4-3\beta)}{32-3\beta^2-24\beta} \cdot \frac{\beta G}{\mathscr{R}x^2} m, \tag{II$_c$}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = \frac{G\beta(1-\beta)}{\mathscr{R}x^2} \cdot \frac{m}{t} \cdot \frac{3\beta^2}{-3\beta^2-24\beta+32}. \tag{III$_c$}$$

(2) *In the envelope*

$$\frac{\mathrm{d}m}{\mathrm{d}x} = \frac{4\pi a}{3\mathscr{R}} x^2 t^3 \frac{\beta}{1-\beta}, \tag{I$_e$}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-\beta}{16\pi c\mathscr{R}(1-\beta)} \cdot \frac{l}{x^2}, \tag{II$_e$}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}x} = \frac{1}{\mathscr{R}} \cdot \frac{\beta(1-\beta)}{t} \cdot \frac{1}{x^2} \left\{\frac{l}{4\pi c(1-\beta)} - mG\right\}. \tag{III$_e$}$$

β is used as a variable, in preference to the total pressure, on account of its smaller variation near the surface of the star.

At the centre the boundary condition $m = 0$ was used, values close to the centre being obtained by the expansions

$$m = \frac{4\pi a}{9\mathscr{R}} \cdot \frac{\beta_c}{1-\beta_c} t_c^3 x^3,$$

$$\beta = \beta_c + \frac{2\pi a G}{3\mathscr{R}^2} t_c^2 \cdot \frac{\beta_c^4}{32-24\beta_c-3\beta_c^2} x^2,$$

$$t = t_c - \frac{2\pi a G}{9\mathscr{R}^2} \cdot \frac{\beta_c^2}{1-\beta_c} \cdot \frac{8-6\beta_c}{32-24\beta_c-3\beta_c^2} t_c^3 x^2,$$

to $x = 0{\cdot}01$.

At the surface the boundary condition was taken as $t = 0$ together with the expansions

$$\beta = \beta_e,$$
$$m = \mu^2 M,$$
$$t = \frac{\beta_e G m_e}{4\mathscr{R}} \left(\frac{1}{x} - 1\right).$$

The condition $\beta = \beta_e$ leads to the mass-luminosity relation

$$l = 4\pi G c m_e (1-\beta_e).$$

This boundary condition is the same as the one used by Hoyle and Fowler (1963) in their study of more massive stars, and is equivalent to our assumption that β is constant in the outer layers of the star, as is at once evident from equation ($\mathrm{III_e}$).

Dividing equation ($\mathrm{III_e}$) by ($\mathrm{II_e}$) we obtain the differential equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{4}{t}\left(1-\beta\right)\left\{\frac{4\pi cGm_{\mathrm{e}}}{l}(1-\beta)-1\right\}$$

for the outermost layers of the star, where m is approximately constant and equal to its value m_{e} at the surface. Integration of this equation gives

$$\frac{1-\beta}{1-\beta-l/4\pi cGm_{\mathrm{e}}} = KT^4,$$

where K is a constant of integration, depending on the actual surface conditions.

As T increases from the outside this shows that $1-\beta$ converges rapidly to the value $l/4\pi cGm_{\mathrm{e}}$, which is the approximate boundary condition used for the present integrations.

Table 1

Principal Characteristics of the Models

β_{c}	β_{f}	β_{e}	x_{f}	q_{f}	$l\times 10^{-38}$	$t_{\mathrm{c}}\times 10^{-19}$	$m_{\mathrm{e}}\times 10^{-34}$
0·95	0·9718	0·9844	0·3084	0·373	0·0254	0·3750	0·645
0·90	0·9404	0·9638	0·3347	0·437	0·0945	0·5641	1·038
0·85	0·9063	0·9381	0·3615	0·502	0·2257	0·7366	1·452
0·80	0·8696	0·9075	0·3885	0·565	0·4480	0·9101	1·925
0·75	0·8301	0·8718	0·4156	0·625	0·8029	1·094	2·492
0·70	0·7878	0·8312	0·4427	0·682	1·351	1·297	3·185
0·65	0·7427	0·7860	0·4696	0·735	2·181	1·526	4·055
0·60	0·6950	0·7362	0·4965	0·783	3·427	1·791	5·170
0·55	0·6444	0·6825	0·5234	0·826	5·293	2·105	6·631
0·50	0·5910	0·6248	0·5501	0·863	8·102	2·486	8·593
0·45	0·5353	0·5642	0·5772	0·896	12·39	2·959	11·31
0·40	0·4771	0·5008	0·6044	0·923	19·05	3·559	15·19
0·35	0·4170	0·4358	0·6317	0·945	29·76	4·348	20·99
0·30	0·3557	0·3694	0·6606	0·964	47·75	5·420	30·13
0·25	0·2935	0·3029	0·6908	0·976	79·97	6·955	45·65
0·20	0·2312	0·2371	0·7234	0·986	143·6	9·302	74·90

II. Construction and Use of Tables

The above system of differential equations was solved on the IBM 1620 computer of the Research School of Physical Sciences, Australian National University, and the results of the computations are given in the following tables.

Table 1 gives:

the values of β at the centre, fitting point, and boundary of the star,
the position of the fitting point x_f,
the fraction of the total mass contained in the core q_f,
the value of $l = \kappa\mu^2 L$,
the value of t at the centre $t_{\mathrm{c}} = \mu RT_{\mathrm{c}}$,
the value of m at the boundary $m_{\mathrm{e}} = \mu^2 M$.

Table 2

VARIATION OF β THROUGHOUT THE CORE

100x	$\beta\times 10^4$															
0	9500	9000	8500	8000	7500	7000	6500	6000	5500	5000	4500	4000	3500	3000	2500	2000
2	9501	9002	8502	8002	7502	7002	6502	6002	5501	5001	4501	4001	3501	3001	2500	2000
4	9504	9007	8507	8008	7508	7007	6507	6006	5506	5005	4504	4004	3503	3002	2502	2001
6	9510	9015	8517	8017	7517	7016	6515	6014	5513	5011	4510	4008	3507	3005	2504	2003
8	9517	9026	8530	8031	7531	7029	6527	6025	5523	5020	4517	4015	3512	3010	2507	2005
10	9527	9040	8546	8048	7548	7046	6543	6039	5535	5031	4527	4023	3519	3015	2511	2008
12	9538	9058	8567	8069	7569	7066	6561	6056	5551	5045	4539	4033	3527	3021	2516	2011
14	9552	9078	8590	8094	7593	7089	6583	6076	5569	5061	4552	4045	3537	3029	2522	2015
16	9567	9102	8618	8123	7622	7116	6609	6099	5589	5079	4568	4058	3548	3038	2528	2020
18	9584	9128	8648	8155	7654	7147	6637	6126	5613	5100	4586	4073	3560	3047	2536	2025
20	9602	9156	8682	8191	7689	7181	6669	6155	5639	5123	4606	4090	3574	3058	2543	2030
22	9621	9187	8720	8231	7729	7219	6704	6187	5668	5148	4628	4108	3589	3070	2552	2036
24	9642	9221	8760	8273	7772	7260	6743	6222	5699	5176	4652	4128	3605	3082	2561	2042
26	9663	9257	8803	8320	7818	7305	6785	6260	5734	5206	4677	4149	3622	3096	2571	2049
28	9685	9294	8849	8370	7868	7353	6830	6301	5770	5238	4705	4172	3641	3110	2582	2056
30	9708	9333	8898	8423	7922	7405	6878	6346	5810	5272	4734	4197	3660	3126	2593	2064
32		9374	8950	8479	7979	7460	6930	6393	5852	5309	4765	4223	3681	3142	2605	2072
34		9416	9003	8539	8040	7519	6985	6443	5896	5348	4798	4250	3703	3159	2617	2080
36			9059	8601	8104	7581	7043	6496	5944	5389	4833	4279	3727	3177	2630	2089
38				8667	8171	7647	7105	6552	5994	5432	4870	4310	3751	3195	2644	2098
40					8243	7717	7170	6612	6046	5478	4909	4341	3776	3214	2658	2107
42					8317	7790	7240	6675	6102	5527	4950	4375	3803	3235	2672	2116
44						7868	7312	6741	6161	5578	4992	4410	3830	3256	2687	2126
46							7389	6811	6223	5631	5037	4447	3859	3277	2703	2136
48								6885	6289	5687	5084	4485	3890	3300	2719	2147
50								6964	6358	5747	5134	4525	3921	3324	2735	2158
52									6431	5809	5186	4567	3954	3348	2753	2169
54										5876	5241	4612	3988	3374	2771	2180
56											5299	4658	4024	3401	2789	2192
58											5360	4707	4062	3429	2809	2205
60												4759	4102	3458	2829	2218
62													4144	3489	2850	2231
64													4189	3521	2872	2245
66														3556	2896	2260
68															2921	2275
70																2292
72																2309

Table 2 gives the variation of β throughout the core, with a step length equal to one-fiftieth of the radius, each column listing the values of β for a separate model.

Tables 1 and 2 can now be used to derive the various physical characteristics of massive stars.

The luminosity of the star is given by the condition of thermal equilibrium, which in our notation can be written

$$\frac{l}{\kappa} = \frac{4\pi t_c^3}{\mu T_c^3} \int_0^{x_f} x^2 \rho \epsilon \, dx. \tag{1}$$

Dividing equation ($\mathrm{II_c}$) by ($\mathrm{III_c}$) we obtain the following differential equation

$$\frac{dt}{d\beta} = \frac{-2(4-3\beta)t}{3\beta^2(1-\beta)}, \tag{2}$$

which can be integrated to give the temperature T as a function of β,

$$T = T_c \left\{ \frac{\beta_c e^{-4/\beta_c}}{(1-\beta_c)} \times \frac{1-\beta}{\beta e^{-4/\beta}} \right\}^{2/3}. \tag{3}$$

Using the equation of state we get

$$\rho = \frac{a\mu}{3\mathscr{R}} \cdot T_c^3 \left\{ \frac{\beta_c e^{-4/\beta_c}}{1-\beta_c} \right\}^2 \frac{1-\beta}{\beta} e^{8/\beta}. \tag{4}$$

TABLE 3

COMPARISON OF AN EXACT INTEGRATION WITH THE INTERPOLATED VALUES

Quantity	Accurate Integration	Interpolation Method
log L	39·0781	39·0781
log T_e	5·1255	5·1274
log T_c	8·3014	8·3021

By this means T and ρ are obtained as functions of β and T_c only. Since the rate of energy generation per unit mass ϵ is a function of T and ρ, we see that the integrand of the luminosity integral (1) can be expressed as a function of β and T_c. In order to evaluate this integral numerically it is therefore necessary to have tables, such as Table 2, giving the variation of β with x.

For a given value of β_c ,the values of l and t_c are known from Table 1.

Since in Table 2, β is given as a function of x in the range 0 to x_f it is possible, using equations (3) and (4) and the appropriate value of ϵ to find a value of T_c which satisfies equation (1).

Once T_c has been determined the radius of the star will be given by the formula

$$R = t_c/\mu T_c,$$

and it is then possible to derive the effective temperature.

This gives then the position of the star in the Herzsprung-Russell diagram.

III. Interpolation

If a model is required for an intermediate value of β_c it is possible, by interpolation in Tables 1 and 2, to obtain a stellar model with reasonable accuracy.

As an example we have, using linear interpolation in Table 2, computed a model of a star composed of pure helium, corresponding to a value of $\beta_c = 0 \cdot 575$ and an energy generation given by

$$\epsilon_{3\alpha} = 1 \cdot 4 \times 10^{11} \frac{\rho^2 Y^3}{T_8^3} e^{-43.2/T_8} \text{ erg/s g.}$$

The values found from the interpolation method were then compared with those obtained from an accurate integration of the differential equations given in Section I of this paper, and it is seen from Table 3 that the agreement is quite good.

The value of log L was found by linear interpolation between the values of log l corresponding to $\beta_c = 0 \cdot 55$ and $\beta_c = 0 \cdot 6$ as given in Table 1.

Table 4

The effect on luminosity and effective temperature of small amounts of carbon in stars composed mainly of hydrogen

β_c	X_C	$T_c \times 10^{-8}$	log T_e	$\log(L/L_\odot)$	β_c	X_C	$T_c \times 10^{-8}$	log T_e	$\log(L/L_\odot)$
	10^{-10}	1·3676	5·0090			10^{-10}	1·0877	4·9115	
0·2	5×10^{-11}	1·4457	5·0212	7·5787	0·6	5×10^{-11}	1·1450	4·9226	5·9565
	10^{-11}	1·6497	5·0496			10^{-11}	1·2917	4·9487	
	0	4·2923	5·2574			0	2·1157	5·0557	
	10^{-10}	1·2849	4·9932			10^{-10}	1·0162	4·8657	
0·3	5×10^{-11}	1·3569	5·0051	7·1006	0·7	5×10^{-11}	1·0678	4·8765	5·5522
	10^{-11}	1·5442	5·0331			10^{-11}	1·1983	4·9015	
	0	3·5281	5·2125			0	1·7322	4·9815	
	10^{-10}	1·2186	4·9732			10^{-10}	0·9318	4·8039	
0·4	5×10^{-11}	1·2857	4·9849	6·7015	0·8	5×10^{-11}	0·9762	4·8140	5·0729
	10^{-11}	1·4596	5·0123			10^{-11}	1·0847	4·8370	
	0	2·9905	5·1682			0	1·3656	4·8871	
	10^{-10}	1·1539	4·9465			10^{-10}	0·8087	4·7081	
0·5	5×10^{-11}	1·2162	4·9579	6·3303	0·9	5×10^{-11}	0·8396	4·7162	4·3971
	10^{-11}	1·3770	4·9849			10^{-11}	0·9040	4·7323	
	0	2·5293	5·1168			0	0·9689	4·7473	

IV. Stars composed of Pure Hydrogen

Models of stars composed of pure hydrogen are given by Boury (1960). As pointed out in this paper, it is to be expected that a certain amount of ^{12}C will have been formed during the period of contraction of these stars.

As shown by Ledoux and Boury (1959) the temperatures in the central regions of these massive stars will have reached 10^8 °K long before these stars have reached the main sequence.

In consequence, at these high temperatures, nuclear reactions of the proton-proton type will set in, followed by the triple alpha process, converting the helium formed in the previous process into carbon.

Only a small fraction of ^{12}C will be formed in this way. But this small amount of carbon will be sufficient, at these high temperatures, to influence the rate of energy production through the carbon cycle, and will influence the equilibrium configuration of the star in a marked manner.

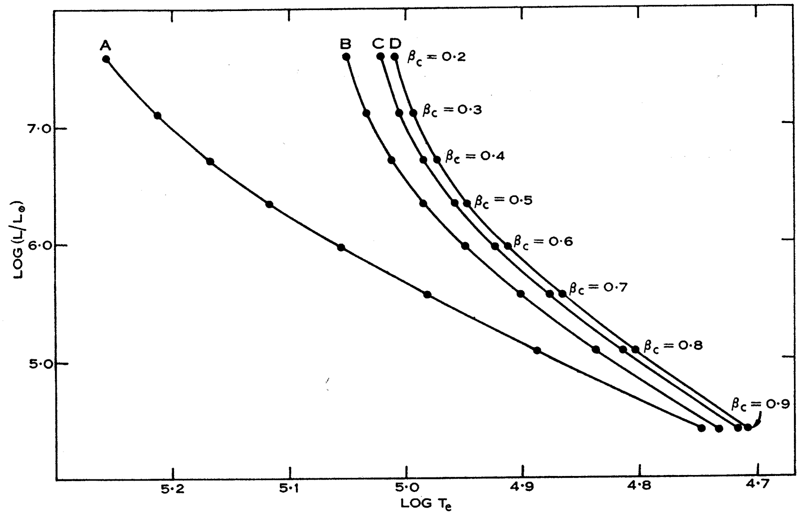

Fig. 1.—Positions of stars composed of pure hydrogen in the Herzsprung-Russell diagram, for various abundances of carbon. Curve A, $X_c = 0$; curve B, $X_c = 10^{-11}$; curve C, $X_c = 5\times10^{-11}$; D, $X_c = 10^{-10}$.

Using the method described above, we have computed models of massive hydrogen stars with small amounts of ^{12}C for the assumed abundances

$$X_c = 10^{-11},\ 5\times10^{-11},\ 10^{-10},\ \text{and } 0,$$

and for the following values of the rate of energy generation (Ledoux 1961)

$$\epsilon = \epsilon_{pp} + \epsilon_{CN},$$

where

(1)
$$\begin{aligned}
\epsilon_{pp} &= 4\cdot19\times10^3\, \alpha\rho X^2\tau^2\, e^{-\tau}\ \text{erg/g s},\\
\tau &= 33\cdot804/T_6^{\frac{1}{3}},\\
\alpha &= 0\cdot25, \text{ when} && T_6 < 8,\\
\alpha &= 0\cdot5, && 8 < T_6 < 13,\\
\alpha &= 0\cdot96, && 13 < T_6 < 20,\\
\alpha &= 0\cdot71, && 20 < T_6.
\end{aligned}$$

(2)
$$\epsilon_{CN} = a\cdot10^{23}\cdot\rho\cdot X\,\gamma\,\tau^2\, e^{-\tau}\ \text{erg/g s},$$

when

(a) If $T_6 < 10$ $\quad a = 1\cdot2,\quad \gamma = X_C,\quad \tau = 136\cdot7/T_6^{\frac{1}{3}},$
(b) If $10 < T_6 < 18$ $\quad a = 6\cdot17,\quad \gamma = X_C + X_N,\quad \tau = 152\cdot3/T_6^{\frac{1}{3}},$
(c) If $18 < T_6$ $\quad a = 6\cdot17,\quad \gamma = X_C + X_N + X_O,\quad \tau = 152\cdot3/T_6^{\frac{1}{3}}.$

The results are given in Table 4 and the position of these stars in the H–R diagram are given in Figure 1.

It is seen from this figure that a small amount of carbon formed during the contraction period has a marked effect on the position of the hydrogen main sequence.

This is due to the fact that when a small amount of ^{12}C is present, the energy generation is mainly due to the carbon cycle and no longer to the proton-proton cycle.

V. References

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