THE NUMBER OF INDEPENDENT VARIABLES FOR AN n-PARTICLE COLLISION*

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Suppose we have n vectors in a space of dimensionality ν, $n \geqslant \nu$, and that we define scalar products S_{ij} between the vectors p_i as follows

$$S_{ij} = p_i p_j = \sum_{\alpha,\beta}^{\nu} g_{\alpha\beta}\, p_{\alpha i}\, p_{\beta i}, \tag{1}$$

where $g_{\alpha\beta} = g_{\beta\alpha}$ and $\det(g_{\alpha\beta}) \neq 0$.

In matrix notation we write (1) as

$$S = \widetilde{P} G P. \tag{2}$$

If the rank of a matrix is denoted by r, we have $r(G) = \nu$, and, in general, $r(P) = \nu$, so that, in general, $r(S) = \nu$.

Thus we can find an orthogonal matrix O such that the symmetric matrix S can be written

$$S = \widetilde{O} \Lambda O, \tag{3}$$

where Λ is the diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{4}$$

Since $r(S) = \nu$, we may write

$$\lambda_{\nu+1} = \lambda_{\nu+2} = \ldots = \lambda_n = 0. \tag{5}$$

Consequently the number of independent parameters in the matrix O is $\frac{1}{2}n(n-1) - \frac{1}{2}(n-\nu)(n-\nu-1)$. Thus the matrix S depends on

$$\tfrac{1}{2}n(n-1) - \tfrac{1}{2}(n-\nu)(n-\nu-1) + \nu$$

independent parameters.

In the physical problem of the scattering of n-particles (neutral scalar bosons) the p_i are the four-momenta ($\nu = 4$) of the colliding particles. Energy-momentum conservation requires that

$$\sum_{i=1}^{n} p_i = 0, \tag{6}$$

and the lengths of the vectors p_i are interpreted as particle masses m_i which we agree to fix.

$$p_i^2 = m_i^2, \qquad i = 1, 2, \ldots, n. \tag{7}$$

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Thus the number N of independent variables describing the process is

$$N = \tfrac{1}{2}(n-1)(n-2) - \tfrac{1}{2}(n-\nu-1)(n-\nu-2) + \nu - n. \tag{8}$$

If the p_i happen to be insufficient in number to span the space (i.e. $n<\nu$) S will clearly be a general symmetric matrix having $\frac{1}{2}n(n+1)$ independent elements. In this case the number of variables describing the scattering process does not depend on ν and is

$$N = \tfrac{1}{2}(n-1)n - n = \tfrac{1}{2}n(n-3), \qquad n<\nu+1. \tag{9}$$

Combining (8) and (9) we have finally

$$N = \tfrac{1}{2}n(n-3) - \tfrac{1}{2}(n-\nu-1)(n-\nu)\theta(n-\nu-1), \tag{10}$$

where

$$\begin{aligned} \theta(x) &= 1, \qquad x>0, \\ &= 0, \qquad x<0. \end{aligned}$$

This problem is mentioned by Chan Hong Mo* and reference is made to the present discussion which was communicated privately by the present author.

* CHAN HONG MO (1962).—*Nuovo Cim.* **23**: 181.