TWO-BODY BOUND STATE AND SCATTERING WITH NON-CENTRAL FORCES

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[*Manuscript received October* 9, 1963]

Summary

A class of local potentials containing central, tensor, and $L.S$ parts is considered, for which the n-p coupled radial equations for the $\pi = -1$ states of any J reduce to uncoupled equations over a region R of r. If R is finite all potentials may be of short range. The possibilities of describing the deuteron and n-p scattering by this model are considered. It is shown that the observed values of the high-energy mixing parameters ϵ_2, ϵ_4 may be fitted without solving the wave equations.

I. A Class of Exactly Soluble Potentials

Nicholson (1962) and Davies and Delves (1963) have shown that the $J = 1$, $\pi = -1$ bound state of the neutron and proton system becomes "exactly soluble" for certain classes of non-central potentials: in the former case for a central plus a tensor potential having an r^{-2} radial dependence, and in the latter case for a central plus tensor plus a spin-orbit potential containing an r^{-2} term.

In this paper we consider the $\pi = -1$ states for any value of J, for a group of central plus tensor plus spin-orbit potentials that includes those of Nicholson and some of those of Davies and Delves. These potentials lead to uncoupled Schrodinger equations, and also lead to values of the mixing parameters used to describe neutron-proton scattering without the need to solve the Schrodinger equations.

Suppose that neutron and proton interact through two-body potentials having central, tensor, and spin-orbit components. The non-relativistic Schrodinger equation for the $\pi = -1$ states of angular momentum J yields the coupled radial equations

$$u''+\left[k^2+W_C+(J-1)W_{LS}-\frac{2(J-1)}{2J+1}W_T-\frac{(J-1)J}{r^2}\right]u+\frac{6\{J(J+1)\}^{\frac{1}{2}}}{2J+1}W_T w=0, \tag{1}$$

$$w''+\left[k^2+W_C-(J+2)W_{LS}-\frac{2(J+2)}{2J+1}W_T-\frac{(J+1)(J+2)}{r^2}\right]w+\frac{6\{J(J+1)\}^{\frac{1}{2}}}{2J+1}W_T u=0, \tag{2}$$

while for the $\pi = +1$ state of momentum J the radial equation is

$$v''+\left[k^2+W_C-W_{LS}+2W_T-\frac{J(J+1)}{r^2}\right]v=0, \tag{3}$$

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where we have written the potentials V as

$$V_C(r) = -(M/\hbar^2)W_C,$$

$$V_T(r) = -(M/\hbar^2)W_T,$$

$$V_{LS}(r) = -(M/\hbar^2)W_{LS}.$$

If we suppose that, over a certain region R of r,

$$w = \beta u, \tag{4}$$

where β is a constant, then (1) and (2) require that, within R,

$$(2J+1)^2(2/r^2+W_{LS})+6W_T[1+\{J(J+1)\}^{\frac{1}{2}}(\beta-1/\beta)] = 0. \tag{5}$$

R might consist of several subregions R_n, each characterized by a constant β_n. The quadratic equation (5) has roots β_1, β_2 satisfying

$$\beta_1\beta_2 = -1, \tag{6}$$

and it is convenient to take $|\beta_1|<1$. If (5) is true, then (1) and (2) have within R two independent solutions (u_1, w_1), (u_2, w_2) in which

$$\left.\begin{aligned} w_1 &= \beta_1 u_1, \\ w_2 &= \beta_2 u_2 = -(1/\beta_1)u_2, \end{aligned}\right\} \tag{7}$$

and u_1, u_2 satisfy the two equations

$$u''_{1,2}+\left[k^2+W_C+(J-1)W_{LS}+\frac{6\beta_{1,2}\{J(J+1)\}^{\frac{1}{2}}-2(J-1)}{2J+1}W_T-\frac{J(J-1)}{r^2}\right]u_{1,2} = 0. \tag{8}$$

The solution of each equation contains two arbitrary constants: if R includes $r = 0$ we particularize on the solutions satisfying

$$u_1(0) = u_2(0) = 0. \tag{9}$$

If R is a finite region then all potentials may be of short range, but if R is $(0,\infty)$ then at least one of W_{LS} and W_T must contain a long-range r^{-2} component over all r. If R is $(0,\infty)$ and if (5) is to hold for more than one value of J, then W_{LS} must contain this term.

II. Ground State

If (4) holds over R in the $J = 1$ bound state of the system, then (5) is

$$2/r^2+W_{LS}+\tfrac{2}{3}W_T[1+2^{\frac{1}{2}}(\beta-1/\beta)] = 0. \tag{10}$$

Any set of potentials satisfying (10) within R causes (1) and (2) for $J = 1$ to reduce to

$$u''_{1,2}+[-\alpha^2+W_C+2^{3/2}\beta_{1,2}W_T]u_{1,2} = 0, \tag{11}$$

where $-\alpha^2$ replaces k^2. Outside R, (1) and (2) hold, and u, w, u', and w' must be continuous at the boundaries of R.

Suppose that R is $(0,\infty)$. Provided that W_C, W_T diminish at least like r^{-2} at large r, u_1 and u_2 are asymptotically

$$u_{1,2} \sim a_{1,2}\,\mathrm{e}^{-\alpha r}+b_{1,2}\,\mathrm{e}^{\alpha r}.$$

Using (7), the ground state is the linear combination

$$\left.\begin{aligned} u_\mathrm{g} &= c_1u_1+c_2u_2, \\ w_\mathrm{g} &= \beta_1c_1u_1-(1/\beta_1)c_2u_2, \end{aligned}\right\} \tag{12}$$

and for u_g, w_g to behave suitably at large r we must have

$$c_1b_1+c_2b_2 = 0,$$

$$\beta_1c_1b_1-(1/\beta_1)c_2b_2 = 0,$$

which is only possible if $b_1 = c_2 = 0$, if $b_2 = c_1 = 0$, or if $b_1 = b_2 = 0$. That is, a bound state is only possible if u_1 is already correctly behaved at $r = 0$ and $r = \infty$, or if u_2 is suitably behaved, or if both are (in which case the bound state would be degenerate). Otherwise, no linear combination can give u_g and w_g correct behaviour. If u, βu is the appropriate solution, then the normalization integral gives for d, the fraction of D-state contained in the ground state, the value

$$d = \beta^2/(1+\beta^2). \tag{13}$$

It is well known that d should be about $0{\cdot}04$ in order to accord with the observed value of the magnetic moment of the deuteron.

Suppose that R is $(0, a)$ and that $W_C = W_T = W_{LS} = 0$ for $r > a$. Within R the state is the linear combination (12). Continuity of wave function and its slope at $r = a$ requires that

$$c_1u_1+c_2u_2 = \mathrm{e}^{-\alpha a},$$

$$\beta_1c_1u_1-(1/\beta_1)c_2u_2 = \lambda\,\mathrm{e}^{-\alpha a}(1+3/\alpha a+3/\alpha^2a^2),$$

and similarly for u_1', u_2'; where $u_1 = u_1(a)$ and so on. From these expressions follow

$$c_{1,2} = \mathrm{e}^{-\alpha a}\frac{\alpha u_{2,1}(a)+u_{2,1}'(a)}{u_{1,2}(a)u_{2,1}'(a)-u_{1,2}'(a)u_{2,1}(a)},$$

$$\lambda = \frac{\alpha(\beta+1/\beta)u_1u_2+\beta u_1u_2'+(1/\beta)u_1'u_2}{(u_1u_2'-u_1'u_2)(1+3/\alpha a+3/\alpha^2a^2)},$$

and the condition that, at $r = a$,

$$(u_1'u_2'+u_1u_2A\alpha)(1+\beta^2)+u_1'u_2(A+\alpha\beta^2)+u_1u_2'(A\beta^2+\alpha) = 0, \tag{14}$$

where

$$A = \alpha+\frac{3}{a}\cdot\frac{2+\alpha a}{3+3\alpha a+\alpha^2a^2}. \tag{15}$$

The fraction of D-state in the ground state is then readily seen to be

$$d = \frac{\beta^2\int_0^a u_1^2\,dr - \frac{2c_2}{c_1}\int_0^a u_1u_2\,dr + \frac{c_2^2}{c_1^2\beta^2}\int_0^a u_2^2\,dr + \frac{c_2^2\lambda^2}{c_1^2}\int_a^\infty e^{-2\alpha r}\left(1+\frac{3}{\alpha r}+\frac{3}{\alpha^2r^2}\right)^2 dr}{(1+\beta^2)\int_0^a u_1^2\,dr + \frac{c_2^2}{c_1^2}\left(1+\frac{1}{\beta^2}\right)\int_0^a u_2^2\,dr + \frac{1}{2\alpha a}e^{-2\alpha a} + \frac{c_2^2\lambda^2}{c_1^2}\int_a^\infty e^{-2\alpha r}\left(1+\frac{3}{\alpha r}+\frac{3}{\alpha^2r^2}\right)^2 dr}. \tag{16}$$

In the particular case that W_C, W_T are constant for $r < a$ and zero otherwise, (10) requires that

$$W_{LS0} = W_{LS}+2/r^2$$

is similarly a square well. The solutions that vanish at $r = 0$ are then

$$u_1 = \sin\gamma_1 r, \qquad u_2 = \sin\gamma_2 r,$$

and it is clear that

$$r_1^2 = W_C+2^{3/2}\beta_1 W_T-\alpha^2 > 0,$$

$$r_2^2 = W_C-2^{3/2}\frac{1}{\beta_1}W_T-\alpha^2 > 0,$$

for otherwise the continuity conditions at $r = a$ cannot be met. Using (10), it follows that

$$\gamma_1^2+\gamma_2^2 = 2W_C-2W_T-3W_{LS0}-2\alpha^2 > 0. \tag{17}$$

Equation (14) represents a stronger condition on these potentials.

It can be seen from (16) that, for a given value of d for the ground state, a range of values of β is possible, provided that R is less than $(0,\infty)$. For a given β, if (10) is satisfied at all, then it is satisfied for a range of potentials

$$W_T = \mu(W_T)_0,$$

$$W_{LS} = -2/r^2+\mu(W_{LS0})_0,$$

where μ takes any value. Thus the equations for u_1, u_2 contain a parameter μ:

$$u_{1,2}''+[-\alpha^2+W_C+2^{3/2}\beta_{1,2}\,\mu(W_T)_0]u_{1,2} = 0,$$

and so a range of functions u_1, u_2 are available. For a given value of β, the functional form $(W_T)_0$, its magnitude μ, and the form and magnitude of W_C are available for variation to produce a normalizable wave function and to fit the observed value of the electric quadrupole moment,

$$Q = \tfrac{1}{5}\pi e\int_0^\infty (uw.2^{3/2}-w^2)r^2\,dr. \tag{18}$$

The satisfaction of these two conditions does not exhaust the freedom in μ, W_T, and W_C, and so in u_1, u_2. So if R is $(0, a)$ there is a range available in d; as given by (16) when the potentials vanish outside R. Conversely, it is possible to obtain a desired d, as well as a bound state and a desired Q, over a range of values of β, accompanied by appropriate ranges of potential functions. As $a \to \infty$, however, the range of β that will yield d narrows to the single value given by (13).

The case treated by Nicholson (1962) was defined by

$$\left.\begin{aligned} R &= (0,\infty),\\ W_{LS}(r) &= 0,\\ W_T(r) &= \frac{-3}{r^2[1+2^{\frac{1}{2}}(\beta-1/\beta)]},\end{aligned}\right\} \tag{19}$$

and it was found that the binding energy, $d = 0\cdot 04$, and the observed Q of the deuteron could be fitted, as an example, by the simple wave function

$$u = N(e^{-\alpha r}-e^{-\delta r}),$$

and corresponding central potential

$$W_C = \frac{(\delta^2-\alpha^2)e^{-(\delta-\alpha)r}}{1-e^{-(\delta-\alpha)r}}+\frac{6.2^{\frac{1}{2}}\beta}{r^2[1+2^{\frac{1}{2}}(\beta-1/\beta)]}, \tag{20}$$

with $\beta = 0\cdot 204$, so that W_C had an r^{-2} repulsive core. By replacing (19) by

$$\left.\begin{aligned} W_{LS} &= W_{LS0}(r)-2/r^2,\\ W_T &= \frac{3W_{LS0}(r)}{2[1+2^{\frac{1}{2}}(\beta-1/\beta)]},\end{aligned}\right\} \tag{21}$$

the r^{-2} term can be removed from both W_C and W_T; and if, further,

$$W_C = \frac{(\delta^2-\alpha^2)e^{-(\delta-\alpha)r}}{1-e^{-(\delta-\alpha)r}}-\frac{3.2^{\frac{1}{2}}W_{0LS}(r)}{[1+2^{\frac{1}{2}}(\beta-1/\beta)]},$$

then the same wave function (20) can be used to achieve the same fit for d and Q. The case treated by Davies and Delves (1963) used a common potential function $W(r)$, with

$$\left.\begin{aligned} W_{LS} &= V_{LS}W(r)-2/r^2,\\ W_T &= V_TW(r),\\ W_C &= V_CW(r).\end{aligned}\right\} \tag{22}$$

Their solution procedure is different and does not involve (4). The constants V_{LS}, V_T, and V_C are independent parameters. By using for $W(r)$ a square well with a hard core, Davies and Delves were able to obtain good fits to d and Q and to approximate to the singlet and triplet effective ranges and the triplet scattering length.

III. Scattering

If (5) is to hold for all J, with W_{LS} and W_T independent of J, then, over R,

$$\frac{2}{r^2}+W_{LS}(r) = 6\gamma W_T(r), \tag{23}$$

with γ a constant, and

$$\beta_J^2+\frac{1+\gamma(2J+1)^2}{\{J(J+1)\}^{\frac{1}{2}}}\beta_J-1 = 0. \tag{24}$$

The roots of (24) satisfy (6), and we choose $|\beta_{J1}|<1$: then $\beta_{J1} \to 0$ as $J \to \infty$. It should be noted that β is independent of k. For $\beta = \beta_{J1,J2}$, (1) and (2) reduce to (8), two uncoupled equations for u_1 and u_2, and we choose the solutions satisfying (9).

In one of the standard scattering notations (Blatt and Biedenharn 1952) there are, for the case of scattering by central, tensor, and spin-orbit potentials, α and β eigenfunctions of the scattering matrix, which take the following asymptotic forms:

$$\left.\begin{aligned} u_\alpha &\sim \cos\epsilon_J \sin[kr-(J-1)\tfrac{1}{2}\pi+\delta_\alpha], \\ w_\alpha &\sim \sin\epsilon_J \sin[kr-(J+1)\tfrac{1}{2}\pi+\delta_\alpha] \\ &= -\tan\epsilon_J\, u_\alpha, \\ u_\beta &\sim -\sin\epsilon_J \sin[kr-(J-1)\tfrac{1}{2}\pi+\delta_\beta], \\ w_\beta &\sim \cos\epsilon_J \sin[kr-(J+1)\tfrac{1}{2}\pi+\delta_\beta] \\ &= \cot\epsilon_J\, u_\beta. \end{aligned}\right\} \tag{25}$$

If R is the whole range $(0,\infty)$, then these relations are of the same form as (4) and we can make the identification

$$\left.\begin{aligned} u_\alpha &= u_1, & w_\alpha &= w_1 \\ u_\beta &= u_2, & w_\beta &= w_2, \\ \beta_{J1} &= -\tan\epsilon_J, & \beta_{J2} &= \cot\epsilon_J, \end{aligned}\right\} \tag{26}$$

and obtain the scattering eigenfunctions as the solutions of equations (8) satisfying (9). In the usual theory the forces are supposed to be of limited range and it then follows that $\epsilon_J \to 0$ as $k \to 0$ for each J. In the present case, however, in which at least the spin-orbit potential contains an r^{-2} term over the range $(0,\infty)$, ϵ_J is independent of k.

If R is less than $(0,\infty)$, say $(0, a)$, and if the potentials are of limited range but not necessarily coterminous with R, then we know from the usual theory that $\epsilon_J \to 0$ as $k \to 0$. The α and β scattering eigenfunctions are then, within R, linear combinations of u_1, u_2:

$$\left.\begin{aligned} u_{\alpha,\beta} &= C_{1\alpha,\beta}u_1+C_{2\alpha,\beta}u_2, \\ w_{\alpha,\beta} &= \beta_1 C_{1\alpha,\beta}u_1-(1/\beta)C_{2\alpha,\beta}u_2, \end{aligned}\right\} \tag{27}$$

while, outside R, $u_{\alpha,\beta}$, $w_{\alpha,\beta}$ satisfy the coupled equations (1) and (2) and have the asymptotic forms (25). If $a \to \infty$, (27) and (25) require that $C_{2\alpha} \to 0$ and $C_{1\beta} \to 0$, and (26) results. We now consider scattering as k increases, with R held finite, say at $(0,a)$. We rewrite (8) in terms of the variable $\rho = kr/k_0$:

$$\frac{d^2}{d\rho^2}u_{1,2}(\rho)+\left\{k_0^2+\frac{k_0^2}{k^2}\left[W_C\left(\frac{k_0}{k}\rho\right)+(J-1)W_{LS}+\frac{6\beta_{1,2}\{J(J+1)\}^{\frac{1}{2}}-2(J-1)}{2J+1}W_T\right]\right.$$

$$\left.-\frac{J(J-1)}{\rho^2}\right\}u_{1,2} = 0.$$

We may look at this as an equation for scattering at a fixed energy k_0: as k increases the potentials become shallower and of longer range, while R, which in ρ-space is

(0, ka/k_0), becomes greater. So as $k \to \infty$, R becomes $(0,\infty)$ in ρ, and (26) again results. The scaling by k/k_0 does not disturb the validity of (5) nor change the value of β_1. It follows that tan ϵ_J approaches $-\beta_1$ asymptotically as k increases.

With this in mind it is interesting to look at the results of p-p phase-shift analysis as quoted, for example, by Scotti and Wong (1963). The parameter ϵ_2 is about $-1\cdot3°$ at 50 MeV and is nearly constant at $-2\cdot3°$ from 100 to 400 MeV. The parameter ϵ_4 is less constant, being about $-1°$ at 150 MeV and perhaps $-1\cdot7°$ at 400 MeV: we will take the value for large k to be $-1\cdot5°$. On the model of the n-p interaction considered in this paper, and assuming charge-independence, we then obtain the estimates $\beta_{21} = 0\cdot040$ and $\beta_{41} = 0\cdot026$. From (20) the constant γ is

$$\gamma = \{1/(2J+1)^2\}[1+\{J(J+1)\}^{\frac{1}{2}}(\beta-1/\beta)].$$

Using $\beta_{21} = 0\cdot040$ we obtain the prediction that for $J = 1$, $\beta_{11} = 0\cdot062$, and from $\beta_{41} = 0\cdot026$ we obtain the estimate $\beta_{11} = 0\cdot075$. If, on the other hand, we assume that there is 4% of D-state in the ground state of the system, and that R is $(0,\infty)$, then (13) requires that $\beta_{11} = 0\cdot204$. Whether, for a smaller region R, there exist potentials that can yield 4% of D-state in the ground state with a β_{11} substantially closer to the value of about $0\cdot07$ derived from the scattering data, is as yet unresolved.

IV. References

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