THERMAL PROPERTIES OF SYSTEMS EXHIBITING OPTIMUM COUNTERCURRENT HEAT EXCHANGE

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[*Manuscript received June* 3, 1963]

Summary

Heat flow in heat exchange systems, operating at constant pressure, is considered in relation to the thermodynamical measure of entropy called by Keenan the availability. The ratio of the maximum attainable mechanical work from two systems (1) and (2) is shown to be equal to the ratio of the two availabilities or $\Delta B_1/\Delta B_2$, and this reasonably approaches unity only when the exchanger involves countercurrent flow. The temperatures and temperature differences may be plotted against the linear dimension along the exchanger. The temperature difference will pass through one or more stationary values associated with a temperature T^*. At such a pivotal point, we may define enthalpies, ΔH_1^*, ΔH_2^* and specific heats (c_1^*, c_2^*) of the two streams in which the following relations hold:

$$g_2/g_1 = c_1^*/c_2^* = \Delta H_1/\Delta H_2 = \Delta H_1^*/\Delta H_2^*,$$

where g_1 and g_2 are the mass flow rates of the two streams.

I. Introduction

Derivation of the maximum amount of mechanical work which can be obtained from a thermal source forms the basis of classical thermodynamics. However, it is not nearly as well realized that these derivations postulate inexhaustible sources or source in which a finite gain or loss of heat produces no temperature rise. As against this, the economic exploitation of fuels and other resources involves, in many cases, the use of exhaustible sources, in which a finite heat change produces a temperature change. Thus, in the internal combustion engine, energy exchange after the combustion of the fuel gases takes place under conditions of an exhaustible source. The heat change in steam systems, whereby steam is raised from combustion gas or from the products of a nuclear generator, also takes place under these same conditions. In these or similar conditions, energy is extracted from a source in which neither the temperature nor the entropy are maintained constant and thus none of four energy functions of classical thermodynamics are strictly applicable. Nevertheless, the source material giving up a quantity δQ of heat will travel through a definite thermal path determined by its specific heat and latent heats. If, in addition, the geometrical parameters or forces are prescribed, δQ becomes an exact differential and under isopiestic conditions is equivalent to $\mathrm{d}H$, the differential of the enthalpy, or under isometric conditions becomes $\mathrm{d}U$, the differential of the internal energy. Under isopiestic conditions, Keenan (1932, 1951), extending the earlier work of Gibbs (1931), has put forward an expression for the maximum amount of work derived from a thermal source under these conditions which he has termed the "*availability*". Under this definition, *availability* becomes a metric quantity of the dimensions of energy.

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Aust. J. Phys., 1964, **17**, 26–36

In the second of the two papers cited above, Keenan has given a number of examples of the derivation of the maximum attainable work from physical and chemical systems, using the concept of availability. This and the following paper will be concerned with the exchange of thermal energy between two different bodies, and the efficiency of the exchange process will be evaluated in terms of the relative availabilities of the sources and sinks. The reactions will occur under approximately isopiestic conditions in practical heat exchange. The efficiency of the exchange process will be shown to be a simple generalization of the Kelvin efficiency.

II. Derivation of Availability

Let the exhaustible thermal source contain a quantity of heat at a temperature T_1, and let T_0 be the temperature of the sink supposed to be inexhaustible in the sense that the addition of heat to the sink will not increase the temperature T_0. On the other hand, as heat is withdrawn from the source, its temperature, depending on the specific heat and latent heat in the temperature range $T_0 \leqslant T \leqslant T_1$, will fall. Let an element of heat δQ be withdrawn at the temperature T_1. The maximum amount of work equivalent to this element of heat in any reversible cycle becomes

$$\begin{aligned}\delta W &= \delta Q(T-T_0)/T \\ &= \delta Q - T_0 \delta Q/T.\end{aligned}$$

If the thermal exchange takes place at a fixed pressure p the value of δQ becomes $\mathrm{d}H$ and we get

$$(\mathrm{d}W)_p = \mathrm{d}H - T_0\,\mathrm{d}H/T, \tag{1}$$

where $(\mathrm{d}W)_p$ is an element of mechanical work done at pressure p. The total amount of work done when all the enthalpy at temperature T_1 is transferred to the temperature T_0 becomes W_p,

$$\begin{aligned}W_p &= \int_{T_0}^{T_1} \mathrm{d}H - T_0 \int_{T_0}^{T_1} \frac{\mathrm{d}H}{T} \\ &= \Delta H_1 - T_0 \Delta S_1,\end{aligned}$$

where ΔH_1 is the enthalpy change between the states at $\{p,\ T_0\}$ and $\{p,\ T_1\}$ and ΔS_1 is the corresponding entropy change.

The maximum amount of work done at constant pressure is defined by Keenan as the increment in the availability, ΔB_1, where

$$\Delta B_1 = \Delta H_1 - T_0\,\Delta S_1. \tag{2}$$

The ratio of the availability to the enthalpy at the same temperature has been called the availability ratio $\mathscr{A}$ (Bosworth 1954). Thus,

$$\mathscr{A}_1 = \Delta B_1/\Delta H_1 = 1 - T_0\,\Delta S_1/\Delta H_1. \tag{3}$$

Since the ratio $\Delta H_1/\Delta S_1$ is the heat loaded average of the temperature we can put

$$\overline{T}_1 = \Delta H_1/\Delta S_1.$$

The availability ratio then becomes the ratio of two temperatures

$$\mathscr{A}_1 = (\overline{T}_1 - T_0)/\overline{T}_1. \tag{4}$$

Had the source been inexhaustible (or the transfer of heat been infinitesimal in comparison with the enthalpy content), $\overline{T}_1$ would then become T_1 and

$$\mathscr{A}_1 = (T_1 - T_0)/T_1,$$

or the availability would become the Kelvin efficiency of the heat transfer between single temperatures T_1 and T_0. The property of the availability for an exhaustible system may thus be regarded as the generalization of the Kelvin efficiency to an extended range of sources and sinks.

This concept of efficiency in terms of availability may be extended to pairs of interacting thermal systems. If a hotter body hands enthalpy ΔH_1 to a second cooler body, which gains an equal amount of enthalpy ΔH_2, then

$$\Delta H_1 = -\Delta H_2.$$

Normally the hotter body will be subject to a fall in temperature which may take a mean value $\overline{T}_1$ while the cooler body is subject to a temperature rise in which its mean value, on the basis above is $\overline{T}_2$. If heat is required to flow naturally in the required direction, then

$$\overline{T}_1 > \overline{T}_2,$$

so that

$$(\overline{T}_1 - T_0)/\overline{T}_1 > (\overline{T}_2 - T_0)/T_2,$$

where T_0, the sink temperature, is less than T_2. The efficiency η of the operation of transferring heat is then the ratio of the maximum work done by the two systems or

$$\eta = \frac{W_{p2}}{W_{p1}} = \frac{\Delta H_2 - T_0 \Delta S_2}{\Delta H_1 - T_0 \Delta S_1} = \frac{\Delta B_2}{\Delta B_1} = \frac{\overline{T}_1(T_0 - \overline{T}_2)}{\overline{T}_2(\overline{T}_1 - T_0)}, \tag{5}$$

where the subscript 1 refers to the hotter body and the subscript 2 to the cooler body.

$\eta \to 0$ only when process 2 takes place isothermally at T_0 and $\eta \to 1$ only when $\overline{T}_2 \to \overline{T}_1$.

III. Heat Flow in a Countercurrent Exchanger

To allow heat to flow naturally from a body A to another body B, it is first necessary that the temperature of A should be greater than that of B. To get a *highly efficient operation*, in the sense already discussed, *the temperature of B should be as little below the temperature of A as possible.* Now, as the heat exchange proceeds, the temperature of A falls and that of B rises. The condition of a small temperature difference between the two bodies concerned can only be maintained if the two bodies are fluids and are caused to flow in opposite directions. Stream B in the early stages and at a low temperature is in thermal contact with stream A at a later stage and thus also at a low temperature. At the same time stream B in the later stages and thus at a higher temperature is in thermal contact with A at early stages and at a higher temperature. Such a thermal contacting system constitutes the well-known *countercurrent heat exchanger*.

The mean temperature difference is, as we have seen, reciprocally related to the efficiency of the exchange, which thus depends on the overall heat transfer coefficient and on the transfer area. As the transfer area increases, the mean temperature difference falls, but even if the transfer area is increased without limit, the temperature difference is still limited by inequalities in the thermal properties, which have been termed by Schack (1933) and by others, the water equivalents of the two streams. The water equivalents are the products of the mass flow rates by the specific heats. If the bodies are subject to phase changes at the temperatures and pressures of the exchange, the effective specific heats will be infinite. The temperature difference between the two streams can only be maintained vanishingly small throughout the exchanger if the water equivalents of the two streams are maintained constant at all points along the exchanger.

No matter how we may alter the shape of the exchanger and no matter how we treat the interacting streams, conditions of continuity demand that, in unit time, the same mass of interacting fluid passes all points along the stream. If g_1 is the mass flow of stream 1, the quantity g_1 will be a stream characteristic independent of the point considered or of the effective exchange area at that point.

IV. Equations of Heat Exchange

The temperatures of the two interacting streams are best specified by denoting the temperature of an arbitrary point in the lower stream by T, while the corresponding temperature of a point in the upper stream is denoted by $(T+\theta)$, with θ the temperature difference. Further, let T_0 be the entrance temperature of the lower stream and let T_∞ be the entrance temperature of the upper stream (Fig. 1). Let θ_0 and θ_∞ be the corresponding temperature difference at the low and the high temperature ends of the exchanger. The values of the temperature differences of θ_0, θ_∞, and θ (at an arbitrary point in the exchanger) are controlled by the thermal properties of the streams and by the flow rates.

Let the upper (or hotter) stream be characterized by the following parameters:

Mass flow rate	g_1
Heat capacity at constant pressure	c_1
Density	ρ_1
Thermal conductivity	k_1
Stream thickness	y_1

and let the corresponding parameters of the lower stream be g_2, c_2, ρ_2, k_2, and y_2 respectively; let h be the heat transfer coefficient per unit width of the surface and let x be the coordinate along the stream.

The heat flowing from a volume $y_1\,dx$ of the upper stream per unit width of the exchange surface and to the volume $y_2\,dx$ of lower stream in unit time becomes

$$h\,dx.\theta.$$

The time taken for the upper stream to pass the length element $\mathrm{d}x$ amounts to

$$(y_1\rho_1/g_1)\mathrm{d}x.$$

The heat lost through $\mathrm{d}x$ in this time is

$$(h\theta y_1\rho_1/g_1)(\mathrm{d}x)^2,$$

and the heat lost per unit mass of the upper stream becomes

$$(h\theta/g_1)\mathrm{d}x.$$

Falls in the temperature of the upper stream are produced both by conduction along the stream and by exchange to the other stream. That due to conduction in

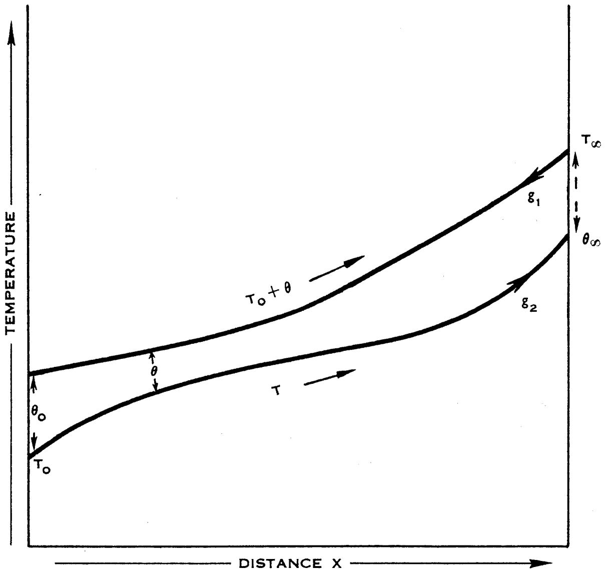

Fig. 1

the time $(y_1\rho_1/g_1)\mathrm{d}x$ amounts to

$$\frac{k_1}{\rho_1 c_1}\cdot\frac{\mathrm{d}^2(T+\theta)}{\mathrm{d}x^2}\cdot\frac{\rho_1 y_1}{g_1}\,\mathrm{d}x$$

or

$$\frac{k_1}{c_1 g_1}\cdot\frac{\mathrm{d}^2(T+\theta)}{\mathrm{d}x^2}\,y_1\,\mathrm{d}x.$$

The total temperature drop in the length $\mathrm{d}x$ due to both causes then becomes

$$\frac{\mathrm{d}(T+\theta)}{\mathrm{d}x}=\frac{1}{c_1 g_1}\left\{h\theta+k_1 y_1\frac{\mathrm{d}^2(T+\theta)}{\mathrm{d}x^2}\right\}. \tag{6}$$

The corresponding temperature rise in the lower stream amounts to

$$\frac{\mathrm{d}T}{\mathrm{d}x}=\frac{1}{c_2 g_2}\left\{h\theta+k_2 y_2\frac{\mathrm{d}^2T}{\mathrm{d}x^2}\right\}. \tag{7}$$

In equations (6) and (7), g_1 and g_2 measure the magnitudes only and not the directions of flow. The quantities g_1 and g_2 are treated as positive. The quantities c_1 and c_2 refer to specific heats at the distance x along the exchanger; if H_1 and H_2 are specific enthalpies at the corresponding points of the two streams, then

$$c_1 = \left(\frac{\partial H_1}{\partial (T+\theta)}\right)_p,$$

$$c_2 = \left(\frac{\partial H_2}{\partial T}\right)_p.$$

Line conduction along the streams in equations (6) and (7) tends to smooth out temperature differences produced by heat exchange and in turn tends to reduce the efficiency of the exchange process. Maximum exchange efficiency can only be obtained after a minimization of the line conduction and this in turn can be reduced without limit by making the exchange areas as large as possible. The first term in equation (6) becomes much larger than the second term, which is neglected, and

$$\mathrm{d}(T+\theta)/\mathrm{d}x = h\theta/c_1g_1 \geqslant 0, \tag{7a}$$
$$\mathrm{d}T/\mathrm{d}x = h\theta/c_2g_2 \geqslant 0.$$

From this equation we may deduce

$$\mathrm{d}\theta/\mathrm{d}T = c_2g_2/c_1g_1 - 1, \tag{8}$$

and

$$\mathrm{d}\theta/\mathrm{d}(T+\theta) = 1 - c_1g_1/c_2g_2. \tag{8a}$$

The quantity θ, which is never negative, may vary with T or $(T+\theta)$ along the exchanger and may pass through one or more minimal values. At any particular minimum of θ let the temperature T take the value T^*. Then at $T = T^*$

$$\mathrm{d}\theta/\mathrm{d}T = 0 \quad \text{and} \quad \mathrm{d}^2\theta/\mathrm{d}T^2 > 0. \tag{9}$$

This last condition is equivalent to

$$\mathrm{d}^2(T+\theta)/\mathrm{d}x^2 > \mathrm{d}^2T/\mathrm{d}x^2,$$

which is only possible if

$$\frac{1}{c_2^3g_2^2}\cdot\frac{\mathrm{d}c_2}{\mathrm{d}T} > \frac{1}{c_1^3g_1^2}\cdot\frac{\mathrm{d}c_1}{\mathrm{d}(T+\theta)}. \tag{10}$$

Let the values of c_1 and c_2 at T^* be c_1^* and c_2^*, and let the corresponding temperature coefficients of these heat capacities be $\dot{c}_1^*$ and $\dot{c}_2^*$. Conditions (8) and (9) reduce to

$$c_2g_2/c_1g_1 - 1 = 0 \quad \text{at } T = T^*,$$

or

$$c_1^*/c_2^* = g_2/g_1, \tag{11}$$

and from condition (10)

$$\dot{c}_2^*/c_2^* > \dot{c}_1^*/c_1^*. \tag{12}$$

The total quantity of heat delivered from the upper stream in falling from the entrance temperature T_∞ becomes

$$g_1 \int_{T_0+\theta_0}^{T\infty} \mathrm{d}H_1, \text{ to be called } g_1\Delta H_1,$$

and this must be equal to the total heat received by the lower stream entering at T_0, namely,

$$g_2 \int_{T_0}^{T\infty-\theta\infty} \mathrm{d}H_2, \text{ to be called } g_2\Delta H_2.$$

From this equality

$$\Delta H_1/\Delta H_2 = g_2/g_1 = c_1^*/c_2^*, \tag{13}$$

irrespective of whether the temperature T^* defined by this equation describes a minimum or maximum value of θ, or whether it is a single-valued or multiple-valued quantity. At least one minimum value is attained if condition (12) obtains, whereas a maximum value is obtained if the logarithmic temperature coefficient of the thermal capacity of the upper stream ($\dot{c}_1/c_1$) is greater than that of the lower ($\dot{c}_2/c_2$).

By making the rate of heat exchange and the surface area sufficiently large, it is possible to arrange for the value of θ at the minimum, if such a minimum exists, to be arbitrarily small. We may now integrate equations (8) and (8a) to give

$$\theta = \frac{g_2}{g_1}\int_{T^*}^{T} \frac{c_2}{c_1}\,\mathrm{d}T - T + T^* \geqslant 0, \tag{14}$$

and

$$\theta = (T+\theta) - T^* - \frac{g_1}{g_2}\int_{T^*}^{T+\theta} \frac{c_1}{c_2}\,\mathrm{d}(T+\theta) \geqslant 0, \tag{14a}$$

on the condition that, at $T = T^*$,

$$\dot{c}_2^*/c_2^* > \dot{c}_1^*/c_1^*.$$

The actual value of θ at any particular temperature T and the temperature T^* at which θ vanishes, will both normally vary with the ratio g_2/g_1. A diagrammatic example of two interacting streams in which both c_1 and c_2 are continuous functions of the temperature throughout the range of the exchange in which the inequalities (12) holds is given in Figure 2(*a*).

If the inequality (12) does not hold for any temperature within the range of the heat exchanger, then it is impossible to have a true mathematical minimum θ. If

$$\dot{c}_2^*/c_2^* < \dot{c}_1^*/c_2^*,$$

there may be found a $T = T^*$ at which the stationary value of θ becomes a maximum. We denote this maximum value, which is always positive, by θ^*. Physical conditions of operation (flow rates) can then be adjusted so that this value of θ shall be as small as possible and consistent with the condition that no value of θ shall become negative. A diagrammatic representation of the temperature distributions in the exchanger is given in Figure 2(*b*).

Equations (14) and (14a) may now be modified to take into account the possible existence of θ^* and become

$$\theta = \frac{g_2}{g_1}\int_{T^*}^{T}\frac{c_2}{c_1}\,\mathrm{d}T - T + T^* + \theta^* \geqslant 0, \tag{15}$$

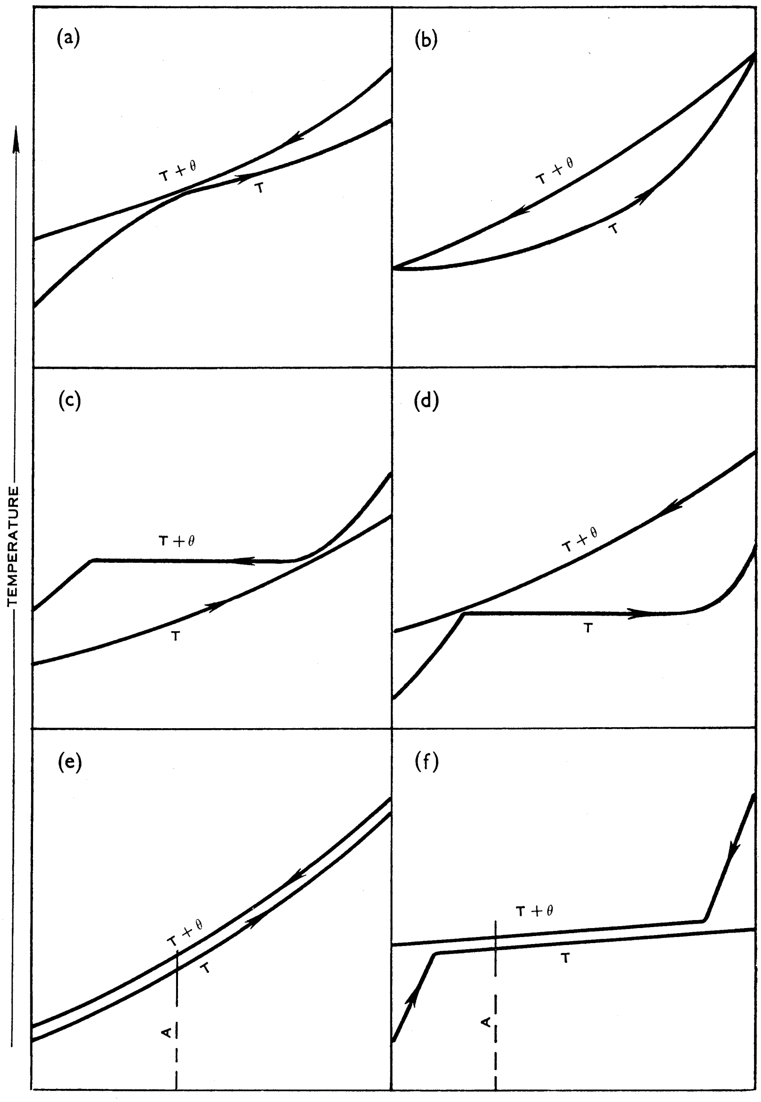

Fig. 2

and

$$\theta = (T+\theta) - T^* - \frac{g_1}{g_2}\int_{T^*+\theta^*}^{T+\theta}\frac{c_1}{c_2}\,\mathrm{d}(T+\theta) \geqslant 0. \tag{15a}$$

Finally, if $\dot{c}_2^*/c_2^* = \dot{c}_1^*/c_1^*$, there is no true maximum or minimum (see Fig. 2(e)).

V. Analysis of Results

We have assumed so far that the heat capacities have varied continuously with the temperature. However, if phase changes occur in the exchanger, either or both c_1 and c_2 may become unbounded at certain temperatures. If c_1 becomes infinite within the operating range of $(T+\theta)$ then, according to equation (7a), the $(T+\theta)$ versus x curve shows a horizontal step. This is represented in Figure 2(c). Similarly, if c_2 becomes infinite in the operating range, the T versus x curve will show a horizontal step as illustrated in Figure 2(d). In either case, real values of T^* with vanishing values of θ^* exist: in Figure 2(c) T^* occurs at the right-hand side of the $(T+\theta)$ step and in Figure 2(d), at the left-hand side of the T step. In these instances, the value of θ is not a function of the flow rate, g_2/g_1.

In systems in which phase changes occur in both streams the dominating step will be the longer one. If the hotter stream passes a larger fraction of its total enthalpy in the form of latent heat of transition then the exchange characteristics will be similar to the curve of Figure 2(c), but if the acceptor stream exchanges a larger fraction in the form of latent heat, the T curve will control the value of T^*.

Finally, consideration must be given to exchange systems in which $\mathrm{d}\theta/\mathrm{d}T$ may vanish over a range of values of T. If the ratio c_1/c_2 is constant at all values of temperature within the range of the exchanger (a simple example occurs when both heat capacities are temperature independent), then equation (8) may be satisfied for any value of the temperature by a suitable flow rate. The $(T+\theta)$ versus x and the T versus x curves will be parallel (Fig. 2(e)). We cannot, however, superimpose the T and $(T+\theta)$ curves, since these, as implied by equation (7a), would be horizontal and there would be no heat exchange. The distance between these parallel lines may be made vanishingly small by increasing the value of h or the exchange area.

The curves of $(T+\theta)$ and T may also be made parallel and sensibly superimposed if the two streams exchange latent heats with only an infinitesimal temperature difference (Fig. 2(f)). A specific example occurs if the exchanger is used to boil water from heat delivered from steam at an infinitesimally elevated pressure (multiple-effect system). At the end of the exchanger, however, where one or both of the streams become a single-phase system, the relevant specific heats will change from infinite to finite values. Such a change, according to equation (7a) is possible only if the value of θ changes from infinitesimal to finite. The upper stream must thus change completely to the high-temperature phase earlier than the lower stream, and, at the opposite end of the exchanger, completely to the lower-temperature phase later than the lower stream. In Figure 2(f) the upper stream is shown entering the system at a temperature above the transition point, while the lower stream enters at a temperature below the transition point and is discharged at the transition point.

In discussing the performance of a heat exchanger in terms of the minimum attainable temperature difference, we must therefore distinguish between three different classes, two of which include subclasses.

Class I, in which a unique T* *exists.*—This class consists of two subclasses: in one, illustrated in Figure 2(a), the T^* is sensitive to flow rate and in the other, illustrated in Figures 2(c) and 2(d), the T^* does not depend on the flow rate within a certain range.

Class II, in which T* *is associated with a maximum* θ^*.—This is illustrated in Figure 2(*b*).

Class III in which a range of T*'s *exist.* This is illustrated in Figures 2(*e*) and 2(*f*).

VI. Thermodynamical Properties of Minimum Temperature Differences

In classes I and III, in which at least one value of T^* exists, the temperature difference between the two streams becomes infinitesimal and at this point (or points), the streams are in thermal equilibrium. We accordingly may write

$$g_1 \int_{T_0+\theta_0}^{T^*} c_1 \, \mathrm{d}T = g_2 \int_{T_0}^{T^*} c_2 \, \mathrm{d}T. \tag{16}$$

The integrals in this equation, since the exchanger operates under approximately isothermal conditions, are the enthalpy changes proceeding from the terminal points to the pivotal temperature T^*. These we denote by ΔH_1^* and ΔH_2^* respectively and equation (16) becomes

$$g_1 \, \Delta H_1^* = g_2 \, \Delta H_2^*, \tag{16a}$$

but, since equation (13) yields

$$g_1 c_1^* = g_2 c_2^*,$$

we have

$$c_1^*/c_2^* = \Delta H_1^*/\Delta H_2^*, \tag{17}$$

which, in effect, defines the value of θ_0 in terms of the pivotal properties.

The quantities ΔH_1^* and ΔH_2^* have perfectly definite meanings for class I systems. For class III systems, the values of these quantities are arbitrary within certain ranges. However, at a specified value T_a^*, in the acceptable range of pivotal temperature, we may define an enthalpy change

$$\int_{T_0+\theta_0}^{T_a^*} c_1 \, \mathrm{d}T \quad \text{as } \Delta H_{1a}^*.$$

The specific heats at T_a^* now become unbounded and equation (17) becomes

$$\Delta H_{1a}^*/\Delta H_{2a}^* = g_2/g_1.$$

In the case of class II systems, there exists at T^* a value of θ which attains a maximum, say, θ^*. Accordingly, the two streams cannot be brought into thermal equilibrium at any point within the exchanger, in spite of the fact that there is instantaneously no net flow at $T = T^*$.

The condition

$$g\Delta H_1^* = g\Delta H_2^*$$

can only be established by the equality of the definite integrals

$$\Delta H_1^* = \int_{T_0+\theta_0}^{T^*+\theta^*} c_1 \, \mathrm{d}T,$$

$$\Delta H_2^* = \int_{T_0}^{T^*} c_2 \, \mathrm{d}T.$$

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