COMPLEX SINGULARITIES IN PRODUCTION AMPLITUDES

By J. Cunningham*

[*Manuscript received November* 25, 1963]

Summary

Complex singularities in the $\pi+N \rightarrow \pi+\pi+N$ amplitude corresponding to vertex contractions of the single-loop Feynman diagram exist on the physical sheet of the amplitude irrespective of whether the theory is formulated in terms of time-ordered or retarded products.

I. Introduction

The amplitude studied in lowest-order perturbation theory which describes the inelastic scattering of a pion on a nucleon is usually the Fourier transform of a time-ordered (T) product:

$$\int \frac{\mathrm{d}^4k_1 \ldots \mathrm{d}^4k_5 \delta(p+k-p'-k'-k'')}{(k_1^2-\mu_1^2+i\epsilon) \ldots (k_5^2-\mu_5^2+i\epsilon)} \delta(p+k_1-k_2)\delta(k+k_2-k_3)$$

$$\times \delta(-p'+k_3-k_4)\delta(-k'+k_4-k_5). \tag{1.1}$$

The four momenta of the incoming nucleon and pion are p and k, while those of the outgoing nucleon and pions are p', k', and k''. The quantities μ_i are the masses of the intermediate particles.

Now in field theory the equation (1.1) can be written in terms of a retarded (R) product rather than a T product. This involves changing the signs of the imaginary parts of the denominators in a well-defined way. The question thus arises whether or not the definition of the Feynman function is thereby altered—in the physical scattering regions the R and T product formulations are identical, but what is important is whether or not we are dealing in both cases with the same sheet of the amplitude. A change in the sign of the imaginary parts may alter the prescription of how we must thread our way round the branch points to reach the physical sheet.

Various authors (Kim 1961; Landshoff and Treiman 1961; Cunningham 1962) have shown, using a variety of different variables, that in the single-loop graph for the process $\pi+N \rightarrow \pi+\pi+N$ complex singularities always exist on the physical sheet due to complex singularities of certain contracted diagrams of the vertex type. All these authors have, however, made assumptions: (1) that the physical singularities arising from various contracted diagrams are the physical singularities which one would obtain by treating each contracted graph as if it were itself a leading graph, (2) that the theory has been formulated in terms of T products.

We shall show that assumption (1) is justified and that assumption (2) need not be made in this particular example.

* Department of Applied Mathematics, University College of North Wales, Bangor, U.K.

II. Contracted Diagrams

The physical singularities of any given graph occur as physical singularities in all higher graphs which possess this graph as a contraction. Such an assumption is implicit in the inductive arguments which support the Mandelstam representation (Eden *et al.* 1961*a*, 1961*b*; Landshoff, Polkinghorne, and Taylor 1961).

Let the amplitude considered be an integral of the structure

$$I = \int_A^B f(q,z)g(q,z)\,\mathrm{d}q, \tag{2.1}$$

where z summarizes the variables which describe the scattering process and q represents the integration variables.

Suppose that we are interested in the contraction which, when treated in its own right as a leading curve, has the form

$$I_{\mathrm{c}} = \int_A^B f(q,z)\,\mathrm{d}q. \tag{2.2}$$

Integrate by parts:

$$I = [F(B,z)g(B,z)-F(A,z)g(A,z)]-\int_A^B F(q,z)g'(q,z)\,\mathrm{d}q, \tag{2.3}$$

where

$$F(q,z) = \int^q f(q,z)\,\mathrm{d}q.$$

Equation (2.3) will remain valid, as we vary z, so long as each term remains an analytic function of z.

I possesses singularities of five types:

(*a*) end point singularities of f,
(*b*) end point singularities of g,
(*c*) pinch singularities of f,
(*d*) pinch singularities of g,
(*e*) singularities due to both f and g.

The first term on the right-hand side of equation (2.3) possesses branch points of the types (*a*) and (*c*) only (also poles corresponding to branch points of I of the type (*b*)); the second term contains all types of singularity listed above.

Now, if we wish to continue equation (2.3) from a region where both sides are regular to a point suspected of being a singularity of type (*a*) or type (*c*), deformations of the path are necessary to avoid encountering other types of singularity. The definition of the physical sheet prescribes the way in which the higher order branch cuts must be threaded. Such deformations are irrelevant in the case of the first term on the right-hand side because any two paths can be deformed into one another without crossing a singularity of the type (*b*), (*d*), or (*e*). Thus, in general, the left-hand integral possesses the singularities of the first term on the right-hand side—and this term may be treated as if singularities other than those of I_c did not exist.

This proves the stated result.

III. Real Continuations

Let us consider an amplitude $F(z)$ which has been transformed, in the usual fashion, to give a multiple integral of a denominator function $D(\alpha,z)$ over a set of Feynman parameters α_i. Let us continue $F(z)$ from a region of the real z-plane where the denominator D does not vanish for α_i real; then we need not deform our original contours, provided our path in the z-plane is real, until we *first* encounter a Landau curve which corresponds to α_i values lying between zero and unity. Furthermore, $F(z)$ must be singular at this point irrespective of the signs of $i\epsilon$ in the denominator.

Consider the denominator $D(\alpha,z)$: with z real, the zeros of $D(\alpha,z)$ are necessarily real or occur in complex conjugate pairs. Let us suppose that we are performing the α_1 integration. If, as we vary z through real values, the denominator, which has been non-zero on our contours of integration, suddenly vanishes, one of two things must have occurred: either (1) a real zero of D has collided with the end point $\alpha_1 = 0$, i.e. $D(0, \alpha_2, \alpha_3, \ldots, \alpha_n, z) = 0$, or (2) a complex conjugate pair of zeros has pinched the undeformed α_1 contour, i.e. $\partial D/\partial\alpha_1 = 0$; in both cases (1) and (2) we insist that $0 \leqslant \alpha_i \leqslant 1$ because we have not deformed our contours. Now, if, when D first vanished, we had been considering the α_r integration we should have concluded that either (1) $D(\alpha_1, \ldots, \alpha_{r-1}, 0, \alpha_{r+1}, \ldots \alpha_n, z) = 0$ or (2) $\partial D/\partial\alpha_r = 0$ for $0 \leqslant \alpha_i \leqslant 1$. Thus, as we vary z in the real plane, when we first reach a point where the denominator vanishes in the region of integration $0 \leqslant \alpha_i \leqslant 1$, it is evident that we have encountered a Landau curve whereon $0 \leqslant \alpha_i \leqslant 1$, since a Landau curve is a locus $\alpha_i\partial D/\partial\alpha_i = 0$ for *all* $i = 1, 2, \ldots, n$.

This proves the first part of the theorem.

Suppose that the above situation arises for $z = \bar{z}$ corresponding to the value $\alpha = \bar{\alpha}$ where $0 \leqslant \bar{\alpha}_i \leqslant 1$. Then, when we look at the α_1 integration, either we have an end point $\alpha_1 = 0$, or a complex conjugate pair of α_1's pinch the contour. When we proceed to the α_2 integration the critical value of α_1 is $\bar{\alpha}_1$, so that we wish to examine the α_2 integration with α_1 fixed at the real value $\bar{\alpha}_1$; in this way it is clear that either $\alpha_2 = 0$ or a complex conjugate pair of α_2's has pinched the α_2 contour. So proceeding, we see that each integration has either an end point or pinch configuration of the zeros of D: thus we conclude that $F(z)$ is singular at $z = \bar{z}$.

It is vital to realize that this result applies only to the *first* singularity encountered, because, to reach a second singularity, contour deformation may be required and the above argument will fail to hold. It is only the first singularity for which the α_i necessarily lie in the range $0 \leqslant \alpha_i \leqslant 1$.

IV. Conclusion

The vertex contractions of the five-point single loop which give rise to the complex singularities referred to in (I) can be reached in the fashion described in (III).

Thus we conclude that the complex singularities are present regardless of the type of product which has been used to formulate the theory.

V. References

CUNNINGHAM, J. (1962).—Ph.D. Thesis, University of Edinburgh.

EDEN, R. J., LANDSHOFF, P. V., POLKINGHORNE, J. C., and TAYLOR, J. C. (1961*a*).—*Phys. Rev.* **122**: 307.

EDEN, R. J., LANDSHOFF, P. V., POLKINGHORNE, J. C., and TAYLOR, J. C. (1961*b*).—*J. Math. Phys.* **2**: 636.

KIM, Y. S. (1961).—*Phys. Rev. Letters* **6**: 313.

LANDSHOFF, P. V., and TREIMAN, S. B. (1961).—*Nuovo Cim.* **19**: 1249.

LANDSHOFF, P. V., POLKINGHORNE, J. C., and TAYLOR, J. C. (1961).—*Nuovo Cim.* **19**: 939.