

AVERAGE RADIATION-PRESSURE FORCES PRODUCED BY SOUND FIELDS

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Summary

Radiation-pressure forces developed within an acoustic system may be calculated from a detailed knowledge of the fields. It is shown that the time average of a generalized radiation-pressure force may be expressed in terms of quantities on an arbitrary mathematical surface enclosing the system.

The resulting surface integral is related to directly measurable circuit-theory parameters involving the impedance, admittance, or scattering matrices of the system. These results are the same as those obtained previously for electromagnetic systems (Smith 1961, 1964). The system must be linear and free of energy loss mechanisms but there is no limitation to the frequency of excitation.

I. INTRODUCTION

Radiation forces exerted in the presence of radiant energy alone occur quite generally. Here we shall be concerned primarily with those resulting from sound fields. A recent review by Kanevskii (1961) discusses the general aspects of forces in sound fields including radiation pressures. A more specific review by Borgnis (1953) deals with acoustic radiation pressure particularly in relation to plane waves. Radiation pressures may be computed in any situation when the radiation field has been found.

The calculation and observation of radiation pressures are of some practical importance. Lord Rayleigh (1878) examined the forces exerted on a small disk suspended in a sound field and derived an expression for the resulting couple by solving the field problem. A device based on the measurement of this couple became known as the Rayleigh disk. A feature of the Rayleigh disk is that it provides a means of absolute measurement of sound intensity since the radiation-pressure force is calculable. Before the advent of the electro-acoustic reciprocity techniques of microphone calibration, the Rayleigh disk was a widely used instrument for absolute calibration purposes (Hunter 1957, p. 318). Other instruments based on the measurement of radiation pressures in the ultrasonic region have been built (e.g. Gabrielli and Iernetti 1963). Westervelt (1951, 1957) has given a formula for computing the radiation force from an incident plane wave on a single scatterer in terms of scattering and absorption coefficients. This formula may be looked upon as the result of an application of the momentum conservation laws to the incident, absorbed, and scattered radiation. The formula expresses the radiation force for this particular situation in terms of parameters (scattering and absorption coefficients) which, although perhaps calculable only from a detailed solution of the field problem, *are in principle directly measurable*.

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The present paper arrives at general expressions for radiation-pressure induced forces in terms of measurable scattering or impedance parameters. A time-averaged generalized force $\bar{F}_x$ produced by radiation pressures is expressed in terms of an integral over a mathematical surface enclosing the system. This surface integral may be reduced to admittance, impedance, or scattering-matrix forms containing parameters which in principle *are directly measurable*. *Absolute* calibration of an instrument which measures such a force may then be made.

The results in terms of the impedance, admittance, or scattering matrix are identical to those obtained in an entirely different context for electrical radiation-pressure forces (Smith 1960, 1961, 1964). It was in fact conjectured (Smith 1960) that the electrical results could be extended to the acoustic context. It appears that the results have even wider generality for wave motion, being of the form of energy conservation on the average.

The results are obtained for an ideal fluid with classical boundary conditions in the usual acoustic region of infinitesimal velocities. No account is taken of viscosity or thermal conduction effects. These approximations are those of the usual linear, loss-free theory of sound.

Sections II and III state basic equations required for describing the sound field. In Section IV the general surface integral form of the average generalized force $\bar{F}_x$ is obtained. A description of the behaviour of the system in terms of impedance, admittance, or scattering matrices is introduced in Section V and used to reduce the general surface integral. Section VI discusses possible applications of the theory to the absolute calibration of intensity measuring instruments. Since the equations are formally the same as those of the electrical situation, procedures based on electrical formulae are possible. An extension of the technique of Cullen (1952) for the absolute calibration of microwave power meters is discussed. Some limitations are referred to in Section VII.

II. BASIC ACOUSTICAL EQUATIONS

For the small departures from static equilibrium contemplated in the theory of sound the Euler equation,

$$D\mathbf{v}/Dt = -(1/\rho)\nabla P, \quad (1)$$

(with ρ the fluid density, P the pressure, $\mathbf{v}$ the velocity, and t the time) may be written in the linearized form

$$\partial\mathbf{v}/\partial t = -(1/\rho_0)\nabla P, \quad (2)$$

where ρ_0 is the equilibrium density.

The equation of continuity

$$\nabla \cdot (\rho\mathbf{v}) + \partial\rho/\partial t = 0 \quad (3)$$

may also be written correct to first-order terms as

$$\rho_0\nabla \cdot \mathbf{v} + \partial\rho/\partial t = 0 \quad (4)$$

(for simplicity it is supposed that ρ_0 is independent of position). A constitutive equation for the sound disturbances completes the system of equations. The constitutive equation is assumed to be of the loss-free form

$$P = P(\rho), \quad (5)$$

shear components of stress being zero. The boundary conditions are those of classical hydrodynamics, that is,

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad (6)$$

with $\mathbf{n}$ a unit normal on a rigid boundary surface.

The condensation s is defined by

$$\rho = \rho_0(1+s). \quad (7)$$

To first-order

$$\rho_0 s = (P - P_0)/c^2, \quad (8)$$

where c is the sound velocity given by

$$c^2 = P'(\rho_0), \quad (9)$$

the subscript 0 denoting an equilibrium value. Equations (2), (4), (7), and (8) lead to the wave equations

$$\{\nabla^2 - (1/c^2)\partial^2/\partial t^2\} \begin{bmatrix} P \\ \mathbf{v} \\ s \end{bmatrix} = 0. \quad (10)$$

For calculating the sound fields, quadratic terms are neglected and the resulting equations are linear. The radiation-pressure forces are quadratic in the sound field amplitudes and the linear approximation is adequate for finding the dominant contribution to these quadratic effects. This linearity allows the synthesis of a general time dependence from a superposition of single Fourier components by way of Fourier theory. For the present, a single Fourier component with time dependence $\exp(i\omega t)$ will be considered. Equations (2), (4), and (8) become

$$i\omega\rho_0\mathbf{u} = -\nabla p, \quad (11)$$

$$(i\rho_0 c^2/\omega)\nabla \cdot \mathbf{u} = p, \quad (12)$$

$$\rho_0 z = p/c^2, \quad (13)$$

where p , $\mathbf{u}$, and z are the complex r.m.s. amplitudes associated with $P - P_0$, $\mathbf{v}$, and s , that is,

$$\left. \begin{aligned} P - P_0 &= \mathcal{R} \cdot (\sqrt{2} \cdot p e^{i\omega t}), \\ \mathbf{v} &= \mathcal{R} \cdot (\sqrt{2} \cdot \mathbf{u} e^{i\omega t}), \\ s &= \mathcal{R} \cdot (\sqrt{2} \cdot z e^{i\omega t}). \end{aligned} \right\} \quad (14)$$

Equations (11), (12), and (13) will be taken as the basic set. Elimination of p or $\mathbf{u}$ in (11) and (12) gives

$$(\nabla^2 + \omega^2/c^2) \begin{bmatrix} p \\ \mathbf{u} \\ z \end{bmatrix} = 0. \quad (15)$$

III. MOMENTUM FLUX DENSITY TENSOR

For computing radiation-pressure forces some form of radiation stress tensor is required. For our purposes it will be satisfactory to make use of the following momentum integral of the Euler equation (e.g. Landau and Lifshitz 1959, Section 7)

$$(\partial/\partial t) \int_V \int \rho v_j \, d\tau = \int_S \Pi_{jk} \, dA_k, \quad (16)$$

with

$$\Pi_{jk} = P\delta_{jk} + \rho v_j v_k, \quad (17)$$

where v_j are Cartesian components of $\mathbf{v}$, dA_k are Cartesian components of surface area enclosing the volume V (*inward* normal, positive), δ_{jk} is the Kronecker delta, and the usual summation over repeated subscripts is assumed. Π_{jk} may then be interpreted as a momentum flux density tensor relating to the rate of transport of momentum across a surface.

The time average $\bar{\Pi}_{jk}$ of the Π_{jk} obtained from the sound field may be regarded as giving rise to radiation stresses, leading to the Brillouin radiation pressure tensor for a plane wave. This tensor will also include any contribution from any mean velocity (sonic wind). Post (1960) has discussed some alternative stress tensors. A similar problem arises in electromagnetic theory where the Maxwell stress tensor is normally used although other tensors have been proposed from time to time (e.g. Livens 1929). The differences are important only when considering details of the volume distribution of forces. For the present purposes only forces on rigid boundaries will be required and there is no difficulty. Because of the boundary conditions (6) only the first term of (17) contributes to the integral in (16) so that for computing forces on boundaries $\bar{P}\delta_{jk}$ is sufficient, i.e. an isotropic pressure $\bar{P}$. ($\bar{P} - P_0$) is known as the excess pressure in acoustic theory.

IV. GENERAL EXPRESSION FOR THE AVERAGE FORCE

In this section a general expression for the time-average generalized force $\bar{F}_x$ will be found initially for sinusoidal excitation. The acoustic system to be considered is supposed to be excited from external sources. $\bar{F}_x$ is the average generalized force corresponding to a generalized coordinate x relating to the configuration of the system. As a simple practical example suppose x is an angle specifying the orientation of a vane, then $\bar{F}_x$ is simply the average torque acting upon the vane.

A small arbitrarily slow change in the system is considered to be generated by a change δx in the parameter x . The total mechanical work done by the system

on the boundaries during this (adiabatic) change is to be computed and equated to the work done by the average generalized force $\bar{F}_x$. The displacements contemplated are displacements of surfaces on which the boundary conditions (6) apply.

Preliminary Results

(i) From equation (11) with $\delta\omega = 0$,

$$i\omega\rho_0\delta\mathbf{u}^* = \nabla(\delta p^*). \quad (18)$$

Therefore

$$\delta\mathbf{u}^* \cdot \nabla p = \{1/(i\omega\rho_0)\nabla(\delta p^*)\} \cdot \nabla p. \quad (19)$$

Further use of (11) gives

$$\delta\mathbf{u}^* \cdot \nabla p = -\mathbf{u} \cdot \nabla(\delta p^*). \quad (20)$$

(ii) Similarly from equation (12)

$$\delta p^* = (-i\rho_0 c^2/\omega)\nabla \cdot (\delta\mathbf{u}^*) \quad (21)$$

and

$$\delta p^* \nabla \cdot \mathbf{u} = (-i\rho_0 c^2/\omega)\nabla \cdot (\delta\mathbf{u}^*) \nabla \cdot \mathbf{u}. \quad (22)$$

Further use of equation (12) gives

$$\delta p^* \nabla \cdot \mathbf{u} = -p \nabla \cdot (\delta\mathbf{u}^*). \quad (23)$$

Equations (20) and (23) embody the dissipationless character of the system and are vital to the proof.

Suppose the system is enclosed by a mathematical surface consisting perhaps of some boundary surface S_0 together with a surface S . The interior of S and S_0 will in general contain further closed boundary surfaces $S_1, S_2, \dots$

Consider

$$\int \int_{S, S_0, S_1, S_2, \dots} p\mathbf{u}^* \cdot d\mathbf{A}, \quad (24)$$

using the convention that the positive direction for the vector element of area $d\mathbf{A}$ is *inward* for S_0 and S and *outward* for $S_1, S_2, \dots$. Because of the boundary conditions (6) on $S_0, S_1, S_2, \dots$

$$\begin{aligned} \int \int_S p\mathbf{u}^* \cdot d\mathbf{A} &= \int \int_{S, S_0, S_1, S_2, \dots} p\mathbf{u}^* \cdot d\mathbf{A}, \\ &= - \int \int \int_V \nabla \cdot (p\mathbf{u}^*) d\tau, \end{aligned} \quad (25)$$

by Gauss's divergence theorem, where V is the volume enclosed by S and S_0 excluding the interiors of $S_1, S_2, \dots$

Then

$$\int_S p \mathbf{u}^* \cdot d\mathbf{A} = - \int_V \int (\mathbf{u}^* \cdot \nabla p + p \nabla \cdot \mathbf{u}^*) d\tau \quad (26)$$

$$= i\omega \int_V \int (\rho_0 \mathbf{u}^* \cdot \mathbf{u} - p p^* / c^2 \rho_0) d\tau, \quad (27)$$

by use of equations (11) and (12).

In the above equation the first term on the right in the integrand is twice the average kinetic energy density of the wave motion and the second term is twice the average potential energy density (thermodynamically, the internal energy density for the adiabatic conditions of sound compression).

A small variation δ of equations (26) and (27) is now considered. $\delta\omega$ is supposed zero, the system being excited from outside V at a fixed angular frequency. [A variation of equation (27) with $\delta\omega \neq 0$ and the left-hand side zero leads to the resonator action, or adiabatic theorem

$$\delta(T/\omega) = 0, \quad (28)$$

where T is the total energy of the free oscillation of the resulting resonator.]

Then

$$\delta \int_S p \mathbf{u}^* \cdot d\mathbf{A} = i\omega \int_V \int (\rho_0 \mathbf{u} \cdot \mathbf{u}^* - p p^* / c^2 \rho_0) d\tau - \int_V \int \delta(\mathbf{u}^* \cdot \nabla p + p \nabla \cdot \mathbf{u}^*) d\tau, \quad (29)$$

where the integral over δV accounts for movements or deformations of the surfaces $S_0, S_1, S_2, \dots$ generated by δx .

Further

$$\int_V \int \delta(\mathbf{u}^* \cdot \nabla p + p \nabla \cdot \mathbf{u}^*) d\tau = \int_V \int \{ \mathbf{u}^* \cdot \nabla(\delta p) + \delta \mathbf{u}^* \cdot \nabla p + \delta p \nabla \cdot \mathbf{u}^* + p \nabla \cdot (\delta \mathbf{u}^*) \} d\tau \quad (30)$$

$$= \int_V \int \{ \mathbf{u}^* \cdot \nabla(\delta p) - \mathbf{u} \cdot \nabla(\delta p^*) + \delta p \nabla \cdot \mathbf{u}^* - \delta p^* \nabla \cdot \mathbf{u} \} d\tau \quad (31)$$

using (20) and (23),

$$= \int_V \int \nabla \cdot (\delta p \mathbf{u}^* - \delta p^* \mathbf{u}) d\tau \quad (32)$$

$$= \int_S (\mathbf{u} \delta p^* - \mathbf{u}^* \delta p) \cdot d\mathbf{A}, \quad (33)$$

using Gauss's theorem and the fact that $\mathbf{u}$ is tangential on $S_0, S_1, S_2, \dots$ because of the boundary conditions (6). The combination of equations (33) and (29) gives

$$\int_S (p\delta\mathbf{u}^* + \delta p^*\mathbf{u}) \cdot d\mathbf{A} = i\omega \int \int \int_{\delta V} (\rho_0\mathbf{u} \cdot \mathbf{u}^* - pp^*/c^2\rho_0) d\tau. \quad (34)$$

The right-hand side of equation (34) may be expressed in terms of the work done by radiation-pressure forces on the boundaries. In Section III it was shown that the average isotropic pressure $\bar{P}$ is operative as a radiation pressure on boundary surfaces. We now compute $\bar{P}$ by averaging the Euler equation (1) correct to terms of second-order in the sound amplitudes. Including the uniform equilibrium pressure P_0 we have

$$\nabla(\bar{P} - P_0) = -\langle \rho D\mathbf{v}/Dt \rangle \quad (35)$$

$$= -\langle \rho \nabla(\frac{1}{2}v^2) \rangle - \langle \rho \partial\mathbf{v}/\partial t \rangle - \langle \rho \mathbf{v} \times \text{curl } \mathbf{v} \rangle. \quad (36)$$

Now consider contributions to second-order in the right-hand side of equation (36).

The first term becomes

$$-\rho_0 \nabla(\frac{1}{2}\langle v^2 \rangle) = -\nabla(\frac{1}{2}\rho_0 \langle v^2 \rangle). \quad (37)$$

The second term is

$$-\rho_0 \langle \partial\mathbf{v}/\partial t \rangle - \rho_0 \langle s \partial\mathbf{v}/\partial t \rangle. \quad (38)$$

However, $\mathbf{v}$ is sinusoidal to first-order at least, thus

$$\rho_0 \langle \partial\mathbf{v}/\partial t \rangle = 0, \quad (39)$$

to second-order.

Also

$$\rho_0 s = (P - P_0)/c^2, \quad (40)$$

to first-order,

and

$$\partial\mathbf{v}/\partial t = -(1/\rho_0)\nabla(P - P_0), \quad (41)$$

to first-order.

Therefore (38) becomes

$$\langle \rho \partial\mathbf{v}/\partial t \rangle = -\langle (P - P_0)\nabla(P - P_0) \rangle / \rho_0 c^2, \quad (42)$$

to second-order,

$$= -\nabla\{\langle (P - P_0)^2 \rangle / 2\rho_0 c^2\}. \quad (43)$$

The third term of equation (36) vanishes to the order considered since

$$\langle \rho \mathbf{v} \times \text{curl } \mathbf{v} \rangle = \rho_0 \langle \mathbf{v} \times \text{curl } \mathbf{v} \rangle \quad (44)$$

and $\mathbf{v}$ is proportional to $\nabla(P - P_0)$ in the linear approximation (equation (11)),

that is, $\mathbf{v}$ is irrotational in the linear theory.

Therefore

$$\langle \rho \mathbf{v} \times \text{curl } \mathbf{v} \rangle = 0 \quad (45)$$

to second-order at least.

Equation (36) finally becomes

$$\nabla(\bar{P} - P_0) = \nabla \left\{ \frac{1}{2} \langle (P - P_0)^2 \rangle / \rho_0 c^2 - \frac{1}{2} \rho_0 \langle v^2 \rangle \right\}, \quad (46)$$

from which

$$\bar{P} - P_0 = \frac{1}{2} \langle (P - P_0)^2 \rangle / \rho_0 c^2 - \frac{1}{2} \rho_0 \langle v^2 \rangle. \quad (47)$$

An equivalent of equation (47) is obtained by Landau and Lifshitz (1959, Section 64) by use of a different method.

Now

$$\langle v^2 \rangle = \mathbf{u} \cdot \mathbf{u}^*$$

and

$$\langle (P - P_0)^2 \rangle = pp^*, \quad (48)$$

therefore

$$\bar{P} - P_0 = \frac{1}{2} pp^* / \rho_0 c^2 - \frac{1}{2} \rho_0 \mathbf{u} \cdot \mathbf{u}^*. \quad (49)$$

Equation (34) may then be written

$$\int_S \int (p \delta \mathbf{u}^* + \delta p^* \mathbf{u}) \cdot d\mathbf{A} = -2i\omega \int \int_{\delta V} (\bar{P} - P_0) d\tau. \quad (50)$$

However, $\int \int_{\delta V} (\bar{P} - P_0) d\tau$ represents the work done by the radiation pressure on the boundary surfaces during the adiabatic displacement δx . This work is to be equated to the work done by the generalized force $\bar{F}_x$ namely,

$$\bar{F}_x \delta x = \int \int_{\delta V} (\bar{P} - P_0) d\tau. \quad (51)$$

Combining (51) with (50) gives

$$2i\omega \bar{F}_x \delta x = - \int_S \int (p \delta \mathbf{u}^* + \delta p^* \mathbf{u}) \cdot d\mathbf{A} \quad (52)$$

$$= \int_S \int (p^* \delta \mathbf{u} + \delta p \mathbf{u}^*) \cdot d\mathbf{A}. \quad (53)$$

Equation (52) or (53) is the general result expressing the average force for sinusoidal excitation in terms of a surface integral over S . Reductions to forms suitable for practical application will be carried out in the next section. It follows from Parseval's

formula of Fourier analysis that the average forces from individual Fourier components are additive, since the right-hand side of equation (53) is quadratic in the amplitudes.

It is of interest to observe that equation (53) is the sound field version of a similar result in electromagnetic theory (Smith 1961) namely,

$$2i\omega\bar{F}_x\delta x = \int_S (\mathbf{E}^* \times \delta \mathbf{H} + \delta \mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{A}, \quad (54)$$

with $p\mathbf{u}^*$ (Landau and Lifshitz 1959, Section 64) playing the role of complex energy flux vector in a similar way to the complex Poynting vector $\mathbf{E} \times \mathbf{H}^*$. The primary difference is that the sound field is a scalar field whilst the electromagnetic field is a vector field.

V. REDUCTION OF THE SURFACE INTEGRAL EXPRESSION FOR $\bar{F}_x$

Suppose there exists a set of independent complex fields $p_r, \mathbf{u}_r$, ($r = 1, 2, \dots$) representing incoming sound waves of angular frequency ω . p_r and $\mathbf{u}_r$ may be expressed in terms of a scalar complex velocity potential ϕ_r by

$$\left. \begin{aligned} \mathbf{u}_r &= \nabla \phi_r, \\ p_r &= -i\omega\rho_0\phi_r, \end{aligned} \right\} \quad (55)$$

which ensures that p_r and $\mathbf{u}_r$ satisfy the basic field equations (11) and (12), provided ϕ_r is a solution of the wave equation (15). Boundary conditions appropriate to the exterior of S and S_0 must be imposed, together with a condition at large distances to correspond to incident waves. Then

$$\left. \begin{aligned} \mathbf{u}_r &= -\nabla \phi_r, \\ p_r &= -i\omega\rho_0\phi_r, \end{aligned} \right\} \quad (56)$$

will represent outgoing wave solutions for the exterior of S and S_0 . A general solution must include the superposition of the fields (55) and (56). We suppose that a superposition of the fields (55) and (56) provides a complete description and that the individual partial fields are selected to be orthonormal in the following sense,

$$\int_S p_r \mathbf{u}_k^* \cdot d\mathbf{A} = \delta_{rk}. \quad (57)$$

The actual p and $\mathbf{u}$ occurring on the surface S may then be written as linear combinations of the $p_r, \mathbf{u}_r$ on S , that is,

$$p = \sum_r V_r p_r, \quad (58)$$

$$\mathbf{u} = \sum_r I_r \mathbf{u}_r. \quad (59)$$

The V_r and I_r above are not independent. Provided special characteristic values of ω are avoided, the Helmholtz theory of the scalar wave equation for the interior of S and S_0 gives a complete solution as soon as the normal component of the velocity $\mathbf{u}$ or the pressure p is specified over the surface. Thus the V_r and I_r of equations (58) and (59) are not independent but are linearly related, that is,

$$V_r = \sum_k Z_{rk} I_k, \quad (60)$$

or in matrix notation

$$\mathbf{V} = \mathbf{Z}\mathbf{I}, \quad (61)$$

where

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ \vdots \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ \vdots \end{bmatrix} \quad (62)$$

and

$$\mathbf{Z} = [Z_{rk}]. \quad (63)$$

$\mathbf{Z}$ is the impedance matrix of the system referred to the set $\mathbf{u}_r$, p_r . Reciprocity (Rayleigh 1878) results in the symmetry of $\mathbf{Z}$, that is,

$$\mathbf{Z}^T = \mathbf{Z}, \quad (64)$$

where index T denotes the transposed matrix. Further, using (57), (58), and (59) ($\dagger$ denoting Hermitian conjugate)

$$\int_S p \mathbf{u}^* \cdot d\mathbf{A} = \mathbf{I}^\dagger \mathbf{V} \quad (65)$$

$$= \mathbf{I}^\dagger \mathbf{Z} \mathbf{I} \quad \text{from (61),} \quad (66)$$

which is equal to the purely imaginary form (27). Therefore $\mathbf{Z}$ is purely imaginary (loss-free condition). The inverse relationship to equation (61) is

$$\mathbf{I} = \mathbf{Y}\mathbf{V}, \quad (67)$$

with

$$\mathbf{Y} = \mathbf{Z}^{-1} = [Y_{rk}]. \quad (68)$$

$\mathbf{Y}$ is the admittance matrix of the system and is symmetric and purely imaginary since $\mathbf{Z}$ is so. By analogy with electric circuit theory $\mathbf{V}$, $\mathbf{I}$, $\mathbf{Z}$, and $\mathbf{Y}$ may be regarded as "terminal parameters" for the enclosed system.

The general result (53) for the average force $\bar{F}_x$ may now be expressed in terms of these terminal parameters. Using equations (58) and (59) in (53),

$$2i\omega \bar{F}_x \delta x = \int_S \left(\sum_r V_r^* p_r^* \sum_k \delta I_k \mathbf{u}_k + \sum_r \delta V_r p_r \sum_k I_k^* \mathbf{u}_k^* \right) \cdot d\mathbf{A} \quad (69)$$

$$= \mathbf{V}^\dagger \delta \mathbf{I} + \mathbf{I}^\dagger \delta \mathbf{V} \quad \text{using (57),} \quad (70)$$

$$= \mathbf{I}^\dagger (\mathbf{Z}^\dagger + \mathbf{Z}) \delta \mathbf{I} + \mathbf{I}^\dagger \delta \mathbf{Z} \mathbf{I} \quad \text{by (61),} \quad (71)$$

$$= \mathbf{I}^\dagger \delta \mathbf{Z} \mathbf{I}, \quad (72)$$

since $\mathbf{Z}$ is symmetric and purely imaginary.

Therefore

$$2i\omega \bar{F}_x = \mathbf{I}^\dagger (\partial \mathbf{Z} / \partial x) \mathbf{I}. \quad (73)$$

Equation (53) may similarly be reduced to

$$2i\omega \bar{F}_x = \mathbf{V}^\dagger (\partial \mathbf{Y} / \partial x) \mathbf{V}. \quad (74)$$

Equations (73) and (74) are the same impedance and admittance matrix forms as obtained for average forces in electrical systems (Smith 1960, 1961). From (70) they may be written alternatively as

$$2i\omega \bar{F}_x = (\partial W / \partial x)_{\mathbf{I}}, \quad (75)$$

$$2i\omega \bar{F}_x = (\partial W^* / \partial x)_{\mathbf{V}}, \quad (76)$$

where

$$W = \int_S p \mathbf{u}^* \cdot d\mathbf{A} = \mathbf{I}^\dagger \mathbf{V}, \quad (77)$$

is the complex power flowing from the sources. The terminal parameters above are by no means unique. Linear transformations of the p_r and $\mathbf{u}_r$ which leave equations (57) and (64) unchanged give other sets of terminal parameters. Sets of p_r , $\mathbf{u}_r$ which satisfy equation (55) are basic in that unit impedance corresponds to the propagating fields of (55). For example, unit impedance for a plane wave corresponds to an actual pressure to longitudinal velocity of $\rho_0 c$, i.e. the intrinsic wave impedance of the medium.

Another reduction of equation (53) in terms of a set of scattering parameters or scattering matrix may be made along the lines of the method used in the corresponding electromagnetic case (Smith 1964). The total sound field is divided into an incident and a scattered field. The incident field is a sum of fields satisfying equations (55), and the scattered field is a sum of fields satisfying equations (56). The total field may then be written

$$p = \sum_r (a_r + b_r) p_r, \quad (78)$$

$$\mathbf{u} = \sum_r (a_r - b_r) \mathbf{u}_r, \quad (79)$$

where a_r , b_r are r.m.s. amplitudes for the incident and scattered waves respectively. The resulting linear relationship between the a_r and b_r may be written

$$\mathbf{b} = \mathbf{S}\mathbf{a}, \quad (80)$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \end{bmatrix}. \quad (81)$$

The matrix $\mathbf{S}$ is the scattering matrix for the set p_r , $\mathbf{u}_r$. $\mathbf{S}$ is symmetric for reciprocal systems, and for the loss-free systems considered in this paper it is unitary also, that is,

$$\mathbf{S}^T = \mathbf{S} \quad (82)$$

and

$$\mathbf{S}^*\mathbf{S} = \mathbf{U}, \quad (83)$$

where $\mathbf{U}$ is a unit matrix.

Equation (82) is the equivalent of (64) and (83) corresponds to the purely imaginary character of $\mathbf{Z}$. The equivalences follow from a comparison of equations (78) and (79) with (58) and (59), which gives

$$\left. \begin{aligned} \mathbf{V} &= \mathbf{a} + \mathbf{b}, \\ \mathbf{I} &= \mathbf{a} - \mathbf{b}, \end{aligned} \right\} \quad (84)$$

from which the usual expressions for the scattering matrix in terms of impedance or admittance matrices (Montgomery, Dicke, and Purcell 1948) follow,

$$\left. \begin{aligned} \mathbf{S} &= (\mathbf{Z} + \mathbf{U})^{-1}(\mathbf{Z} - \mathbf{U}), \\ \mathbf{S} &= (\mathbf{Y} + \mathbf{U})^{-1}(\mathbf{U} - \mathbf{Y}), \end{aligned} \right\} \quad (85)$$

or conversely,

$$\left. \begin{aligned} \mathbf{Z} &= (\mathbf{U} + \mathbf{S})(\mathbf{U} - \mathbf{S})^{-1}, \\ \mathbf{Y} &= (\mathbf{U} + \mathbf{S})^{-1}(\mathbf{U} - \mathbf{S}). \end{aligned} \right\} \quad (86)$$

Substitution for p , $\mathbf{u}$, as given by equations (78) and (79), in the force equation (53) gives

$$2i\omega\bar{F}_x\delta x = \int_S \int_r \{ \Sigma_r (a_r^* + b_r^*) p_r^* \Sigma_k (\delta a_k - \delta b_k) \mathbf{u}_k + \Sigma_r (\delta a_r + \delta b_r) p_r \Sigma_k (a_k^* - b_k^*) \mathbf{u}_k^* \} \cdot d\mathbf{A}, \quad (87)$$

$$= \Sigma_r \{ (a_r^* + b_r^*)(\delta a_r - \delta b_r) + (\delta a_r + \delta b_r)(a_r^* - b_r^*) \} \quad (88)$$

by equation (57),

that is,

$$i\omega\bar{F}_x\delta x = \mathbf{a}^\dagger \delta \mathbf{a} - \mathbf{b}^\dagger \delta \mathbf{b}. \quad (89)$$

Using equation (80), equation (89) may be written

$$i\omega\bar{F}_x\delta x = \mathbf{a}^\dagger(\mathbf{U} - \mathbf{S}^\dagger\mathbf{S})\delta\mathbf{a} - \mathbf{b}^\dagger\delta\mathbf{S}\mathbf{a}, \quad (90)$$

that is,

$$i\omega\bar{F}_x\delta x = -\mathbf{b}^\dagger\delta\mathbf{S}\mathbf{a}, \quad (91)$$

hence

$$i\omega\bar{F}_x = -\mathbf{b}^\dagger(\partial\mathbf{S}/\partial x)\mathbf{a}. \quad (92)$$

Equation (92) is the required scattering matrix expression and is identical to the result obtained for electromagnetic systems (Smith 1964). Equation (92) may also be obtained by substitution from (84) and (86) into (73) or (74).

VI. APPLICATION

An important aspect of the theory is its possible application to the absolute calibration of sound intensity measuring instruments which depend on radiation pressures for their operation. The Rayleigh disk, which is an instrument of this type, is calibrated by a calculation of the field distribution. The present theory in principle enables an absolute calibration to be made in terms of a *measurement* of scattering matrix elements. Moreover, there is no frequency limit and the instrument complexity may be such that a field distribution calculation is impractical. The situation has been discussed in further detail in the electrical context (Smith 1960, 1961, 1964) and the same considerations apply. For the *single scattering* produced by a single obstacle in a *plane wave field* the theory of Westervelt (1957) would suffice, but the present theory, since it does not relate to wave momentum conservation, covers much broader possibilities. In the audible frequency range reciprocity calibrations of microphones have supplanted absolute instruments of the Rayleigh disk type. However, it is interesting to note that reciprocity calibrations are not *fundamentally* absolute since they refer acoustic quantities to electrical quantities. In practice, convenient absolute electrical standards are readily available but their establishment involves some instrument for which the electrical analogues of the present theory could be applied (e.g. current balance, electrostatic voltmeter).

The particular case of a two-port transmission power meter may be examined in some detail. The input is supposed to be a single wave travelling in an input channel 1. The energy flows through the instrument which has a pointer indicating a deflection x , and out of an output channel 2 to a load (non-reflecting termination). A previous discussion based on equation (92) for the electrical case (Smith 1964) is immediately applicable. The average deflection force may be expressed as

$$i\omega\bar{F}_x = -P_2\{(\partial S_{12}/\partial x)/S_{12} - (S_{22}/S_{12}^2)(\partial S_{11}/\partial x)\}, \quad (93)$$

where P_2 is the acoustic power flow. Further, equation (93) leads to a formula obtained by Cullen (1952) for the calibration of microwave transmission power meters

$$\bar{F}_x = -(P_2\pi/\lambda\omega)\{(\partial x_2/\partial x)_{x_1=0} + (\partial x_2/\partial x)_{x_1=\lambda/4}\}, \quad (94)$$

where λ is the wavelength in the output channel. The derivatives $(\partial x_2/\partial x)$ are

defined in terms of a measurement which preserves the standing wave pattern in the input channel as the position of a fully reflecting termination at a distance x_2 along the output channel is changed. The type of fully reflecting termination, e.g. $u = 0$ or $p = 0$, does not matter. Cullen's formula (equation (94)) is thus applicable to acoustic systems.

The simplest transmission power meter measures the total tension in the channel boundaries. Let x be a coordinate specifying the length of a section of the channel. In equation (93)

$$\left. \begin{aligned} S_{11} &= 0 \\ S_{12} &= \exp\{i\theta - 2\pi ix/\lambda\}, \\ &(\theta \text{ a real constant}) \end{aligned} \right\} \quad (95)$$

giving

$$\bar{F}_x = P_2/c. \quad (96)$$

This same tension occurs in electromagnetic transmission lines and waveguides. It is half the tension exerted by total reflection. The force is independent of frequency, a condition making for an ideal instrument.

VII. LIMITATIONS

The development leading to equation (53) has been based on the linear loss-free theory of sound. The restriction to the linear theory is not important for the usual range of sound amplitudes encountered. However, the effects of the loss-free approximation are probably more serious and more difficult to estimate. The equation of state (5) and the Euler equation (1) ignore any effects of viscosity or thermal conduction. The boundary conditions on the rigid walls have been taken to be those of classical hydrodynamics. The forces calculated are consequently those relating to a highly idealized system. A similar idealization in electromagnetic theory is often warranted, but the validity of this procedure for acoustic systems has not been demonstrated. Generalizations to more general elastic systems which include solids and inhomogeneous media might be expected to follow. However, when deformations of the transmitting medium are generated by δx , the choice of the correct momentum flux density tensor for the wave motion is most important.

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