GAUGE-INDEPENDENT THEORY OF SYMMETRY. I

By L. J. TASSIE* and H. A. BUCHDAHL*

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Summary

The invariance of a system under a given transformation of coordinates is usually taken to mean that its Lagrangian is invariant under that transformation. Consequently, whether or not the system is invariant will depend on the gauge used in describing the system. By defining invariance of a system to mean the invariance of its equations of motion, a gauge-independent theory of symmetry properties is obtained for classical mechanics in both the Lagrangian and Hamiltonian forms. The conserved quantities associated with continuous symmetry transformations are obtained. The system of a single particle moving in a given electromagnetic field is considered in detail for various symmetries of the electromagnetic field, and the appropriate conserved quantities are found.

I. INTRODUCTION

The symmetry properties of a system and the related conservation theorems are of great importance in both classical and quantum physics. In the usual treatment a system is taken to be invariant under a particular operation if the Lagrangian or the Hamiltonian is invariant in form under that operation. The momentum p_i, canonically conjugate to a Cartesian coordinate x_i, is then the generator of translations along the x_i-axis and the component of canonical angular momentum $l_i = x_j p_k - x_k p_j$ $(i, j, k$ cyclic) is the generator of rotations about the x_i-axis.

In general, the canonical momentum $\mathbf{p}$ and the kinetic momentum $m\mathbf{v}$ are not identical, for

$$m\mathbf{v} = \mathbf{p} - e\mathbf{A}/c. \tag{1.1}$$

Here $\mathbf{A}$ may be the vector potential of an electromagnetic field; but even when there is no such field the appearance of the second term on the right of (1.1) may be brought about by a gauge transformation. This means, however, that any symmetry of the system as *formally* defined above may be destroyed by a gauge transformation. For example, if a Lagrangian is invariant under rotations about a certain axis then it will no longer have this property after the gauge transformation,

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \operatorname{grad} \chi, \tag{1.2}$$

has been made, whenever χ *is not* invariant under rotations about the axis in question. (Whether a particle is charged or not is of no account in this context, since in the latter case e may simply be regarded as an arbitrary constant.)

* Department of Theoretical Physics, School of General Studies, Australian National University, Canberra.

Now a gauge transformation is a purely formal device which has no physical consequences. It would therefore surely be better to have a formalism in which the symmetry properties of a system are gauge-independent. When it is recalled that the transformation (1.2) is tantamount to the addition of a total time derivative to the Lagrangian, this formalism may be expected to centre around the freedom one has of adding a total time derivative to any Lagrangian without affecting thereby the equations of motion. After all, physically it is natural to think of invariance under a symmetry operation to mean, in the first place, invariance of the equations of motion. Proceeding along these lines it then turns out that the canonical momenta are not necessarily the generators of translations, nor are the canonical angular momenta necessarily the generators of rotations.

The problem under consideration arises in classical as well as in quantum mechanics, whether of particles or of fields. Since its essential features already appear in the simplest case, namely that of classical particle mechanics, the present paper concerns itself exclusively with the latter. Sections II–IV deal with the Lagrangian and Hamiltonian formulations, whilst various explicit examples are provided in Section V.

II. Lagrangian Formulation

A system shall be defined as being invariant under a particular transformation of the coordinates if the equations of motion are invariant under that transformation. Their invariance is assured if the change in the Lagrangian is the total time derivative of a function of the coordinates and the time. Thus, let the given transformation be written

$$q_k' = g_k'(q,t), \tag{2.1}$$

the inverse of which shall be

$$q_k = g_k(q',t). \tag{2.2}$$

Then if $L(\dot{q},q,t)$ is the given Lagrangian, the function $L(\dot{q}',q',t) = L(\dot{g}(q,t),g(q,t),t)$ shall differ from $L(\dot{q},q,t)$ as follows:

$$L(\dot{q}',q',t) = L(\dot{q},q,t)+\dot{W}(q,t), \tag{2.3}$$

where W is an arbitrary function of its arguments, and a dot always denotes a total time derivative. In the usual symmetry theory the term $\dot{W}$ is absent from (2.3).

Since in classical mechanics conserved quantities are associated only with continuous transformations, it suffices to consider infinitesimal transformations,

$$q_k' = q_k+\epsilon u_k(q,t), \tag{2.4}$$

to find conserved quantities. In (2.4) the u_k are functions characterizing the given transformation and ϵ is an infinitesimal parameter. If (2.4) is a symmetry transformation there must, by definition, exist a function $F(q,t)$ such that

$$L(\dot{q}',q',t) = L(\dot{q},q,t)+\epsilon\dot{F}(q,t). \tag{2.5}$$

From (2.4)

$$\dot{q}_k' = \dot{q}_k+\epsilon\dot{u}_k. \tag{2.6}$$

Inserting this into (2.5), it becomes

$$\sum_k \left(\frac{\partial L}{\partial q_k} u_k + p_k \dot{u}_k\right) = \dot{F},$$

where

$$p_k = \partial L/\partial \dot{q}_k. \tag{2.7}$$

Since the motion satisfies Lagrange's equations $\dot{p}_k = \partial L/\partial q_k$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_k p_k u_k - F\right) = 0, \tag{2.8}$$

that is, the quantity

$$G(\dot{q}, q, t) = \sum_k p_k u_k - F \tag{2.9}$$

is conserved.

It remains to determine how the generalized momenta transform under the transformation defined by (2.1) and (2.3). Write, for convenience,

$$\partial q_i'/\partial q_j = a_{ij}, \qquad \partial q_i/\partial q_j' = b_{ij}, \qquad \sum_j a_{ij} b_{jk} = \sum_j a_{ji} b_{kj} = \delta_{ik}. \tag{2.10}$$

Then

$$p_k' = \frac{\partial}{\partial \dot{q}_k'}\left\{L(\dot{q}', q', t)\right\} = \sum_j b_{jk} \frac{\partial}{\partial \dot{q}_j}\{L(\dot{q}, q, t) + \dot{W}(q, t)\}.$$

Since

$$\dot{W} = \frac{\partial W}{\partial t} + \sum_j \frac{\partial W}{\partial q_j} \dot{q}_j,$$

one has

$$\partial \dot{W}/\partial \dot{q}_j = \partial W/\partial q_j,$$

so that

$$p_k' = \sum_j b_{jk}\left(p_j + \frac{\partial W}{\partial q_j}\right) = \sum_j b_{jk} p_j + \frac{\partial W(q, t)}{\partial q_k'}. \tag{2.11}$$

III. Hamiltonian Formulation

Although in classical mechanics symmetry properties are treated more simply in the Lagrangian formalism, it is still worth considering the Hamiltonian formalism in this context, as this provides a guide to the treatment of symmetry properties in quantum mechanics.

A canonical transformation can be specified by a generator $C(q, p', t)$, namely,

$$p_k = \partial C/\partial q_k, \qquad q_k' = \partial C/\partial p_k', \tag{3.1}$$

the transformed Hamiltonian being given by

$$H'(p', q', t) = H(p, q, t) + \partial C/\partial t. \tag{3.2}$$

If the transformation is to be the mere transformation of coordinates (2.1) the generator C must have the generic form

$$C = \sum_k p'_k g'_k(q,t) - w(q,t). \tag{3.3}$$

A system whose Hamiltonian is H is invariant under a coordinate transformation $q_k \to q'_k = g'_k(q,t)$ if there exists a generator C of the form (3.3) such that

$$H(p',q',t) = H(p,q,t) + \partial C/\partial t. \tag{3.4}$$

When such a generator exists (3.2) and (3.4) together imply

$$H'(p',q',t) = H(p',q',t), \tag{3.5}$$

so that the canonical equations of motion written in terms of the primed variables are exactly the same as those written in terms of the unprimed variables. Accordingly, the problem of determining whether a system is invariant under (2.1) is tantamount to finding out whether there exists a function $w(q,t)$ such that (3.3) and (3.4) hold.

From (3.3) and the first member of (3.1)

$$p_k = \sum_j a_{jk} p'_j - \frac{\partial w}{\partial q_k}.$$

Multiplying throughout by b_{ki} and summing over k,

$$p'_i = \sum_k b_{ki}\left(p_k + \frac{\partial w}{\partial q_k}\right). \tag{3.6}$$

Comparison with (2.11) then shows that

$$w(q,t) = W(q,t). \tag{3.7}$$

If one wants to remain within the Hamiltonian framework, $w(q,t)$ may be obtained by using (2.1), (3.6), and (3.3) to get expressions for the derivatives of w from (3.4). In general it is probably easier to find w from equations (2.3) and (3.7).

IV. Infinitesimal Canonical Transformations

Any infinitesimal transformation can be written as

$$p'_k = p_k - \epsilon\frac{\partial}{\partial q_k}\{G(p,q,t)\}, \qquad q'_k = q_k + \epsilon\frac{\partial}{\partial p_k}\{G(p,q,t)\}. \tag{4.1}$$

Then

$$H'(p',q',t) = H(p,q,t) + \epsilon\frac{\partial}{\partial t}\{G(p,q,t)\}. \tag{4.2}$$

If $J(p,q,t)$ is any function associated with the motion

$$J(p',q',t) = J(p,q,t) + \epsilon[J,G], \tag{4.3}$$

where on the right the Poisson bracket is as usual defined as

$$[J,G] = \sum_k \left(\frac{\partial J}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial J}{\partial p_k}\frac{\partial G}{\partial q_k}\right). \tag{4.4}$$

If the canonical transformation is a coordinate transformation (2.4), the generator will be

$$G = \sum_k p_k u_k - F, \tag{4.5}$$

for with this, equations (4.1) correctly give the new momenta in terms of the old. Invariance under this infinitesimal canonical transformation requires that (3.5) should hold, and therefore that

$$H'(p',q',t) = H(p,q,t)+\epsilon[H,G].$$

Comparison with (4.2) gives

$$(\partial G/\partial t)+[G,H] = \dot{G} = 0.$$

Thus, the system is invariant under the infinitesimal coordinate transformation generated by (4.1) and (4.5) if the generator G is a constant of the motion. This result is in harmony with the result embodied in equations (2·8) and (2·9).

V. Examples

(a) *Invariance under Translations*

$\mathbf{r}(= x,y,z)$ shall be the position vector of a particle of charge e moving in an external electromagnetic field whose potentials are $\mathbf{A},\Phi$. The Lagrangian is, with $\mathbf{v} = \dot{\mathbf{r}}$,

$$L = \tfrac{1}{2}m|\mathbf{v}|^2+(e/c)\mathbf{v}\,.\,\mathbf{A}-e\Phi. \tag{5.1}$$

Consider an infinitesimal translation along the x-axis, $x' = x+\epsilon$.

Then

$$L(\mathbf{v}',\mathbf{r}',t) = L(\mathbf{v},\mathbf{r},t)+\epsilon e\left(\frac{1}{c}\mathbf{v}\,.\,\frac{\partial \mathbf{A}}{\partial x} - \frac{\partial \Phi}{\partial x}\right). \tag{5.2}$$

Invariance under this translation requires the existence of a function $F_x(\mathbf{r},t)$ such that

$$e\left(\frac{1}{c}\mathbf{v}\,.\,\frac{\partial \mathbf{A}}{\partial x} - \frac{\partial \Phi}{\partial x}\right) = \frac{\mathrm{d}F_x}{\mathrm{d}t} = \frac{\partial F_x}{\partial t}+\mathbf{v}\,.\,\operatorname{grad} F_x.$$

Thus one requires, with $e' = e/c$,

$$\partial F_x/\partial t = -e\partial\Phi/\partial x, \qquad \operatorname{grad} F_x = e'\partial\mathbf{A}/\partial x. \tag{5.3}$$

These entail

$$\partial\mathbf{H}/\partial x = \partial(\operatorname{curl}\mathbf{A})/\partial x = 0 \tag{5.4}$$

and

$$\frac{\partial \mathbf{E}}{\partial x} = -\frac{\partial}{\partial x}\left(\operatorname{grad}\Phi+\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{5.5}$$

When equations (5.4) and (5.5) are satisfied, equations (5.3) give the total differential of F

$$\begin{aligned}\mathrm{d}F_x &= e'\left(\frac{\partial A_x}{\partial x}\mathrm{d}x+\frac{\partial A_y}{\partial x}\mathrm{d}y+\frac{\partial A_z}{\partial x}\mathrm{d}z-c\frac{\partial \Phi}{\partial x}\mathrm{d}t\right)\\ &= e'(\mathrm{d}A_x+H_z\,\mathrm{d}y-H_y\,\mathrm{d}z+cE_x\,\mathrm{d}t).\end{aligned}\tag{5.6}$$

Integrating in y, z, t space along a continuous curve which consists of straight segments joining the points $(0,0,0)$, $(y,0,0)$, $(y,z,0)$, (y,z,t) one gets F_x in the form

$$\begin{aligned}F_x(x,y,z,t) &= e'\left\{A_x(x,y,z,t)+\int_0^y H_z(y_1,0,0)\,\mathrm{d}y_1\right.\\ &\qquad\left.-\int_0^z H_y(y,z_1,0)\,\mathrm{d}z_1+c\int_0^t E_x(y,z,t_1)\,\mathrm{d}t_1\right\}\\ &= e'A_x+\mathscr{F}_x(y,z,t),\end{aligned}\tag{5.7}$$

say, the argument x having been omitted from the field strengths since these do not depend on x. (See also the remark following equation (5.12).) The conserved quantity, which is also the generator of translations along the x-axis leaving the system invariant, is

$$G_x = p_x-F_x = m\dot{x}-\mathscr{F}_x.\tag{5.8}$$

When one has invariance under translations along both the x-axis and the y-axis, $\mathbf{E}$ and $\mathbf{H}$ must be independent of both x and y, and then H_z is constant (independent of x, y, z, t). Proceeding as before, one now has two conserved quantities

$$G_x = p_x-F_x = m\dot{x}-\mathscr{F}_x, \qquad G_y = p_y-F_y = m\dot{y}-\mathscr{F}_y,\tag{5.9}$$

where

$$\begin{aligned}\mathscr{F}_x &= e'\left\{H_z y-\int_0^z H_y(z_1,0)\,\mathrm{d}z_1+c\int_0^t E_x(z,t_1)\,\mathrm{d}t_1\right\},\\ \mathscr{F}_y &= e'\left\{-H_z x+\int_0^z H_x(z_1,0)\,\mathrm{d}z_1+c\int_0^t E_y(z,t_1)\,\mathrm{d}t_1\right\}.\end{aligned}\tag{5.10}$$

Evidently simultaneous invariance under translations along all three axes requires that $\mathbf{E}$ and $\mathbf{H}$ are constant everywhere and at all times. One then has three conserved quantities

$$\mathbf{G} = \mathbf{p}-\mathbf{F} = m\mathbf{v}-\vec{\mathscr{F}},\tag{5.11}$$

with

$$\vec{\mathscr{F}} = e'(\mathbf{r}\times\mathbf{H}+c\mathbf{E}t).\tag{5.12}$$

$\vec{\mathscr{F}}$ is the time integral of the Lorentz force. Note that if the integrations in (5.7) and (5.10) are carried out along the world line of the particle, instead of as described earlier, then $\mathscr{F}_x$ and $\mathscr{F}_y$ are the time integrals of the x and y components of the Lorentz force acting on the particle.

For a free particle $\vec{\mathscr{F}} = 0$ and the conserved generators of translations along the coordinate axes are $(m\dot{x},\ m\dot{y},\ m\dot{z}) = m\mathbf{v}$. This vector is sometimes called the *kinetic* momentum, to distinguish it from the *canonical* (or generalized) momentum $\mathbf{p}$, that is to say, from

$$\mathbf{p} = m\mathbf{v}+e'\mathbf{A}. \tag{5.13}$$

In the presence of an electromagnetic field the situation is not so simple. Interest centres not so much around generators of translations in general (i.e. generators which merely take x into $x+\epsilon$, etc.) but around generators of translations under which the system is invariant. Such a generator will be called an *invariant generator* of translations; and it will be recalled that it is conserved. As has been seen, the number of invariant generators one can have in any particular case depends upon the configuration of the field. An invariant generator, when it exists, need be neither a kinetic nor a canonical component of momentum. Accordingly it will also be referred to as a component of *symmetry momentum.* This terminology notwithstanding, when the system is not invariant under translations along a certain axis the corresponding component of the symmetry momentum is not defined.

It may be worth returning to the case when one has invariance under translations along just two axes, which were above taken to be the x-and y-axes. Then, whatever the initial gauge may have been, one can choose a new gauge so that the invariant generator for translations along *one* of the axes becomes the component of canonical momentum along that axis. If, for instance, the latter is the x-axis one need only choose χ in (1.2) to be such that

$$e'\partial\chi/\partial x = -\mathscr{F}_x. \tag{5.14}$$

It is not always possible, however, to choose the gauge so that both invariant generators are the corresponding components of canonical momentum. If this were to be possible one would have to have, in addition to (5.14),

$$e'\partial\chi/\partial y = -\mathscr{F}_y,$$

that is, the condition

$$\partial\mathscr{F}_x/\partial y - \partial\mathscr{F}_y/\partial x = 0 \tag{5.15}$$

would have to be satisfied. In view of (5.10) the left-hand member of (5.15) has the value $2e'H_z$. The required end can therefore be achieved only if $H_z = 0$. Alternatively, the same conclusion follows from the fact that

$$[G_x, G_y] = -e'H_z, \tag{5.16}$$

whereas if G_x and G_y were both canonical components of momentum their Poisson bracket would have to vanish.

(b) *Invariance under Rotations*

Let a particle be moving in a uniform magnetic field (constant in time), the direction of which may, without loss of generality, be taken to be along the z-axis. Then the equations of motion are invariant under rotations about any axis parallel to the z-axis. The gauge may be chosen so that the Lagrangian will be invariant under rotations about any particular one of this set of parallel axes but then it will not be invariant under rotations about any other axis of the set.

In an arbitrary gauge the Lagrangian is again given by (5.1), where now

$$\operatorname{grad}\Phi+\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}=0, \qquad \operatorname{curl}\mathbf{A}=\mathbf{H}=(0,0,H). \tag{5.17}$$

For an infinitesimal rotation about the z-axis

$$x'=x-\epsilon y, \qquad y'=y+\epsilon x, \qquad z'=z. \tag{5.18}$$

Then

$$\begin{aligned}\Delta L &= L(\mathbf{v}',\mathbf{r}',t)-L(\mathbf{v},\mathbf{r},t)\\ &= e'\epsilon\left\{x\left(\dot{x}\frac{\partial A_x}{\partial y}+\dot{y}\frac{\partial A_y}{\partial y}+\dot{z}\frac{\partial A_z}{\partial y}\right)-y\left(\dot{x}\frac{\partial A_x}{\partial x}+\dot{y}\frac{\partial A_y}{\partial x}+\dot{z}\frac{\partial A_z}{\partial x}\right)\right.\\ &\qquad \left.+\dot{x}A_y-\dot{y}A_x+c\left(y\frac{\partial \Phi}{\partial x}-x\frac{\partial \Phi}{\partial y}\right)\right\}.\end{aligned}$$

Keeping equations (5.17) in mind one confirms without difficulty that

$$\Delta L = e'\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\{(xA_y-yA_x)-\tfrac{1}{2}H(x^2+y^2)\}. \tag{5.19}$$

The transformation (5.18) is of the form (2.4) with

$$u_x=-y, \qquad u_y=x, \qquad u_z=0. \tag{5.20}$$

Evidently the function F in (2.5) is here

$$F=e'\{(xA_y-yA_x)-\tfrac{1}{2}H(x^2+y^2)\}. \tag{5.21}$$

According to (2.9) the quantity

$$G_z=-(yp_x-xp_y)-F$$

is conserved. Explicitly

$$G_z=x(p_y-e'A_y)-y(p_x-e'A_x)+\tfrac{1}{2}e'H(x^2+y^2),$$

or

$$G_z=m(\mathbf{r}\times\mathbf{v})_z+\tfrac{1}{2}e'H(r^2-z^2). \tag{5.22}$$

G_z is obviously gauge-invariant.

(c) *Inversions*

Consider inversions through the origin, i.e. the transformation

$$\mathbf{r}'=-\mathbf{r}, \tag{5.23}$$

for a charged particle moving in an electromagnetic field. There is of course no corresponding infinitesimal transformation, and the problem is to construct the generator C of inversions under which the system is invariant, the prescription being such that it holds independently of the gauge.

For illustrative purposes the problem in hand will be solved within the Hamiltonian formalism. Because of (5.23) one has here $b_{ik} = -\delta_{ik}$, so that (3.6) gives

$$\mathbf{p}' = -\mathbf{p} - \operatorname{grad} w(\mathbf{r}, t). \tag{5.24}$$

According to (3.3) and (5.23) the generator C has the form

$$C = -\mathbf{p}' \,.\, \mathbf{r} - w(\mathbf{r}, t). \tag{5.25}$$

Since

$$H(\mathbf{p}, \mathbf{r}, t) = (\tfrac{1}{2}m)|\mathbf{p} - e'\mathbf{A}|^2 + e\Phi,$$

equations (5.23) to (5.25) and (3.4) then give the relation from which the derivatives of w may be read off

$$(\tfrac{1}{2}m)\{|\mathbf{p} + \operatorname{grad} w + e'\mathbf{A}(-\mathbf{r}, t)|^2 - |\mathbf{p} - e'\mathbf{A}(\mathbf{r}, t)|^2\} + e\{\Phi(-\mathbf{r}, t) - \Phi(\mathbf{r}, t)\} = -\partial w/\partial t.$$

The factor multiplying $\mathbf{p}$ must vanish, and so

$$\operatorname{grad} w = -e'\{\mathbf{A}(\mathbf{r}, t) + \mathbf{A}(-\mathbf{r}, t)\}, \tag{5.26}$$

$$\partial w/\partial t = e\{\Phi(\mathbf{r}, t) - \Phi(-\mathbf{r}, t)\}. \tag{5.27}$$

From (5.26) and (5.27) one confirms easily that w exists if, and only if,

$$\mathbf{H}(-\mathbf{r}, t) = \mathbf{H}(\mathbf{r}, t), \qquad \mathbf{E}(-\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t). \tag{5.28}$$

In other words equations (5.28) are the conditions for the integrability of the four equations (5.26) and (5.27). From (3.1), (5.25), and (5.26)

$$\mathbf{p}' = -\mathbf{p} - e'\{\mathbf{A}(\mathbf{r}, t) + \mathbf{A}(-\mathbf{r}, t)\}. \tag{5.29}$$

To integrate (5.26) and (5.27), let their right-hand members be denoted by $\mathscr{A}$ and $\mathscr{A}_0$ respectively. Proceeding as in Section V(a), choose as path of integration a curve composed of straight segments, so that

$$W = \int_0^x \mathscr{A}_x(x_1, y, z, t)\, \mathrm{d}x_1 + \int_0^y \mathscr{A}_y(0, y_1, z, t)\, \mathrm{d}y_1 + \int_0^z \mathscr{A}_z(0, 0, z_1, t)\, \mathrm{d}z_1. \tag{5.30}$$

There is no integral with respect to t since $\mathscr{A}_0(0,0,0,t) = 0$. One could alternatively integrate along the world line π of the particle. Then one merely recovers the result

$$W(= w) = \int_\pi \Delta L\, \mathrm{d}t. \tag{5.31}$$

One might indeed start with this since, by hypothesis, the value of the integral is independent of the path, and the latter may be deformed into that used in equation (5.30).

VI. Discussion

It has been shown that statements which are valid in any gauge can be made about the symmetry properties of a system. An interesting result of this treatment is that the invariant generators of infinitesimal translations are in general not components of the canonical momentum.