INTEGRAL EQUATIONS FOR THREE-PARTICLE INTERACTION AMPLITUDES IN THE NON-RELATIVISTIC CASE

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[*Manuscript received July 20, 1964*]

Summary

By using the technique of summing an infinite series of non-relativistic graphs, accurate integral equations are obtained for the three-particle interaction amplitudes.

A method of solving the problem of interaction of three particles is proposed in this paper. An accurate integral equation for the three-nucleon interaction amplitude is obtained using the method, which is based on non-relativistic field theory.

The only assumption made in the work is that two-body forces are essential. However, this does not limit the proposed method, which allows three-body nuclear forces to be taken into account. The method employs the two-body interaction Hamiltonian

$$V(t)=\sum_{\bar{k}_1\bar{k},\,\bar{p}} a^*(\bar{k},t)a^*(\bar{p}-\bar{k},t)V(\bar{k},\bar{k}')a(\bar{k}',t)a(\bar{p}-\bar{k}',t), \tag{1}$$

where t is the time, $a^*(\bar{k},t)$ and $a^*(\bar{p}-\bar{k},t)$ are creation operators for nucleons with momenta $\bar{k}$ and $\bar{p}-\bar{k}$, and $a(\bar{k}',t)$ and $a(\bar{p}-\bar{k}',t)$ are annihilation operators for the nucleons with momenta $\bar{k}'$ and $\bar{p}-\bar{k}'$, and

$$V(\bar{k},\bar{k}')=\int \exp[\mathrm{i}(\bar{k}-\bar{k}')\bar{r}]\; V(\bar{r})\mathrm{d}\bar{r}, \tag{2}$$

where $V(\bar{r})$ is a potential of interaction between particles. In this case, as was shown in previous papers (Komarov and Popova 1963*a*, 1963*b*) the accurate three-particle amplitude can be represented as a sum of contributions from an infinite series of non-relativistic perturbation theory Feynmann graphs corresponding to the process under study (Fig. 1). The entire sum of the diagrams in the right half of Figure 1 can be represented in the form of Figure 2, since the iteration of this last equation makes it possible to reconstitute the aggregate of diagrams shown in Figure 1. In fact, because the three particles are identical, there is a sum of three diagrams which differ only in the permutation of the particles in the final state.

Thus an accurate integral equation for the three-particle interaction amplitude contains half the number of variables compared with the corresponding Schrödinger equation for an arbitrary potential. For the sake of simplicity the equations are derived for the zero-spin particles. The equations can be generalized for real non-zero-spin particles without any essential difficulties, and the method applied in Komarov and Popova (1963*a*, 1963*b*, 1964) can be used for this purpose.

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By determining the contributions from the separate graphs of Figure 1 one can obtain in analytical form the integral equation for the amplitude of the reaction under study. It is clear from the figure that the main elements of these graphs are the two-nucleon scattering blocks B, C, and D, the vertices A corresponding to the decay of a dinucleon into two nucleons and nucleon propagation lines. As indicated by Komarov and Popova (1963*a*) the contribution from the nucleon–nucleon scattering block is

$$\mathrm{i}(4\pi/m)\,a(\bar{f},\bar{f}',E),$$

where $a(\bar{f},\bar{f}',E)$ is a Green's function of two scattering nucleons, $\bar{f}$ and $\bar{f}'$ are the particle relative motion momenta before and after scattering and E is the relative energy of these particles. In Komarov and Popova (1963*a*) the graph summation method was used to obtain the integral equation for the function $a(\bar{f},\bar{f}',E)$ analogous to the Schrödinger equation

$$a(\bar{f},\bar{f}',E) = (m/4\pi)V(\bar{f},\bar{f}')+(m/8\pi)\int V(\bar{f},\bar{q})\,a(\bar{q},\bar{f},E)(mE-q^2-\mathrm{i}\tau)^{-1}\mathrm{d}\bar{q}. \quad (3)$$

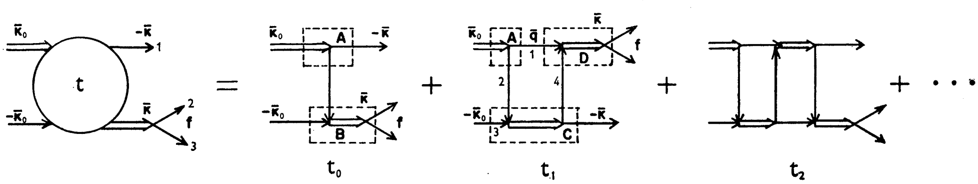

Fig. 1.—The infinite series of perturbation theory Feynmann graphs corresponding to three-particle interaction.

In the case $fr_0<1$ $(f=(mE)^{\frac{1}{2}})$, i.e. in the zero and linear approximations in the range r_0 of nuclear forces, the s-wave part of the Green's function for the scattering nucleon pair only depends on the relative energy E. Hence the contribution from the two-particle scattering block is proportional to the nucleon–nucleon scattering amplitude on mass sheet. In the zero-order approximation in r_0 the contribution from the scattering block is

$$a(f) = -1/(\alpha+\mathrm{i}f), \quad (4)$$

where α is the triplet scattering nucleon–nucleon length. In the linear approximation in r_0 the contribution is

$$a(f) = \tfrac{1}{2}r_0-(1+\alpha r_0)/(\alpha+\mathrm{i}f). \quad (5)$$

The value $G^{(\bar{f}'')}_{d\to 2N}$ corresponding to the vertex of the deuteron decay into two nucleons and depending on relative motion momentum $\bar{f}''$ can be determined by the residue of the pole part of the Green's function $a_{\mathrm{pol}}(\bar{f}'',E)$,

$$G^{(\bar{f}'')}_{d\to 2N} = (\mathrm{i}/m)[8\pi\alpha\,\mathrm{Res}\,a_{\mathrm{pol}}(\bar{f}'',E)|_{E=E_0}]^{\frac{1}{2}},$$

where $E_0=\alpha^2/m$ is the deuteron binding energy. In the zero-order approximation in r_0 and in the linear approximation in r_0 the deuteron decay vertex has the form respectively:

$$G^{0}_{d\to 2N} = (\mathrm{i}/m)(8\pi\alpha)^{\frac{1}{2}}, \quad (6)$$

$$G^{1}_{d\to 2N} = (\mathrm{i}/m)[8\pi\alpha(1+\alpha r_0)]^{\frac{1}{2}}. \quad (7)$$

The nucleon propagation line in the non-relativistic graphs corresponds to the function

$$\gamma(p) = \mathrm{i}/\{E_1 - E(p)\},$$

where E_1 is the kinetic energy of the virtual nucleon obtained from the equation for the total energy conservation at the vertex and $E(p) = p^2/2m$, where $\bar{p}$ is the momentum of the given virtual line calculated from the equation for the momentum conservation at the vertex.

Having determined the contributions from separate elements of the graphs of Figure 1 we can calculate with the aid of the techniques used in Komarov and Popova (1963*a*, 1963*b*) the accurate integral equation for the amplitude $t(\bar{k}_0, \alpha, \bar{k}, \bar{f})$ of nucleon–deuteron inelastic scattering with the production of two interacting nucleons 1 and 2 in the final state with the relative momentum $\bar{f}$; $\bar{k}_0$ denotes the relative momentum of the nucleon and deuteron in the initial state and $\bar{k}$ that of the nucleon 3 and two interacting nucleons 1 and 2.

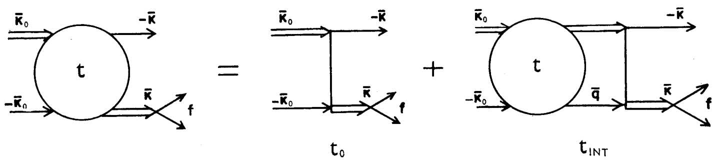

Fig. 2.—The accurate graph equation corresponds to the integral equation for the three-particle interacting amplitude t.

In this equation the contribution from the pole graph t_0 is a free term given by the product of the contributions from the deuteron decay vertex A, the nucleon–nucleon scattering block B and the nucleon propagation line. It is clear from Figure 1 that the contribution in the pole graph from the deuteron decay vertex depends on the momentum $\bar{f}'' = \bar{k} - \frac{1}{2}\bar{k}_0$ of the relative motion of nucleons produced in the deuteron decay and has the form

$$G_{d\to 2N}^{(f'')} = (\mathrm{i}/m)[8\pi\alpha \operatorname{Res} a_{\mathrm{pol}}(\bar{f}'', E)|_{E=E_0}]^{\frac{1}{2}}.$$

The propagation line contribution in the case under study is

$$\gamma(\bar{k}_0, \bar{k}) = \mathrm{i}m[\alpha^2 + \tfrac{1}{2}k^2 - \tfrac{1}{4}k_0^2 + \tfrac{1}{2}(\bar{k}_0 - \bar{k})^2]^{-1},$$

and the nucleon–nucleon scattering block contribution depends on the momenta $\bar{f}$ and $\bar{f}'$ of the relative motion of nucleons issuing from the block and entering it, as well as on the relative energy E, which equals, in the case under study, the quantity f^2/m, and hence must have the form $4\pi\mathrm{i}a(\bar{f},\bar{f}')/m$.

Thus the contribution from the entire pole graph t_0 or the Born term in the integral equations is

$$t_0(\bar{k}_0, \alpha, \bar{k}, \bar{f}) = (8/3)G_{d\to 2N}^{(f'')}\, a(\bar{f}, \bar{f}')\gamma(\bar{k}_0, \bar{k}). \tag{8}$$

The integral term in the desired equation for the nucleon–deuteron inelastic scattering amplitude is given by the contribution from the graph t_{INT} of Figure 2. Obviously the contribution from this term can be determined by investigation of the rectangle graph t_1. The contribution from the rectangle graph t_1 is given by the integral over the momentum $\bar{q}$ and energy ϵ of the virtual particle 1.

$$t_1(\bar{k}_0, a, \bar{k}, \bar{f}) = \frac{m}{8\pi}\int d\epsilon \int \frac{d\bar{q}}{(2\pi)^3} a(\bar{f}, \bar{f}_1')a(\bar{k}', \bar{k}'', E')G^{(\bar{f}'')}_{d\to 2N}\left[\epsilon - \frac{q^2}{2m} + i\tau\right]^{-1} \times$$
$$\left[\frac{k_0^2}{4m} - \frac{a^2}{m} - \epsilon - \frac{(\bar{k}_0 - \bar{q})}{2m} + i\tau\right]^{-1}\left[\frac{k^2}{4m} - \epsilon + \frac{f^2}{m} - \frac{(\bar{k} - \bar{q})^2}{2m} + i\tau\right]^{-1}. \quad (9)$$

The expressions in the square brackets in this equation are the contributions from the propagation lines 1, 2, and 4 respectively (Fig. 1, t_1). $a(\bar{k}', \bar{k}'', E')$ is the scattering amplitude of the virtual nucleon 2 and real nucleon 3. This value defines the contribution from the block "C" of the t_1. The value of the relative energy $E' = f'^2/m$ of the particles 2 and 3 can be determined from the equation

$$E' = \tfrac{3}{4}k_0^2/m - a^2/m - \tfrac{1}{4}q^2/m - \epsilon, \quad (10)$$

and the relative momenta of the incoming and issuing particles $\bar{k}'$ and $\bar{k}''$ are equal to $\bar{k}_0 - \frac{1}{2}\bar{q}$ and $\bar{k} - \frac{1}{2}\bar{q}$ respectively. $a(\bar{f}', \bar{f})$ is the scattering amplitude for particles 1 and 4 and corresponds to the contribution from the block "D" of the t_1, where $\bar{f}'$ and $\bar{f}$ are relative momenta of the particles before and after scattering. The relative energy of the particles 1 and 4 is equal to f'^2/m and can be determined from the equation

$$f'^2/m = \tfrac{3}{4}k_0^2/m - a^2/m - \tfrac{3}{4}k^2/m; \qquad |\bar{f}'| = |\bar{q} - \tfrac{1}{2}\bar{k}|.$$

Now it is important to investigate the integrand of equation (9). It contains as function of the ϵ three poles in the complex plane when

$$\epsilon = \epsilon_1 = \tfrac{1}{2}q^2/m - i\tau,$$
$$\epsilon = \epsilon_2 = \tfrac{1}{4}k_0^2/m - a^2/m - \tfrac{1}{2}(\bar{k}_0 - \bar{q})^2/m + i\tau,$$
$$\epsilon = \epsilon_3 = \tfrac{1}{4}k^2/m + f^2/m - \tfrac{1}{2}(\bar{k} - \bar{q})^2/m + i\tau.$$

Besides that, the integrand contains singularities in ϵ-plane arising from singularities of the function $a(\bar{k}', \bar{k}'', E')$. As function of energy E' this function contains a pole when $E' = a^2/m$, a right cut from $E' = 0$ to $+\infty$ along the real axis, and a left cut from $E' = -\frac{1}{4}\mu/m$ to $-\infty$ along the real axis on the physical sheet. Here μ has a value $1/r_0$ of the order of magnitude of the pion mass. The ϵ-plane and E'-plane are connected by equation (10). If this is borne in mind it can be seen that all singularities on the ϵ-plane will be as follows.

In Figure 3 the E'-pole is in the point

$$\epsilon' = \tfrac{3}{4}k_0^2/m - \tfrac{1}{4}q^2/m,$$

the E' right cut is from the point

$$\epsilon_l = \tfrac{3}{4}k_0^2/m - a^2/m - \tfrac{1}{4}q^2/m$$

to $-\infty$, and the left E' cut is from the point

$$\epsilon_r = \tfrac{3}{4}k_0^2/m - a^2/m - \tfrac{1}{4}q^2/m + \tfrac{1}{4}\mu^2/m$$

to $+\infty$.

If the function $a(\bar{k}', \bar{k}'', E')$ has been defined on the upper sides of the cuts in the E'-plane, this function is defined on the lower sides of those cuts in the ϵ-plane. In this case it is possible to move the integral contour over ϵ into the lower half-plane forming a distance of $i\delta$ between the contour and real axis and enclose it round the lower half-plane having the ϵ_1-pole inside the contour. Because the integrand decreases when ϵ increases, the calculation of the integral over ϵ reduces to the calculation of the residue of the integrand function at the point $\epsilon_1 = \frac{1}{2}q^2/m$. It is clear that the same meaning of the integral over ϵ can be obtained if the function $a(\bar{k}', \bar{k}'', E')$ had been defined on the lower sides of the cuts in the E'-plane.

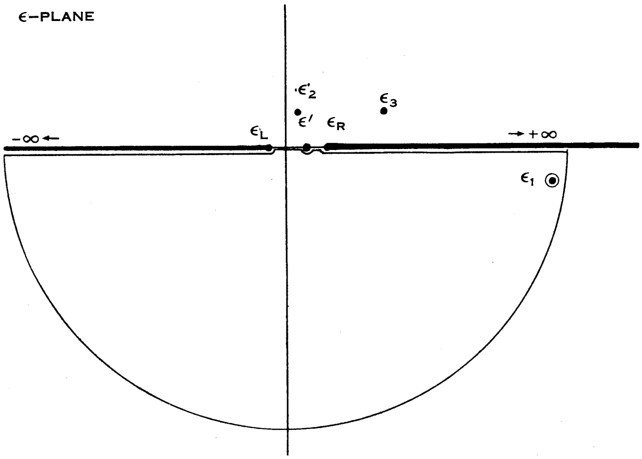

Fig. 3.—Singularities in the ϵ-plane.

Consequently, the contribution from the graph t_1 is

$$t_1(\bar{k}_0, a, \bar{k}, f) = (8/3)^2\, 4\pi \int \frac{d\bar{q}}{(2\pi)^3}\, \tfrac{3}{4} G^{(\bar{f}'')}_{d\to 2N}\, a(\bar{f}_1', \bar{f})\, a(\bar{k}', \bar{k}'', E') \times$$
$$[a_t^2 + (\tfrac{1}{2}\bar{k}_0 - \bar{q})^2]^{-1}[-f^2 + (\tfrac{1}{2}\bar{k} - \bar{q})^2]^{-1}, \qquad (11)$$

where

$$E' = \tfrac{3}{4}k_0^2/m - a^2/m - \tfrac{3}{4}q^2/m.$$

It is possible to pick out from the integrand of expression (11) the part

$$(8/3) G^{(\bar{f}'')}_{d\to 2N}\, a(\bar{k}', \bar{k}'', E')[a^2 + (\tfrac{1}{2}\bar{k}_0 - \bar{q})^2]^{-1},$$

which is equal to the contribution (8) from the pole-graph to $(\bar{k}_0, a, \bar{q}, \bar{k}'')$ where the deuteron and nucleon with the momenta $\bar{k}_0$ and $-\bar{k}_0$ respectively are the incoming particles and three nucleons are the emerging particles, $\bar{k}''$ denotes the relative momentum of two of these nucleons and momentum $\bar{q}$ denotes their onward movement.

Thus the contribution from the rectangle graph t_1 can be rewritten in the form

$$t_1(\bar{k}_0, a, \bar{k}, \bar{f}) = (8/3) 4\pi \int \frac{d\bar{q}}{(2\pi)^3}\, t_0(\bar{k}_0, a, \bar{q}, \bar{k}'')\, a(\bar{f}, \bar{f}_1) \cdot \gamma(\bar{k}, \bar{q}), \qquad (12)$$

where

$$\gamma(\bar{k}, \bar{q}) = [-f^2+(\tfrac{1}{2}\bar{k}-\bar{q})^2]^{-1}.$$

Hence the integral term in the equation for the desired amplitude has the form (after replacement in the expression (12) for t_1 of $t_0(\bar{k}_0, a, \bar{q}, \bar{k}'')$ by the accurate values of this amplitude, $t(\bar{k}_0, a, q', k''))$

$$t_{\mathrm{INT}}(\bar{k}_0, a, \bar{k}, \bar{f}) = (8/3)4\pi \int \frac{\mathrm{d}\bar{q}}{(2\pi)^3} \cdot t(\bar{k}_0, a, \bar{q}', \bar{k}'')a(\bar{f}, \bar{f}_1')\gamma(\bar{k}, \bar{q}). \tag{13}$$

Now the contribution from the pole graph t_0 (equation (8)) and from the integral term t_{INT} (equation (13)) has been obtained, and the total integral equation for the three-particle interaction amplitude describing the reaction in which there are a deuteron and a nucleon in the initial state and three interacting nucleons in the final state can be presented in the form

$$t(\bar{k}_0, a, \bar{k}, \bar{f}) = (8/3)G_{d\to 2N}^{(\bar{f}'')} a(\bar{f}, \bar{f}')\gamma(\bar{k}_0, \bar{k}) + 8\pi \int \frac{\mathrm{d}\bar{q}}{(2\pi)^3} t(\bar{k}_0, a, \bar{q}, \bar{k}'')a(\bar{f}, \bar{f}_1') \cdot \gamma(\bar{k}, \bar{q}). \tag{14}$$

In the case of $fr_0 \ll 1$, i.e. in the zero or linear approximation in r_0, equation (14) must become much simpler since the contributions from the scattering blocks and the deuteron decay vertex must be proportional to the scattering amplitude of two nucleons on mass sheet.

It is easy to obtain the zero-approximation for the integral equation (14) using the meanings of deuteron decay vertex and nucleon–nucleon scattering block in this approximation. Equations (4) and (6) give

$$t^{(0)}(\bar{k}_0, a, \bar{k}, \bar{f}) = (8/3)(8\pi a)^{\frac{1}{2}}(a+\mathrm{i}f)^{-1}\gamma(\bar{k}_0, \bar{k}) + 8\pi \int \frac{\mathrm{d}\bar{q}}{(2\pi)^3} t^0(\bar{k}, a, \bar{q}, \bar{k}'')(a+\mathrm{i}f)^{-1}\gamma(\bar{k}, \bar{q}). \tag{15}$$

This equation coincides directly with the Skornyakov–Ter-Martirosyan (1956) equation obtained from the investigation of the Schrödinger equation for the non-relativistic three-body problem in the above approximation. It should be noted that obtaining equation (14) for the arbitrary approximation on the basis of the Schrödinger equation by the method proposed in Skornyakov and Ter-Martirosyan (1956) is impossible, and equation (15) follows as a particular case (under the assumption of zero range of nuclear forces) from the accurate integral equation (14) derived in this paper.

In the linear approximation in r_0, equation (14) has the form (the expressions (5) and (7) for the block and vertex are used now)

$$t^{(1)}(\bar{k}_0, a, \bar{k}, \bar{f}) = (8/3)\left[\tfrac{1}{2}r_0 - \frac{1+ar_0}{a+\mathrm{i}f}\right]\Big\{[8\pi a(1+ar_0)]^{\frac{1}{2}} \cdot \gamma(\bar{k}_0, \bar{k}) +$$
$$3\pi \int \frac{\mathrm{d}\bar{q}}{(2\pi)^3} t^{(1)}(\bar{k}_0, a, \bar{q}, \bar{k}'')\gamma(\bar{k}, \bar{q})\Big\}. \tag{16}$$

As indicated above, equation (14) was obtained under the assumption that two nucleons 1 and 2, with relative momentum $\bar{f}$, interact in the final state. Since the interaction of nucleons 1 and 3 or 2 and 3 is also possible in the final state the total nucleon–deuteron inelastic scattering amplitude is

$$T(\bar{k}_0, a, \bar{k}, \bar{f}) = t(\bar{k}_0, a, \bar{k}, \bar{f}) + t(\bar{k}_0, a;\ -\tfrac{1}{2}\bar{k}-\bar{f}, \tfrac{3}{4}\bar{k}-\tfrac{1}{2}\bar{f}) + t(k_0, a;\ -\tfrac{1}{2}k+\bar{f}, \tfrac{3}{4}\bar{k}+\tfrac{1}{2}\bar{f}). \quad (17)$$

Obviously, the above method can be used to obtain the integral equation for the nucleon–deuteron elastic scattering amplitude $t'(\bar{k}_0, a, \bar{k}, a)$, where $|\bar{k}_0| = |\bar{k}|$ and the integral equation for the scattering amplitude of three free nucleons $t''(\bar{k}_0, \bar{f}_0, \bar{k}, \bar{f})$, where $\bar{f}_0$ is the relative momentum of a pair of nucleons scattered in the initial state,

$$t'(\bar{k}_0, a, \bar{k}, a) = \tfrac{2}{3}\pi[G^{(\bar{f}'')}_{d\to 2N}]^2\gamma(\bar{k}_0, \bar{k}) + (3/\pi)\int\frac{d\bar{q}}{(2\pi)^3}t'(\bar{k}_0, a, \bar{q}, a)G^{(\bar{f}_1')}_{d\to 2N}\gamma(\bar{k}, \bar{q}), \quad (18)$$

$$t''(\bar{k}_0, \bar{f}_0, \bar{k}, \bar{f}) = \tfrac{2}{3}\pi a(\bar{f}_0, \bar{f}'')a(\bar{f}, \bar{f}')\gamma(\bar{f}_0, \bar{k}_0, \bar{k}) + 2\pi^2\int\frac{d\bar{q}}{(2\pi)^3}t''(\bar{k}_0, \bar{f}_0, \bar{q}, \bar{k}'')a(\bar{f}, \bar{f}_1')\gamma(\bar{f}, \bar{k}, \bar{q}), \quad (19)$$

where $\gamma(\bar{f}_0, \bar{k}_0, \bar{k}) = [-f_0^2 + (\tfrac{1}{2}\bar{k}_0 - \bar{k})^2]^{-1}$, and $\gamma(\bar{f}, \bar{k}, \bar{q}) = [-f^2 + (\tfrac{1}{2}k - q)^2]^{-1}$.

The obtained accurate integral equations (14), (18), and (19) are valid for the description of every kind of three-particle interaction, for example, (d,α), (d,t), and others.

Acknowledgments

The authors wish to express their appreciation to Dr. K. A. Ter-Martirosyan for useful discussions. One of the authors (V.V.K.) thanks Professor E. W. Titterton for his comments and help with English.

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