SHORT COMMUNICATIONS

MOTION OF A CHARGED PARTICLE IN A NON-UNIFORM MAGNETIC FIELD*

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The motion of an electron in a magnetic field of constant gradient has been analysed by Seymour (1959), who derives expressions for the x and y coordinates of the electron in terms of elliptic integrals.

In the course of devising non-electrostatic methods of electron energy spectrometry, it was discovered that analysis was greatly simplified by defining the electron trajectory to be a superior trochoid (i.e. the locus of a point on the produced radius of a circle rolling on a straight line, as shown in Fig. 1), thence to derive the magnetic field variation necessary to produce the trajectory. Such magnetic fields are not unduly complex mathematically and, furthermore, they are closely approximated by those with "hyperbolic" variation of the form $B_z = k/x$, such fields being readily obtained by means of plane inclined magnetic pole faces. The closeness of the correlation of the magnetic fields and the electron trajectories is illustrated later.

Magnetic Field to Produce Trochoidal Trajectory (MKS units are used throughout)

List of Symbols:

a	Radius of trochoid generating arm	metre
b	Radius of rolling circle	metre
B_z	Magnetic flux density in z-direction	Wb/m^2
(e/m)	Specific electronic charge	C/kg
u	Speed of electrons	m/s
v_x, v_y	Velocity components in x- and y-directions	m/s
V	Accelerating voltage for electrons	volt
x	Distance of electron from y-axis	metre
y	Distance of electron from x-axis	metre
θ	Angular displacement of generating circle	radian
ω	Angular velocity of generating circle	radian/s
Ω	Angular velocity of electrons	radian/s

The parametric equations of the superior trochoid shown in Figure 1 are:

$$x = a(1-\cos\theta), \tag{1}$$

$$y = b\theta + a\sin\theta. \tag{2}$$

* Manuscript received July 27, 1964.

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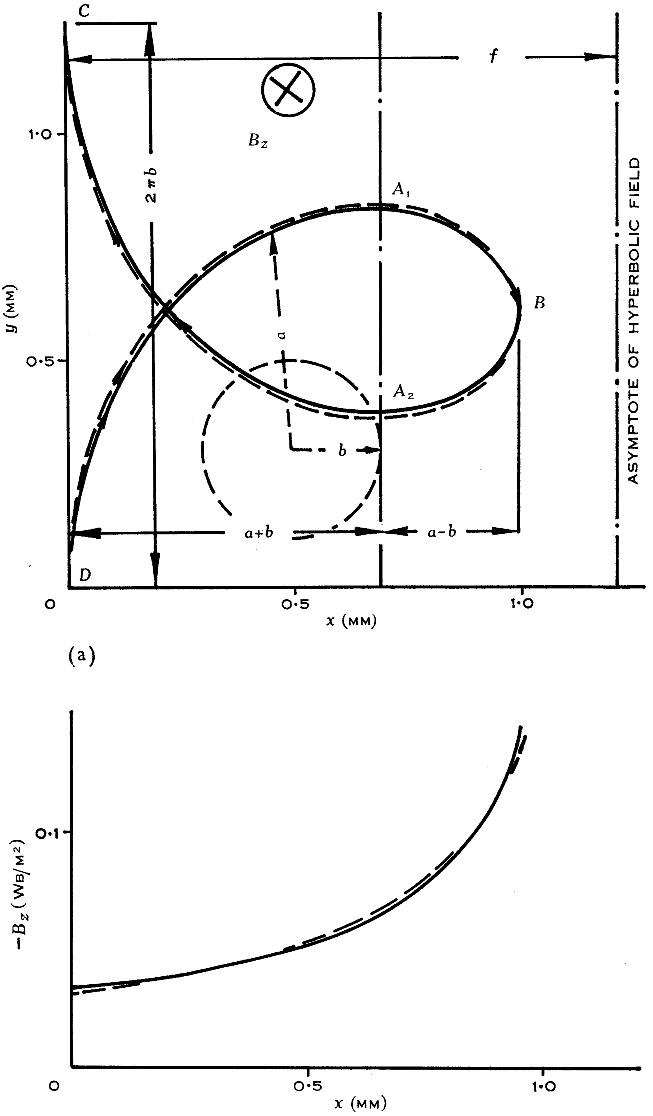
These equations define the trajectory of the electron.



(a)

(b)

Fig. 1.—(*a*) Electron paths; (*b*) magnetic fields. Electron energy, 10 keV.
——— Trochoid path and associated $B_z(x)$.
— — — Hyperbolic $B_z(x)$ and associated path.

Differentiating equations (1) and (2) twice with respect to time yields:

$$\dot{v}_x = \dot{\omega} v_x/\omega + \omega v_y - \omega^2 b, \tag{3}$$

$$\dot{v}_y = \dot{\omega} v_y/\omega - \omega v_x. \tag{4}$$

The non-relativistic equations of motion of an electron moving at constant speed in a magnetic field $B_z(x)$ are:

$$\dot{v}_x = -(e/m)v_y B_z, \tag{5}$$

$$\dot{v}_y = +(e/m)v_x B_z. \tag{6}$$

In order that equations (3) and (4) should represent physically possible electron motion, they must be identical with (5) and (6) respectively. Replacing $(e/m)B_z$ by Ω and equating (3) with (4) and (5) with (6), gives:

$$\Omega v_x = (\dot{\omega}/\omega)v_y - \omega v_x, \tag{7}$$

$$-\Omega v_y = (\dot{\omega}/\omega)v_x + \omega v_y - \omega^2 b. \tag{8}$$

Equations (7) and (8) may be integrated and, noting that the electron's speed u is constant and equal to $(a+b)\omega$, an explicit expression for the magnetic field may be derived in terms of the trochoidal parameters thus:

$$\frac{e}{m}B_z = \Omega = -u\frac{a(a+b\cos\theta)}{[(a+b)^2+2ab\cos\theta]^{3/2}}, \tag{9}$$

or, in terms of x,

$$B_z = -\frac{u}{(e/m)}\cdot\frac{a(a+b)-bx}{[(a+b)^2-2bx]^{3/2}}. \tag{10}$$

This explicit magnetic flux variation to produce the trochoidal trajectory is shown in Figure 1(b).

Electron Trajectory in a Hyperbolic Field

Additional Symbols Used in this Analysis:

f	Distance from the y-axis to the asymptote of the magnetic field	metre
k	A parameter giving the flux density at a distance x from the y-axis by means of the relation $B_z = k/(x-f)$	Wb/m

Substituting for B_z in equation (6) gives

$$v_x\,\mathrm{d}v_y/\mathrm{d}x = k(e/m)\{v_x/(x-f)\}. \tag{11}$$

Integrating equation (11),

$$v_{y_1} - u = k(e/m)\ln\{(f-x_1)/f\}. \tag{12}$$

The broken line in Figure 1 shows a typical electron trajectory. Certain points on this curve, namely, A_1, A_2, and B, will henceforth be called "principal points". The x coordinates of these points are easily found by means of equation (12), since for A_1 and A_2, $v_{y_1} = 0$, and for B, $v_{y_1} = -u$. Thus for A_1 and A_2

$$x_A = f\{1-\exp(-u/k(e/m))\}. \tag{13}$$

Similarly for B,

$$x_B = f\{1-\exp(-2u/k(e/m))\}. \tag{14}$$

Considering the trochoidal electron trajectory, the x components of the points A_1 and A_2 may be found by equating the y velocity component to zero, giving

$$x_A = a+b. \tag{15}$$

The point B is clearly given by

$$x_B = 2a. \tag{16}$$

Eliminating x_A and x_B from equations (13)–(16) yields:

$$f = (a+b)^2/2b, \tag{17}$$

$$k = u[(e/m)\ln\{(a+b)/(a-b)\}]^{-1}. \tag{18}$$

Correlation between the Two Methods

Equations (17) and (18) relate two electron trajectories, one trochoidal and the other produced by a hyperbolic magnetic field, both of which have the same x coordinates for their principal points A_1, A_2, and B. The trajectory in the hyperbolic field, which has been found completely using a Runge-Kutta digital computer program, can be seen to have virtually the same y coordinates at these points also, and in fact the two trajectories are practically coincident (there being a $1\cdot2\%$ difference between them in respect of the distance A_1A_2 and a $0\cdot7\%$ difference in the distance CD for a value of $a/b = 2\cdot5$). The importance of equations (17) and (18) is that they form a bridge between two different analytical approaches.

Other Properties of Non-uniform Magnetic Fields

In the course of devising the above-mentioned electron energy spectrometer, the following properties of the non-uniform fields were discovered analytically.

(*a*) A field region of finite size in which electrons of a particular energy enter and leave along a common axis will provide angular dispersion for adjacent energies according to the relation

$$\delta\alpha = \frac{\pi ab}{(a+b)^2}\cdot\frac{\delta V}{V}, \tag{19}$$

where $\delta\alpha$ = dispersion angle at the exit plane in radians, and V = energy of electrons in volts.

(*b*) The trajectories of all mono-energetic electrons traversing a hyperbolic magnetic field are geometrically similar, irrespective of their point of entry into the field, but the size of a loop of the trajectory is proportional to the distance of the point of entry from the asymptote of that field (Fig. 2).

(*c*) As a corollary to this, a parallel pencil of electrons entering a finite field region with an energy such as to leave the field on the same axis as it entered, will be divergent in the plane of the trajectory at the exit. The equivalent negative focal length of the system in this plane is

$$l = \frac{(a+b)^4}{4\pi ab^2} = \frac{f}{2[1-\exp\{-2u/k(e/m)\}]}. \tag{20}$$

As far as the author is aware, these properties have not been verified experimentally, although an electron energy spectrometer using a non-uniform magnetic field produced by inclined magnetic poles has been designed (Mathams 1963*a*, 1963*b*).

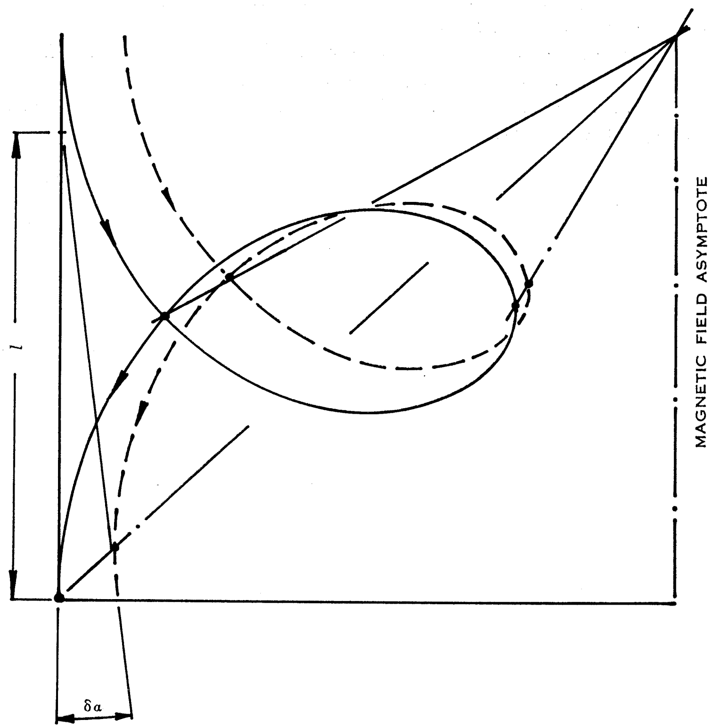


Fig. 2.—Similar trajectories of two electrons of equal energies.

Conclusions

The definition of an electron trajectory as a superior trochoid and the subsequent calculation of the non-uniform magnetic field to produce this trajectory leads to much simpler equations than the converse process of computing a trajectory in a defined non-uniform field. Furthermore, magnetic fields having variations very close to the exact form are readily produced by inclined pole faces. Correlating equations have been derived which relate these magnetic fields, and certain of their electron-optical properties have been stated.

A proposed application of the dispersive property of the finite magnetic field is an energy spectrometer for high voltage electron microscopy. There may also be applications in the study of high temperature plasmas.

This work was carried out at A.E.I. Research Laboratory, Aldermaston, England, and the author is indebted to many former colleagues for their guidance. In particular, he would mention the late Mr. D. P. R. Petrie and Mr. T. Mulvey, who gave valuable advice, and Dr. R. Bullough and Mr. A. C. Heath, who devised

the three-dimensional Runge-Kutta program for computing complete trajectories in hyperbolic fields.

References

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