# EQUILIBRIUM OF A PINCH DISCHARGE WITH A TRANSVERSE ROTATING MAGNETIC FIELD

By H. A. Blevin* and R. B. Miller*

[*Manuscript received March* 19, 1965]

*Summary*

The electron density distribution in a linear pinch discharge with a transverse rotating magnetic field is calculated for partially ionized plasmas. Numerical examples are given for distributions in the plasma with and without externally applied axial magnetic fields, and with different degrees of ionization.

## I. Introduction

It has been shown (Blevin and Thonemann 1961, 1962) that a magnetic field rotating about the axis of a cylindrical plasma column can produce large azimuthal current densities in the plasma, provided that the angular frequency of the rotating field lies between the ion and electron cyclotron frequencies. Blevin and Thonemann also calculated the equilibrium electron density distribution when an axial magnetic field is applied so that the radial Lorentz force confines the plasma away from the tube walls. In the present paper we consider the manner in which the equilibrium electron density is modified when an axial current flows through the plasma.

## II. Electron Density Distribution

For a plasma column of finite length the experiments described by Blevin and Thonemann indicated that end effects caused a distortion of the rotating magnetic field and modified the equilibrium plasma distribution. This complication could be avoided by using either toroidal discharges of large aspect ratio, or linear discharges for which the ratio of length to diameter is very large.

Consider a cylindrical plasma column, which is sufficiently long so that end effects can be neglected, to which an axial electric field $E_z$ and magnetic fields $B_a$, $B_0$ are applied as shown in Figure 1. It is assumed that there is no axial variation of any quantity. Provided that the angular frequency $\omega$ of the rotating field lies between the ion and electron cyclotron frequencies and the electron collision frequency is less than the electron cyclotron frequency, then the electrons rotate in synchronism with the rotating field (Blevin and Thonemann 1961), and it follows that (using CGS electromagnetic units)

$$j_\theta = -ner\omega, \tag{1}$$

$$B_z(r) = B_a - 2Ne\omega, \tag{2}$$

$$\mathrm{d}N/\mathrm{d}(r^2) = -\pi n, \tag{3}$$

where $n$ is the electron density, $e$ is the electronic charge, $j_\theta$ is the azimuthal current

* Department of Physics, University of New England, Armidale, N.S.W.

density, and $N$ is the number of electrons per unit length of the discharge which are *outside* a cylinder of radius $r$, that is,

$$N = 2\pi \int_r^\infty nr\,\mathrm{d}r.$$

When the plasma is in equilibrium $\nabla p = \mathbf{j}\times\mathbf{B}$. Putting $p = nkT$, where $T = T_\mathrm{i}+T_\mathrm{e}$ is the sum of the ion and electron temperatures and is assumed to be constant throughout the plasma, this equilibrium condition becomes

$$kT\,\mathrm{d}n/\mathrm{d}r = j_\theta B_z - j_z B_\theta. \tag{4}$$

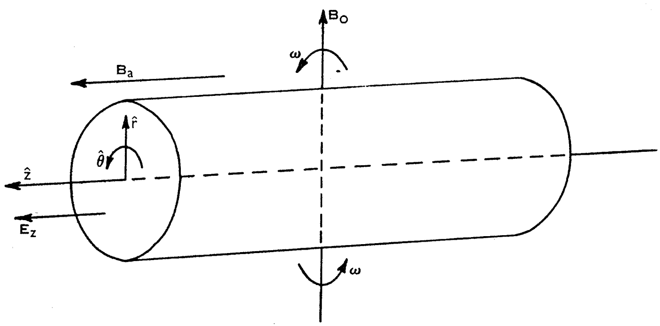


Fig. 1.—Configuration of fields applied to the cylindrical plasma.

Also, neglecting displacement currents, Maxwell's equations give $\nabla\times\mathbf{B} = 4\pi\mathbf{j}$ and, taking the $z$ component of this equation,

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = 4\pi j_z. \tag{5}$$

Substituting (1), (2), (3), and (5) in (4)

$$\frac{\mathrm{d}^2N}{\mathrm{d}(r^2)^2} = -\frac{e\omega}{2kT}\frac{\mathrm{d}N}{\mathrm{d}(r^2)}(B_\mathrm{a}-2Ne\omega)+\frac{2\pi^2 j_z}{kTr^2}\int_0^r j_z r\,\mathrm{d}r. \tag{6}$$

To proceed further, $j_z$ is required as a function of radial position. For a highly ionized plasma the resistivity is nearly independent of electron density (Spitzer 1956), and in the first instance we will assume that

(i) $j_z$ is constant throughout the plasma, and

(ii) the radius of the plasma column is $a$.

Thus the total axial current $I_z = \pi a^2 j_z$. Putting $(r/a)^2 = x$, (6) becomes (for $x < 1$)

$$\frac{\mathrm{d}^2N}{\mathrm{d}x^2}+\frac{B_\mathrm{a}a^2e\omega}{2kT}\frac{\mathrm{d}N}{\mathrm{d}x}-\frac{(ae\omega)^2}{2kT}\frac{\mathrm{d}N^2}{\mathrm{d}x}-\frac{I_z^2}{kT} = 0,$$

which can be integrated using the boundary conditions $N = 0$ and $\mathrm{d}N/\mathrm{d}x = 0$ when $x = 1$, to give

$$\frac{\mathrm{d}N}{\mathrm{d}x}+\frac{B_{\mathrm{a}}a^2e\omega}{2kT}N-\frac{(ae\omega)^2}{2kT}N^2+\frac{I_z^2}{kT}(1-x) = 0, \tag{7}$$

which is an integrable form of Riccati's equation (McLachlan 1950).

Let

$$z = \exp\left\{-\frac{ae\omega}{2kT}\int(ae\omega N-\tfrac{1}{2}B_{\mathrm{a}}a)\,\mathrm{d}x\right\} \tag{8}$$

and

$$X = 1+(B_{\mathrm{a}}^2a^2/8I_z^2)-x.$$

Then (7) reduces to

$$\frac{\mathrm{d}^2z}{\mathrm{d}X^2}-\frac{1}{2}\left(\frac{I_z ae\omega}{kT}\right)^2Xz = 0$$

with the solution

$$z = A_0X^{\frac{1}{2}}\mathrm{I}_{\frac{1}{3}}(\phi)+A_1X^{\frac{1}{2}}\mathrm{I}_{-\frac{1}{3}}(\phi), \tag{9}$$

where

$$\phi = \frac{\surd 2}{3}\left(\frac{I_z ae\omega}{kT}\right)X^{3/2};$$

$A_0$, $A_1$ are constants of integration; and $\mathrm{I}_{\frac{1}{3}}(\phi)$, $\mathrm{I}_{-\frac{1}{3}}(\phi)$ are the modified Bessel functions. From (8) and (9),

$$N = \frac{B_{\mathrm{a}}}{2e\omega}+\frac{kT}{X(ae\omega)^2}\left[1+3\phi\left\{\frac{A_0\mathrm{I}'_{\frac{1}{3}}(\phi)+A_1\mathrm{I}'_{-\frac{1}{3}}(\phi)}{A_0\mathrm{I}_{\frac{1}{3}}(\phi)+A_1\mathrm{I}_{-\frac{1}{3}}(\phi)}\right\}\right],$$

where the prime denotes differentiation with respect to $\phi$. Using the boundary condition $N = 0$ when $x = 1$ and the recurrence relations for Bessel functions, it follows that

$$A_0/A_1 = -k_0,$$

where

$$k_0 = \{\mathrm{I}_{-\frac{1}{3}}(\phi_0)+\mathrm{I}_{\frac{2}{3}}(\phi_0)\}\{\mathrm{I}_{\frac{1}{3}}(\phi_0)+\mathrm{I}_{-\frac{2}{3}}(\phi_0)\}^{-1},$$

and

$$\phi_0 = \phi(x=1) = B_{\mathrm{a}}^3a^4e\omega/48I_z^2kT.$$

Put

$$\chi = \{k_0\mathrm{I}_{-\frac{2}{3}}(\phi)-\mathrm{I}_{\frac{2}{3}}(\phi)\}\{k_0\mathrm{I}_{\frac{1}{3}}(\phi)-\mathrm{I}_{-\frac{1}{3}}(\phi)\}^{-1},$$

then

$$N = \frac{B_{\mathrm{a}}}{2e\omega}+\frac{3kT}{(ae\omega)^2}\phi\chi.$$

Using (2) and (3),

$$B_z = (-2^{3/2}/a) I_z X^{\frac{1}{2}} \chi, \tag{10}$$

$$n = (I_z^2/\pi a^2 kT) X (1-\chi^2), \tag{11}$$

and from (5),

$$B_\theta = (2I_z/a) x^{\frac{1}{2}}, \qquad \text{for } x < 1. \tag{12}$$

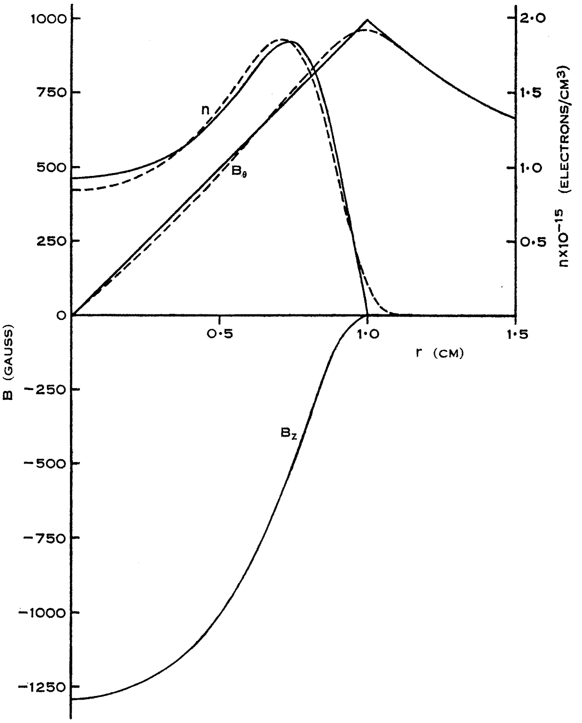


Fig. 2.—Electron density and magnetic field distributions for different values of ionization (—— 100%, - - - 16%, at the axis) with $T = 10^5$ °K, $I_z = 5000$ A, $B_a = 0$, and $N_0 = 4 \cdot 05 \times 10^{15}$ electrons/cm.

Equations (10), (11), and (12) give the magnetic field distributions (apart from the rotating field, which does not enter into the pressure balance equation) and the electron density distribution in the plasma. The field distributions can be grouped into three categories.

(i) No externally applied axial field, $B_a = 0$.

(ii) Axial field reversed in the central region of the discharge. Equation (2) shows that this occurs when $B_a < 2N_0 e\omega$, where $N_0$ $(= N_{r=0})$ is the number of electrons per unit length of the plasma column.

(iii) Axial field in the same direction throughout the discharge. This occurs for $B_a > 2N_0 e\omega$.

Examples of each of these distributions are shown in Figures 2, 3, and 4 using some arbitrarily chosen values of the discharge parameters $B_a$, $I_z$, $T$, and $N_0$ (in fact

$N_0$ has been chosen to make $a = 1$ cm in each case). Perhaps the most unrealistic feature of these results is the assumed discontinuity in the electrical conductivity at $r = a$ and, as a consequence of this, the large pressure gradients at the plasma boundary. In practice the current density in the outer regions of the discharge where $n$ has small values is determined either by electron run-away or, if there is an appreciable number of neutral molecules present, by electron collisions with gas molecules. In the latter case the electron density distribution can be determined from (6) by inserting the appropriate expression for the current density $j_z$ as a function of radial position.

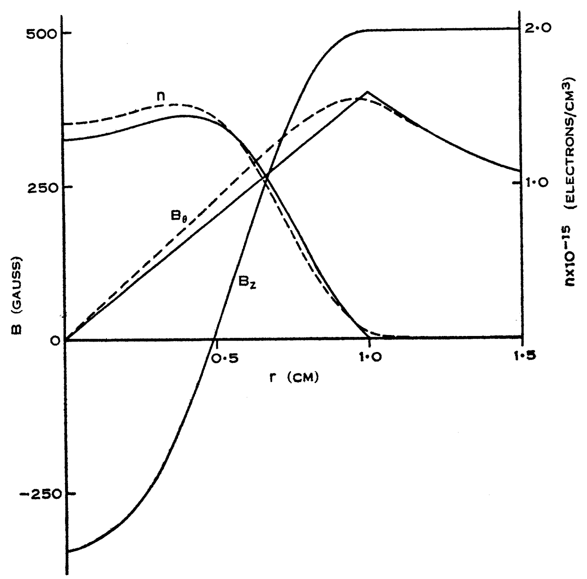


Fig. 3.—Electron density and magnetic field distributions for different values of ionization (—— 100%, - - - 49%, at the axis) with $T = 10^5$ °K, $I_z = 2000$ A, $B_a = 500$ G, and $N_0 = 2\cdot 64\times 10^{15}$ electrons/cm.

For a partially ionized gas the resistivity $\eta$ is approximately given by (Allis 1956) $\eta = (m/ne^2)(\nu_i+\nu_n)$, where $\nu_i$, $\nu_n$ are the appropriate momentum transfer collision frequencies for electron–ion and electron–neutral collisions respectively, and $m$ is the electron mass. In a *hydrogen* plasma $\nu_n$ is approximately constant for electron energies above about 3 eV, and the collision frequencies are given by (Delcroix 1960)*

$$\nu_n \approx 2\times 10^{-7} n_n,$$

$$\nu_i \approx 2\cdot 6 n T_e^{-3/2} \ln \Lambda,$$

where $n_n$ is the neutral gas density and $\ln \Lambda$ is a slowly varying function of $n$ and

* Although we will use these values in later numerical examples, it should be noted that this value of $\nu_n$ is a rather high approximation.

$T_e$ (Spitzer 1956). Substituting these values in the expression for $\eta$, then

$$j_z = \frac{E_z}{\eta} = \frac{e^2 T_e^{3/2} E_z}{2\cdot 6 m \ln\Lambda}\left\{\frac{n}{\alpha n_n + n}\right\},$$

where $\alpha \approx 7\cdot 7\times 10^{-8} T_e^{3/2}/\ln\Lambda$.

If $\ln\Lambda$ is approximated by a constant with a value of 10 for the electron densities and temperatures considered in the examples of Figures 2, 3, and 4, then using (3)

$$j_z = -KE_z \frac{\mathrm{d}N}{\mathrm{d}(r^2)}\Big/\left(\alpha n_n - \frac{1}{\pi}\frac{\mathrm{d}N}{\mathrm{d}(r^2)}\right), \tag{13}$$

where $K$ and $\alpha$ are now functions of $T_e$ alone. For $\alpha n_n \ll n$ this reduces to the former case $j_z =$ constant, while for $\alpha n_n \gg n$ the current density is proportional to $n$.

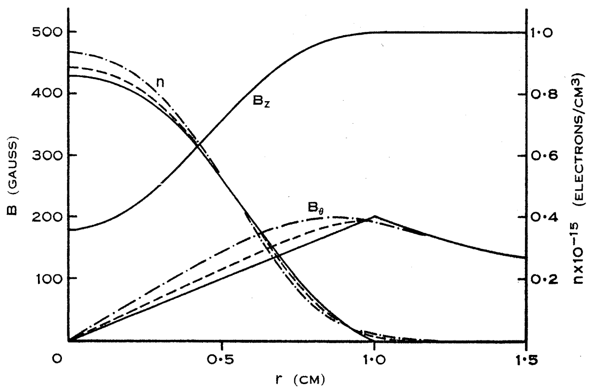


Fig. 4.—Electron density and magnetic field distributions for different values of ionization (—— 100%, --- 70%, —•— 44%, at the axis) with $T = 10^5$ °K, $I_z = 1000$ A, $B_a = 500$ G, and $N_0 = 1\cdot 00\times 10^{15}$ electrons/cm.

Substituting (13) into (6) gives

$$\frac{\mathrm{d}^2N}{\mathrm{d}(r^2)^2} + \frac{e\omega}{2kT}\frac{\mathrm{d}N}{\mathrm{d}(r^2)}(B_a - 2Ne\omega) - \frac{2\pi^2(KE_z)^2\mathrm{d}N/\mathrm{d}(r^2)}{kTr^2\{\alpha n_n - (1/\pi)\mathrm{d}N/\mathrm{d}(r^2)\}}\int_0^r \frac{\{\mathrm{d}N/\mathrm{d}(r^2)\}\,.\,r\,.\,\mathrm{d}r}{\alpha n_n - (1/\pi)\mathrm{d}N/\mathrm{d}(r^2)} = 0,$$

which can be solved numerically for given discharge parameters.

Examples of the resulting magnetic field and electron density distributions are shown in Figures 2, 3, and 4 for several values of $\alpha n_n$. The percentage ionization (at any particular position in the plasma) can only be determined from $\alpha n_n$ when $T_e$ is known, and the values indicated on the diagrams were obtained by taking $T_e = T_i = \frac{1}{2}T$. Although the neutral gas density has been assumed to be independent of $r$ in these calculations, it is no more difficult to include a dependence of $n_n$ on radial position. For comparison with the results obtained for $j_z =$ constant, the values of

$T$, $I_z$, $B_a$, and $N_0$ were held constant in each case. This necessitates a change in the value of $E_z$ for different amounts of neutral gas.

These results show that appreciable neutral particle densities are required to affect the magnetic field profiles to any great extent. For example, in the case considered in Figure 4 the ionization at the tube axis must be less than $\sim 70\%$ for the magnetic field components to change by more than 10% from the case where neutral gas is not present. In addition, the effect of neutral particles on the field profiles depends critically on the nature of the density distribution. This is shown in Figure 2, where the fields are only slightly changed for an ionization as low as 10%. In this case $\alpha n_n$ becomes comparable to $n$ only in a small region near the plasma boundary. For higher temperature plasmas (high $\alpha$) a lower percentage of neutral molecules is required to produce a comparable effect on the magnetic field distributions.

## III. Conclusions

In a pinch discharge with a transverse rotating magnetic field, a measurement of the magnetic fields $B_\theta$, $B_z$ as a function of radius would enable the plasma temperature $T$ to be determined, and also the electron density as a function of radius. For partially ionized plasmas the neutral gas density could also be estimated when the percentage ionization is sufficiently low so that the resistivity of the plasma is appreciably modified by the presence of neutral gas molecules.

## IV. Acknowledgments

This work forms part of a program which has been supported by the Australian Institute of Nuclear Science and Engineering. One of us (R.B.M.) gratefully acknowledges the financial assistance received from an I.C.I.A.N.Z. scholarship.

## V. References

Allis, W. P. (1956).—"Handbuch der Physik." Vol. 21, pp. 383–444. (Springer–Verlag: Berlin.)

Blevin, H. A., and Thonemann, P. C. (1961).—U.K.A.E.A. Research Report CLM–R12.

Blevin, H. A., and Thonemann, P. C. (1962).—*Nucl. Fusion*, 1962 Suppl., Part I, p. 55.

Delcroix, J. L. (1960).—"Introduction to the Theory of Ionized Gases." (Interscience: New York.)

McLachlan, N. W. (1950).—"Ordinary Non-Linear Differential Equations." (Oxford Univ. Press.)

Spitzer, L. (1956).—"Physics of Fully Ionised Gases." (Interscience: New York.)