SHORT COMMUNICATIONS

UNSTEADY HEAT TRANSFER IN CHANNEL FLOW*

By I. J. Kumar† and Balkrishan†

The study of heat transfer in channel flow at high fluid velocities is important owing to its application in the design of rocket nozzles and jet pipes. The problem retains its importance even at moderately high velocities where, for the study of heat transfer processes, one may consider the heating gas as an incompressible fluid, e.g. in the flow of gases through rocket combustion chambers and the initial heating of a heat regenerator. The study also finds an application in the theory of internal ballistics for calculation of heat transfer to a gun barrel during firing. No analytic solution can be derived for such problems, because the flow and energy equations for the fluid in such cases are coupled nonlinearly owing to convective terms. Though a number of steady-state problems of the above type have been studied with boundary-layer approximation, very few exact solutions are known.

Recently, Johnson (1961) has studied an exact solution to the problem of heat transfer in parallel fluid flow over a conducting half-space, the fluid being considered viscous and incompressible and with the assumption of continuity of flux and temperature at the solid–fluid interface. Here we discuss a solution, suitable for small values of time, to the problem of heat transfer between two solid walls $|z| > l$ and an in-flowing incompressible fluid in the channel $|z| < l$, assuming the conditions of flux continuity and convective heat transfer across the interface. It is further assumed that fluid is set in motion impulsively with a uniform velocity along the channel. Since the velocity distribution in the above problem satisfies a diffusion type of equation for which a solution is available in the literature, such a solution can be substituted in the energy equation for the fluid to obtain the temperature–time history.

For the problem stated above, the flow and energy equations for the fluid and the solid are:

$$\frac{\partial u}{\partial t}-\nu\frac{\partial^2 u}{\partial z^2}=0 \qquad |z|<l \qquad t>0, \tag{1}$$

$$\rho_1 c_1\frac{\partial T_1}{\partial t}-K_1\frac{\partial^2 T_1}{\partial z^2}=\mu\left(\frac{\partial u}{\partial z}\right)^2 \qquad |z|<l \qquad t>0, \tag{2}$$

$$\rho_2 c_2\frac{\partial T_2}{\partial t}-K_2\frac{\partial^2 T_2}{\partial z^2}=0 \qquad |z|>l \qquad t>0. \tag{3}$$

Initial Conditions:

$$u=u_0, \tag{4}$$

$$T_1=T_0, \tag{5}$$

$$T_2=0. \tag{6}$$

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† Defence Science Laboratory, Delhi, India.

Boundary Conditions:

From the symmetry of the problem, we may write the boundary conditions for $z > 0$ as

$$u = 0 \qquad z = l \qquad t > 0, \tag{7}$$

$$\frac{\partial u}{\partial z} = 0 \qquad z = 0 \qquad t > 0, \tag{8}$$

$$K_1 \frac{\partial T_1}{\partial z} = K_2 \frac{\partial T_2}{\partial z} \qquad z = l \qquad t > 0, \tag{9}$$

$$K_1 \frac{\partial T_1}{\partial z} + H(T_1 - T_2) = 0 \qquad z = l \qquad t > 0, \tag{10}$$

$$\frac{\partial T_1}{\partial z} = 0 \qquad z = 0 \qquad t > 0. \tag{11}$$

In all the above, the subscripts 1 and 2 refer to the fluid and solid respectively, and subscript 0 refers to the initial value. The remaining symbols have their usual meaning.

The solution of the partial differential equation (1) with boundary and initial conditions (7), (8), and (4) is (Carslaw and Jaeger 1948)

$$u(z,t) = u_0\left[1 - \sum_{n=0}^{\infty} (-1)^n \left\{\operatorname{erfc}\left(\frac{(2n+1)l - z}{2(\nu t)^{\frac{1}{2}}}\right) + \operatorname{erfc}\left(\frac{(2n+1)l + z}{2(\nu t)^{\frac{1}{2}}}\right)\right\}\right]. \tag{12}$$

Substituting this in (2), we get

$$\frac{\partial T_1}{\partial t} - k_1 \frac{\partial^2 T_1}{\partial z^2} = \frac{u_0^2}{c_1 \pi t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \sum_{r=1}^{4} \exp\left(-\frac{\xi_{r,mn}^2}{4\nu t}\right), \tag{13}$$

where

$$\left.\begin{aligned} \xi_{1,mn}^2 &= 2z^2 + 2lz(2m+2n+2) + l^2\phi_{mn}^2, \\ \xi_{2,mn}^2 &= 2z^2 + 2lz(2m-2n) + l^2\phi_{mn}^2, \\ \xi_{3,mn}^2 &= 2z^2 - 2lz(2m+2n+2) + l^2\phi_{mn}^2, \\ \xi_{4,mn}^2 &= 2z^2 - 2lz(2m-2n) + l^2\phi_{mn}^2, \end{aligned}\right\} \tag{14}$$

and

$$\phi_{mn}^2 = (2m+1)^2 + (2n+1)^2.$$

Let $\bar{T}$ be the Laplace transform of T, defined as

$$\bar{T} = \int_0^\infty T \exp(-pt)\, dt. \tag{15}$$

The Laplace transforms of (13) and (3) are therefore given respectively as

$$\frac{d^2\bar{T}_1}{dz^2}-q_1^2\bar{T}_1 = -\overline{\frac{u_0^2}{c_1 k_1 \pi t}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\sum_{r=1}^{4}\exp\left(-\frac{\xi_{r,mn}^2}{4\nu t}\right)} \tag{16}$$

and

$$\frac{d^2\bar{T}_2}{dz^2}-q_2^2\bar{T}_2 = 0, \tag{17}$$

the bar over the right-hand side of (16) meaning the Laplace transform of the expression under the bar, where

$$\left.\begin{aligned} q^2 &= p/k, \\ k &= K/\rho c. \end{aligned}\right\} \tag{18}$$

Taking the Laplace transform of the double sum of line sources on the right-hand side of (16) is justified, since the double series can be easily proved to be uniformly convergent by application of Dini's test and the integral test for convergence of double series (Bromwich 1942). Equation (16) can therefore be put in the form

$$\frac{d^2\bar{T}_1}{dz^2}-q_1^2\bar{T}_1 = -\frac{2E}{k_1\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\sum_{r=1}^{4}\mathrm{K}_0\left(\frac{q_1\xi_{r,mn}}{P_r^{\frac{1}{2}}}\right), \tag{19}$$

where

$$E = u_0^2/c_1 T_0 \qquad \text{(Eckert number)},$$

$$P_r = \nu/k_1 \qquad \text{(Prandtl number)},$$

and K_0 is the modified Bessel function of the second kind of order zero.

The solutions of (19) and (17) with boundary conditions (9)–(11) are

$$\frac{\bar{T}_1}{T_0} = \frac{1}{p}\left(1-\frac{h\cosh q_1 z}{\Delta p \sinh q_1 l}\right)$$
$$+\frac{2E}{k_1\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\sum_{r=1}^{4}\int_l^z \frac{1}{q_1}\mathrm{K}_0\left(\frac{q_1\xi_{r,mn}}{P_r^{\frac{1}{2}}}\right)\sinh q_1(\beta-z)\,d\beta, \tag{20}$$

$$\frac{\bar{T}_2}{T_0} = \frac{hK^*}{p\,\Delta p}\exp\{-q_2(z-l)\}, \tag{21}$$

where

$$\Delta p = q_1+hK^*+h\coth q_1 l,$$
$$h = H/K_1,$$
$$K^* = (K_1/K_2)(k_2/k_1)^{\frac{1}{2}}.$$

To obtain the values of T_1/T_0 and T_2/T_0 for small values of time, we expand the integrands (after Carslaw and Jaeger 1948) for large values of p. The series for $\bar{T}_1$ and $\bar{T}_2$ are quite complicated, and we therefore retain the first few terms only, whose inversions with respect to Laplace transformation provide a solution useful for small values of time. It may, however, be remarked that the exponential terms of the type $\exp[-q_1\{(2n+1)l+z\}]$ have been neglected for $n > 1$, and the inversions for all the terms retained are available at Appendix V of Carslaw and Jaeger (1959). Thus we get

$$\frac{T_1}{T_0} = 1-\sum_{s=1}^{2}\sum_{n=0}^{1}\frac{(-1)^n}{K^*+1}\left\{\operatorname{erfc}\left(\frac{z_{ns}}{2(k_1 t)^{\frac{1}{2}}}\right)-\exp\{h(K^*+1)z_{ns}+k_1 th^2(K^*+1)^2\}\right.$$
$$\left.\times\operatorname{erfc}\left(\frac{z_{ns}}{2(k_1 t)^{\frac{1}{2}}}+h(K^*+1)(k_1 t)^{\frac{1}{2}}\right)\right\}$$
$$-\sum_{s=1}^{2}\frac{2}{(K^*+1)^2}\left\{\operatorname{erfc}\left(\frac{z_{1s}}{2(k_1 t)^{\frac{1}{2}}}\right)-2(K^*+1)h\left(\frac{k_1 t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{z_{1s}^2}{4k_1 t}\right)\right.$$
$$-\{1-h(K^*+1)z_{1s}-2h^2(K^*+1)^2k_1 t\}$$
$$\times\exp\{h(1+K^*)z_{1s}+k_1 th^2(K^*+1)^2\}$$
$$\left.\times\operatorname{erfc}\left(\frac{z_{1s}}{2(k_1 t)^{\frac{1}{2}}}+h(K^*+1)(k_1 t)^{\frac{1}{2}}\right)\right\}$$
$$+\frac{2EP_r^{\frac{1}{2}}}{\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\sum_{r=1}^{4}\int_l^z\left(\frac{\beta+\xi_{r,mn}-z}{2k_1 t}\right)^{\frac{1}{2}}\exp\left(-\frac{(\beta+\xi_{r,mn}-z)^2}{8k_1 t}\right)$$
$$\times K_{\frac{1}{4}}\left(\frac{\beta+\xi_{r,mn}-z}{8k_1 t}\right)d\beta, \tag{22}$$

where $\quad z_{ns} = (2n+1)l+(-1)^s z;$

$$\frac{T_2}{T_0} = \frac{K^*}{K^*+1}\left\{\operatorname{erfc}\left(\frac{(k_1/k_2)^{\frac{1}{2}}(z-l)}{2(k_1 t)^{\frac{1}{2}}}\right)-\exp\{h(K^*+1)(k_1/k_2)^{\frac{1}{2}}(z-l)+k_1 th^2(K^*+1)^2\}\right.$$
$$\left.\times\operatorname{erfc}\left(\frac{(k_1/k_2)^{\frac{1}{2}}(z-l)}{2(k_1 t)^{\frac{1}{2}}}+h(K^*+1)(k_1 t)^{\frac{1}{2}}\right)\right\}$$
$$-\frac{2K^*}{(K^*+1)^2}\left\{\operatorname{erfc}\left(\frac{(k_1/k_2)^{\frac{1}{2}}(z-l)+2l}{2(k_1 t)^{\frac{1}{2}}}\right)-2h(K^*+1)\left(\frac{k_1 t}{\pi}\right)^{\frac{1}{2}}\right.$$
$$\times\exp\left(-\frac{\{(k_1/k_2)^{\frac{1}{2}}(z-l)+2l\}^2}{4k_1 t}\right)$$
$$-[1-h(K^*+1)\{(k_1/k_2)^{\frac{1}{2}}(z-l)+2l\}-2h^2(K^*+1)^2 k_1 t]$$
$$\times\exp[h(K^*+1)\{(k_1/k_2)^{\frac{1}{2}}(z-l)+2l\}+k_1 th^2(K^*+1)^2]$$
$$\left.\times\operatorname{erfc}\left(\frac{(k_1/k_2)^{\frac{1}{2}}(z-l)+2l}{2(k_1 t)^{\frac{1}{2}}}+h(K^*+1)(k_1 t)^{\frac{1}{2}}\right)\right\}. \tag{23}$$

To obtain the heat transfer rate at the surface, $H(T_1-T_2)$, we have to get T_1 and T_2 at $z = l$. For this purpose, it is convenient to put $z = l$ in equations

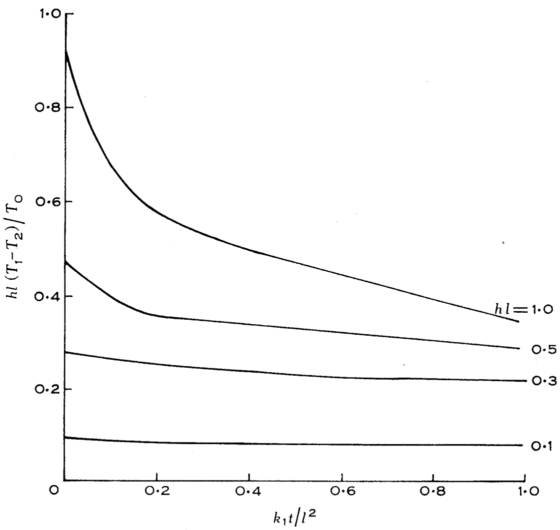

Fig. 1.—Surface heat transfer rate $hl(T_1-T_2)/T_0$ plotted against small values of k_1t/l^2, for various values of hl.

(20) and (21) and then to apply the inversion theorem. This gives

$$\begin{aligned}\frac{T_1}{T_0}\bigg|_{z=l} &= 1-\frac{1}{K^*+1}\Big(1-\exp\{h^2(K^*+1)^2k_1t\}\,\mathrm{erfc}\{h(K^*+1)(k_1t)^{\frac{1}{2}}\}\Big)\\ &\quad-\frac{2}{K^*+1}\left\{\mathrm{erfc}\left(\frac{l}{(k_1t)^{\frac{1}{2}}}\right)-\exp\{2lh(K^*+1)+k_1th^2(K^*+1)^2\}\right.\\ &\qquad\left.\times\,\mathrm{erfc}\left(\frac{l}{(k_1t)^{\frac{1}{2}}}+h(K^*+1)(k_1t)^{\frac{1}{2}}\right)\right\}\\ &\quad-\frac{2}{(K^*+1)^2}\left\{\mathrm{erfc}\left(\frac{l}{(k_1t)^{\frac{1}{2}}}\right)-2h(K^*+1)\left(\frac{k_1t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{l^2}{k_1t}\right)\right.\\ &\qquad-\{1-2hl(K^*+1)-2h^2(K^*+1)^2k_1t\}\\ &\qquad\times\exp\{2hl(K^*+1)+h^2(K^*+1)^2k_1t\}\\ &\qquad\left.\times\,\mathrm{erfc}\left(\frac{l}{(k_1t)^{\frac{1}{2}}}+h(K^*+1)(k_1t)^{\frac{1}{2}}\right)\right\}; \qquad (24)\end{aligned}$$

$$\begin{aligned}\frac{T_2}{T_0}\bigg|_{z=l} &= \frac{K^*}{K^*+1}\Big(1-\exp\{h^2(K^*+1)^2k_1t\}\,\mathrm{erfc}\{h(K^*+1)(k_1t)^{\frac{1}{2}}\}\Big)\\ &\quad-\frac{2K^*}{(K^*+1)^2}\left\{\mathrm{erfc}\left(\frac{l}{(k_1t)^{\frac{1}{2}}}\right)-2h(K^*+1)\left(\frac{k_1t}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{l^2}{k_1t}\right)\right.\\ &\qquad-\{1-2hl(K^*+1)-2h^2(K^*+1)^2k_1t\}\\ &\qquad\times\exp\{2hl(K^*+1)+h^2(K^*+1)^2k_1t\}\\ &\qquad\left.\times\,\mathrm{erfc}\left(\frac{l}{(k_1t)^{\frac{1}{2}}}+h(K^*+1)(k_1t)^{\frac{1}{2}}\right)\right\}. \qquad (25)\end{aligned}$$

Figure 1 exhibits the non-dimensional heat transfer rate $hl(T_1-T_2)/T_0$ plotted against values of k_1t/l^2 for several values of the non-dimensional heat transfer coefficient hl. These results have been computed for $l=1$, $(k_1/k_2)^{\frac{1}{2}}=0\cdot 1$, and $K^*=0\cdot 1$.

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