EXCITATION OF CYCLOTRON ELECTROMAGNETIC WAVES IN A MAGNETOACTIVE PLASMA BY A STREAM OF CHARGED PARTICLES, INCLUDING TEMPERATURE EFFECTS IN THE STREAM

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Summary

The dispersion equation for cyclotron electromagnetic waves in a system comprising a charged particle stream injected at an angle to the static magnetic field of a magnetoactive plasma is derived for a general wave-normal angle θ when the temperature of the stream has been taken into account. The expression for the growth rate is derived in general for all cyclotron waves.

I. Introduction

The radiative instability problem of a stream–magnetoactive plasma system has been studied by a number of authors during the past few years (Zheleznyakov 1960*a*, 1960*b*; Stepanov and Kitsenko 1961; Neufeld and Wright 1964; Fung 1966*a*, 1966*b*). The main features of these treatments have been that:

(1) The ambient plasma is taken to be cold and magnetoactive.

(2) The distribution functions of the stream considered are:

(2.1) a delta function distribution in momentum space for both components of momentum $p_\perp$ and $p_\parallel$ where $p_\perp \neq 0$ and $p_\parallel \neq 0$ (here we assign the direction parallel to the static magnetic field to be the longitudinal direction and the one perpendicular to it the transverse direction);

(2.2) a distribution function where there is dispersion of particles over the momenta $p_\perp$ and $p_\parallel$, and $p_\perp^0 = 0$ whereas $p_\parallel^0$ is non-zero ($p_\perp^0$, $p_\parallel^0$ are values of momenta where the distribution curve shows the maximum);

(2.3) as in case (2.2) but where $p_\perp^0 \neq 0$.

(3) The assumed emission or wave-normal angle θ falls into three classes:

(3.1) strictly longitudinal propagation, that is, $\theta = 0°$ or $180°$;

(3.2) θ close to $0°$ or $180°$;

(3.3) general θ.

Zheleznyakov (1960*a*, 1960*b*) considered the instability problem in the combinations (2.1), (3.1), and (2.3), (3.1). Neufeld and Wright (1964) developed and interpreted the case (2.1), (3.1). Stepanov and Kitsenko (1961) discussed the case (2.2), (3.2) and some particular cases of (2.2), (3.3). Fung (1966*a*) derived the growth rate for the case (2.1), (3.3) (evaluation of the growth rate and application of the theory has also been attempted by Fung (1966*a*, 1966*b*) for two different types of emissions in radio astronomy).

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When the distribution function of the stream is as type (2.2), most radiating particles in the system acquire zero or very small values of $p_{\perp}$ and hence the excitation of cyclotron waves is not important in such a system. On the other hand, the excitation of Čerenkov and longitudinal plasma waves will be more pronounced in the system with such a "longitudinal stream" (a stream being injected along the static magnetic field). As far as excitation of cyclotron waves is concerned, the combination (2.3), (3.3), that is, $p_{\perp}^{0}$ and $p_{\parallel}^{0}$ non-zero, seems to be most important, for it probably represents the practical situation. It is the purpose of the present paper to derive the dispersion equation, thus leading to the calculation of the growth rate, for the case (2.3), (3.3).

II. Theory

We consider a cold magnetoactive ambient plasma being specified by the collisionless Appleton–Hartree formula

$$n_j^2 = 1 - \frac{x(1-x)}{1-x-\frac{1}{2}y^2\sin^2\theta \mp \{\frac{1}{4}y^4\sin^4\theta + (1-x)^2y^2\cos^2\theta\}^{\frac{1}{2}}}, \tag{1}$$

where n_j is the refractive index, $x = \omega_{\mathrm{p}}^2/\omega^2$, $y = \omega_H/\omega$, $\omega_{\mathrm{p}}^2 = 4\pi Ne^2/m_0$ is the angular plasma frequency squared, ω is the angular wave frequency, $\omega_H = ZeH_0/m_0c$ is the angular plasma gyro frequency, θ is the wave-normal angle, N is the particle density of the plasma, e is the electronic charge, Z is the number of electronic charges in the charged particle, m_0 is the rest mass of the charged particle, and c is the speed of light in a vacuum.

Instead of a strictly "monoenergetic" charged particle stream, we consider a stream having momentum spread in both components $p_{\perp}$ and $p_{\parallel}$. $p_{\perp}^{0}$ and $p_{\parallel}^{0}$ are supposed to be the values of momentum components where the distribution curve reaches its maximum. More precisely, the unperturbed particle distribution function $f_0(\bar{p})$ of the stream is given by

$$f_0(\bar{p})\,\mathrm{d}\bar{p} = \frac{1}{A}\exp\left\{-\frac{(p_{\parallel}-p_{\parallel}^{0})^2}{a_{\parallel}^2} - \frac{(p_{\perp}-p_{\perp}^{0})^2}{a_{\perp}^2}\right\}\mathrm{d}\bar{p}, \tag{2}$$

where $A = 2\pi^{3/2}\,a_{\perp}^2\,a_{\parallel}\,G_0$ is the normalization constant of f_0, $a_{\perp}^2 = 2m_0\,\kappa T_{\perp}$, $a_{\parallel}^2 = 2m_0\,\kappa T_{\parallel}$, m_0 is the rest mass of the radiating particle,* κ is the Boltzmann constant, and $T_{\perp}$ and $T_{\parallel}$ are the transverse and longitudinal temperatures respectively. For the normalization constant

$$G_0 = \int_0^\infty \zeta\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta, \quad \text{where} \quad \zeta = p_{\perp}/a_{\perp} \quad \text{and} \quad \zeta_0 = p_{\perp}^{0}/a_{\perp}.$$

With an ambient plasma specified by (1) and a distribution function of radiating particles as described by (2), we will now derive the dispersion equation for electromagnetic waves in the stream–magnetoactive plasma system, employing the method used by Zheleznyakov (1960*a*, 1960*b*), Neufeld and Wright (1964), and Fung (1966*a*).

The relativistic expressions of dielectric tensor components for growing electromagnetic waves in the stream are (Fung 1966*a*)

* For simplicity, we consider one species of radiating particles only.

$$
\left.
\begin{aligned}
\epsilon_{xx} &= 1-4\pi \iint \frac{\omega_0^2 m_0 \omega_H s^2 \mathrm{J}_s \mathrm{J}_s'(\omega m - k_\parallel p_\parallel) f_0}{\omega^2 k_\perp R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad -2\pi \iint \frac{\omega_0^2 m_0^2 \omega_H^2 s^2 \mathrm{J}_s^2 p_\perp (k_\parallel^2 - \omega^2/c^2) f_0}{\omega^2 k_\perp^2 R^2} \mathrm{d}p_\perp \mathrm{d}p_\parallel, \\
\epsilon_{yy} &= 1-4\pi \iint \frac{\omega_0^2 p_\perp \mathrm{J}_s'^2 (\omega m - k_\parallel p_\parallel) f_0}{\omega^2 R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad -4\pi \iint \frac{\omega_0^2 p_\perp^2 k_\perp \mathrm{J}_s' \mathrm{J}_s'' (\omega m - k_\parallel p_\parallel) f_0}{\omega^2 m_0 \omega_H R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad -2\pi \iint \frac{\omega_0^2 p_\perp^3 \mathrm{J}_s'^2 (k_\parallel^2 - \omega^2/c^2) f_0}{\omega^2 R^2} \mathrm{d}p_\perp \mathrm{d}p_\parallel, \\
\epsilon_{zz} &= 1-\frac{\omega_0^2}{\omega^2} - 4\pi \iint \frac{\omega_0^2 p_\parallel^2 k_\perp \mathrm{J}_s \mathrm{J}_s' (\omega m - k_\parallel p_\parallel) f_0}{\omega^2 m_0 \omega_H R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad -2\pi \iint \frac{\omega_0^2 p_\perp p_\parallel \mathrm{J}_s^2 (p_\parallel \omega/mc^2 + 2k_\parallel) f_0}{\omega^2 R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad -2\pi \iint \frac{\omega_0^2 p_\perp p_\parallel^2 \mathrm{J}_s^2 (k_\parallel^2 - \omega^2/c^2) f_0}{\omega^2 R^2} \mathrm{d}p_\perp \mathrm{d}p_\parallel, \\
\epsilon_{xy} &= -\mathrm{i}\frac{\omega_0^2}{\omega^2}\left\{2\pi \iint \frac{m_0 \omega_H s \mathrm{J}_s \mathrm{J}_s' (\omega m - k_\parallel p_\parallel + p_\perp^2 \omega/mc^2) f_0}{k_\perp R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \right. \\
&\quad +2\pi \iint \frac{p_\perp s \mathrm{J}_s'^2 (\omega m - k_\parallel p_\parallel) f_0}{R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad +2\pi \iint \frac{p_\perp s \mathrm{J}_s \mathrm{J}_s'' (\omega m - k_\parallel p_\parallel) f_0}{R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad \left. +2\pi \iint \frac{m_0 p_\perp^2 s \omega_H \mathrm{J}_s \mathrm{J}_s' (k_\parallel^2 - \omega^2/c^2) f_0}{k_\perp R^2} \mathrm{d}p_\perp \mathrm{d}p_\parallel \right\}, \\
\epsilon_{xz} &= -2\pi\left\{2 \iint \frac{\omega_0^2 p_\parallel s \mathrm{J}_s \mathrm{J}_s' (\omega m - k_\parallel p_\parallel) f_0}{\omega^2 R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \right. \\
&\quad \left. + \iint \frac{\omega_0^2 m_0 \omega_H s p_\perp p_\parallel \mathrm{J}_s^2 (k_\parallel^2 - \omega^2/c^2) \mathrm{f}_0}{\omega^2 k_\perp R^2} \mathrm{d}p_\perp \mathrm{d}p_\parallel \right\}, \\
\epsilon_{yz} &= \mathrm{i}\frac{2\pi\omega_0^2}{\omega^2}\left\{\iint \frac{p_\parallel \mathrm{J}_s \mathrm{J}_s' (\omega m - k_\parallel p_\parallel) f_0}{R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \right. \\
&\quad + \iint \frac{p_\perp p_\parallel k_\parallel (\mathrm{J}_s'^2 + \mathrm{J}_s \mathrm{J}_s'') (\omega m - k_\parallel p_\parallel) f_0}{m_0 \omega_H R} \mathrm{d}p_\perp \mathrm{d}p_\parallel \\
&\quad \left. + \iint \frac{p_\perp^2 p_\parallel \mathrm{J}_s \mathrm{J}_s' (k_\parallel^2 - \omega^2/c^2) f_0}{R^2} \mathrm{d}p_\perp \mathrm{d}p_\parallel \right\},
\end{aligned}
\right\} \quad (3)
$$

where $R = (\omega m - k_{\parallel} p_{\parallel} - s m_0 \omega_H)$; $\omega_0^2 = 4\pi N' Z^2 e^2/m_0$ is the square of the angular plasma frequency of the stream; $m = (m_0^2 + p_\perp^2/c^2 + p_\parallel^2/c^2)^{\frac{1}{2}}$ is the relativistic mass of the particle in the stream, m_0 being the rest mass of the charged particle; N' is the particle density of the stream; $k_\parallel$, $k_\perp$ are wave vector components along and perpendicular to the static magnetic field respectively; s is the harmonic number; J_s, J_s', J_s'' are Bessel's function, its first derivative, and its second derivative with respect to its argument respectively, the argument being $a = k_\perp p_\perp/m_0 \omega_H$; and f_0 is given by expression (2). The integration with respect to $p_\perp$ is carried out from 0 to ∞ and that with respect to $p_\parallel$ is carried out from $-\infty$ to $+\infty$.

It should be noted that for normal Doppler waves, the term $s m_0 \omega_H$ is positive in the Doppler relation $(\omega m - k_\parallel p_\parallel - s m_0 \omega_H)$; for a negatively charged radiator, e and hence ω_H are negative, and s must also be negative in this case. When we consider anomalous cyclotron waves, the sign of s is opposite to that for normal cyclotron waves.

Let
$$\left.\begin{aligned} \xi' &= (p_\parallel - p_\parallel^0)/a_\parallel\,, \\ \beta_j &= \{\omega\tilde{m} - k_\parallel p_\parallel^0 - s\omega_H m_0\}/k_\parallel a_\parallel\,, \\ \delta(p_\perp, p_\parallel) = \delta(\zeta, \xi') &= -\{\omega m - \omega\tilde{m} - k_\parallel(p_\parallel - p_\parallel^0)\}/k_\parallel a_\parallel\,, \end{aligned}\right\} \tag{4}$$

where the superscript $\sim$ indicates that the corresponding value is taken at the point $p_\perp = p_\perp^0$, $p_\parallel = p_\parallel^0$. Note that the present quantity ξ' is equivalent to the quantity ξ defined by Zheleznyakov. This is in order to distinguish it from the normalized frequency $\xi = \omega/|\omega_H|$, which is introduced later.

Let us now consider an integral of the form

$$I(\zeta, \beta_j) = \int_C \frac{g(\xi', \zeta)\exp(-\xi'^2)}{\beta_j - \delta(\xi', \zeta)}\,\mathrm{d}\xi'\,. \tag{5}$$

The contour of integration runs along the real axis of $p_\parallel$ from $-\infty$ to $+\infty$, bypassing from above or below the singularities of the integrand.

The integrand of the above integral will have singularities at two points lying on the real axis specified by $\xi'_{1,2}$ such that

$$\beta_j - \delta(\xi'_{1,2}, \zeta) = 0\,, \tag{6}$$

or

$$\omega(m_0^2 + p_\parallel^2/c^2 + p_\perp^2/c^2)^{\frac{1}{2}} - k_\parallel p_\parallel - s\omega_H m_0 = 0\,.$$

Note that we have changed the variables in (4) in order to deal with the denominator in (3); in fact, equation (6) is equivalent to $R = 0$, which is the Doppler equation.

It has been pointed out (Zheleznyakov 1960*b*) that when $|\xi'| \leqslant 1$ and $|\zeta - \zeta_0| \leqslant 1$ we have

$$|\beta_j| \gg |\delta(\xi', \zeta)|, \qquad |\xi'_{1,2}| \gg |\xi'|\,, \tag{7}$$

and the integral (5) can be simplified to

$$I(\zeta,\beta_j) \simeq \frac{1}{\beta_j}\int_{-\infty}^{\infty} g(\xi')\exp(-\xi'^2)\,\mathrm{d}\xi' + \frac{1}{\beta_j^2}\int_{-\infty}^{\infty} g(\xi')\,\delta(\xi')\exp(-\xi'^2)\,\mathrm{d}\xi'$$

$$-\mathrm{i}\pi \sum_{l=1,2} \delta_l \frac{g(\xi_l')\exp(-\xi_l'^2)}{(\partial\delta/\partial\xi')_l}, \tag{8}$$

with an accuracy up to terms of order $1/\beta_j^2$. In this expression $\delta_l = +1$ if the contour of integration in (5) bypasses the singularity ξ_l from below, and $\delta_l = -1$ if the contour bypasses the singularity ξ_l from above.

The mass of the radiating particle can be expressed in terms of ξ', ζ as

$$m(\xi',\zeta) = \tilde{m}\left\{1+\frac{2a_{\parallel}p_{\parallel}^0}{\tilde{m}^2c^2}\xi'+\frac{a_{\parallel}^2}{\tilde{m}^2c^2}\xi'^2+\frac{2a_{\perp}^2\zeta_0}{\tilde{m}^2c^2}\left(\zeta-\zeta_0\right)+\frac{a_{\perp}^2}{\tilde{m}^2c^2}\left(\zeta-\zeta_0\right)^2\right\}^{\frac{1}{2}}. \tag{9}$$

Let us assume that $m(\xi',\zeta)$ changes little in the range $|\xi'| \leqslant 1$, $|\zeta-\zeta_0| \leqslant 1$. We can now express $m(\xi',\zeta)$ in the form

$$m(\xi',\zeta) \simeq \tilde{m}\left\{1+\frac{a_{\parallel}p_{\parallel}^0}{\tilde{m}^2c^2}\xi'+\frac{a_{\parallel}^2}{2\tilde{m}^2c^2}\xi'^2+\frac{a_{\perp}^2\zeta_0}{\tilde{m}^2c^2}\left(\zeta-\zeta_0\right)+\frac{a_{\perp}^2}{2\tilde{m}^2c^2}\left(\zeta-\zeta_0\right)^2\right\}. \tag{10}$$

It is easy to see that the angular plasma frequency can be written as

$$\omega_0^2(\xi',\zeta) \simeq \tilde{\omega}_0^2\left\{1-\frac{a_{\parallel}p_{\parallel}^0}{\tilde{m}^2c^2}\xi'-\frac{a_{\parallel}^2}{2\tilde{m}^2c^2}\xi'^2-\frac{a_{\perp}^2\zeta_0}{\tilde{m}^2c^2}\left(\zeta-\zeta_0\right)-\frac{a_{\perp}^2}{2\tilde{m}^2c^2}\left(\zeta-\zeta_0\right)^2\right\}. \tag{11}$$

In writing down expressions (10) and (11), we have taken

$$\left|\frac{a_{\parallel}p_{\parallel}^0}{\tilde{m}^2c^2}\right| \ll 1; \qquad \left|\frac{a_{\parallel}^2}{\tilde{m}^2c^2}\right| \ll 1; \qquad \left|\frac{a_{\perp}^2\zeta_0}{\tilde{m}^2c^2}\right| \ll 1; \qquad \left|\frac{a_{\perp}^2}{\tilde{m}^2c^2}\right| \ll 1. \tag{12}$$

Since the denominators $R = \omega m - k_{\parallel}p_{\parallel} - sm_0\omega_H$ in the equations (3) can be written as $k_{\parallel}a_{\parallel}\{\beta_j-\delta(\xi',\zeta)\}$ and the quantities ω_0 and m are expressed as functions of ξ' (given in (10) and (11)), all the integrals in (3) fall into the type specified by (5), when integration is carried out with respect to ξ'. Let us note that, on account of the inequalities given in (7), the integral $I(\zeta,\beta_j)$ can be approximated

$$I(\zeta,\beta_j) \simeq \frac{1}{\beta_j}\int_{-\infty}^{\infty} g(\xi')\exp(-\xi'^2)\,\mathrm{d}\xi'. \tag{13}$$

In the expressions for the dielectric tensor components, it has been pointed out (Fung 1966*a*) that the largest terms are the ones containing $(\omega m - k_{\parallel}p_{\parallel} - sm_0\omega_H)^{-2}$ and the number 1. Confining ourselves to this approximation, the integration with respect to ξ' can be performed readily on substituting (4), (5), (10), (11), and (13) into (3).

$$
\left.\begin{aligned}
\epsilon_{xx} &= 1-\frac{s^2\tilde{\gamma}\omega_H^2 W}{k_\perp^2 D^2 G_0}\Big\{R_1\int_0^\infty \mathrm{J}_s^2\,\zeta Z\,\mathrm{d}\zeta - R_2\int_0^\infty \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)Z\,\mathrm{d}\zeta \\
&\qquad -R_3\int_0^\infty \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)^2 Z\,\mathrm{d}\zeta\Big\},\\
\epsilon_{yy} &= 1-\frac{a_\perp^2 W}{\tilde{m}^2 D^2 G_0}\Big\{R_1\int_0^\infty \mathrm{J}_s'^2\,\zeta^3 Z\,\mathrm{d}\zeta - R_2\int_0^\infty \mathrm{J}_s'^2\,\zeta^3(\zeta-\zeta_0)Z\,\mathrm{d}\zeta \\
&\qquad -R_3\int_0^\infty \mathrm{J}_s'^2\,\zeta^3(\zeta-\zeta_0)^2 Z\,\mathrm{d}\zeta\Big\},\\
\epsilon_{zz} &= 1-\frac{a_\parallel^2 W}{\tilde{m}^2 D^2 G_0}\Big\{R_4\int_0^\infty \mathrm{J}_s^2\,\zeta Z\,\mathrm{d}\zeta - \zeta_0 R_5\int_0^\infty \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)Z\,\mathrm{d}\zeta \\
&\qquad -\tfrac{1}{2}R_5\int_0^\infty \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)^2 Z\,\mathrm{d}\zeta\Big\},\\
\epsilon_{xy} &= -\frac{\mathrm{i}sm_0\,\omega_H\,a_\perp W}{k_\perp \tilde{m}^2 D^2 G_0}\Big\{R_1\int_0^\infty \mathrm{J}_s\mathrm{J}_s'\,\zeta^2 Z\,\mathrm{d}\zeta - R_2\int_0^\infty \mathrm{J}_s\mathrm{J}_s'\,\zeta^2(\zeta-\zeta_0)Z\,\mathrm{d}\zeta \\
&\qquad -R_3\int_0^\infty \mathrm{J}_s\mathrm{J}_s'\,\zeta^2(\zeta-\zeta_0)^2 Z\,\mathrm{d}\zeta\Big\},\\
\epsilon_{xz} &= -\frac{sm_0\,\omega_H\,p_\parallel^0 W}{k_\perp \tilde{m}^2 D^2 G_0}\Big\{\Big(R_1-\frac{a_\parallel^2}{2\tilde{m}^2c^2}\Big)\int_0^\infty \mathrm{J}_s^2\,\zeta Z\,\mathrm{d}\zeta - R_2\int_0^\infty \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)Z\,\mathrm{d}\zeta \\
&\qquad -R_3\int_0^\infty \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)^2 Z\,\mathrm{d}\zeta\Big\},\\
\epsilon_{yz} &= \frac{\mathrm{i}a_\perp p_\parallel^0 W}{\tilde{m}^2 D^2 G_0}\Big\{\Big(R_1-\frac{a_\parallel^2}{2\tilde{m}^2c^2}\Big)\int_0^\infty \mathrm{J}_s\mathrm{J}_s'\,\zeta^2 Z\,\mathrm{d}\zeta - R_2\int_0^\infty \mathrm{J}_s\mathrm{J}_s'\,\zeta^2(\zeta-\zeta_0)Z\,\mathrm{d}\zeta \\
&\qquad -R_3\int_0^\infty \mathrm{J}_s\mathrm{J}_s'\,\zeta^2(\zeta-\zeta_0)^2 Z\,\mathrm{d}\zeta\Big\},
\end{aligned}\right\}\quad (14)
$$

where $Z = \exp\{-(\zeta-\zeta_0)^2\}$.

As the argument of the Bessel function and its derivative $a = k_\perp p_\perp/m_0\,\omega_H = k_\perp\zeta/a_\perp m_0\,\omega_H$ is a function of ζ, all the J_s, J_s', and J_s'' have to be kept inside the integrals. In (14), we define

$$
\left.\begin{aligned}
W &= \tilde{\omega}_0^2(c^2k_\parallel^2-\omega^2)/\omega^2c^2,\\
D &= \omega-k_\parallel v_\parallel - s\gamma\omega_H,\\
R_1 = 1-a_\parallel^2/4\tilde{\omega}^2c^2,&\qquad R_2 = a_\perp^2\zeta_0/\tilde{m}^2c^2,\qquad R_3 = a_\perp^2/2\tilde{m}^2c^2,\\
R_4 = \frac{1}{2}-\frac{3a_\parallel^2}{8\tilde{m}^2c^2}-\frac{5p_\parallel^{02}}{4\tilde{m}^2c^2}+\frac{p_\parallel^{02}}{a_\parallel^2},&\qquad \text{and}\qquad R_5 = \frac{a_\perp^2}{\tilde{m}^2c^2}\Big(\frac{1}{2}+\frac{p_\parallel^{02}}{a_\parallel^2}\Big).
\end{aligned}\right\}\quad (15)
$$

Following Fung's (1966*a*) method, the dispersion equation for electromagnetic cyclotron waves in the stream–magnetoactive plasma system can be expressed as

$$A\left(\frac{c^2k^2}{\omega^2}-n_j^2+1\right)^2+B\left(\frac{c^2k^2}{\omega^2}-n_j^2+1\right)+C=0\,, \tag{16}$$

where
$$A=\epsilon_{xx}\sin^2\theta+\epsilon_{zz}\cos^2\theta+2\epsilon_{xz}\sin\theta\cos\theta\,,$$
$$B=2\sin\theta\cos\theta(\epsilon_{xy}\epsilon_{yz}-\epsilon_{yy}\epsilon_{xz})+\epsilon_{xz}^2-\epsilon_{xx}\epsilon_{zz}$$
$$-\cos^2\theta(\epsilon_{yy}\epsilon_{zz}+\epsilon_{yz}^2)-\sin^2\theta(\epsilon_{xx}\epsilon_{yy}+\epsilon_{xy}^2)\,,$$
$$C=\epsilon_{zz}(\epsilon_{xx}\epsilon_{yy}+\epsilon_{xy}^2)+\epsilon_{xx}\epsilon_{yz}^2+2\epsilon_{xy}\epsilon_{yz}\epsilon_{xz}-\epsilon_{yy}\epsilon_{xz}^2\,,$$

and ϵ_{ik} are given by (14).

Employing the real $-k$ approach, we write

$$\omega=\tilde{\omega}+\delta'\,,\qquad \text{with}\qquad |\omega|\gg|\delta'|\,, \tag{17}$$

where $\tilde{\omega}$, the "characteristic frequency", is real and δ' is complex; the imaginary part of δ' gives us the growth rate. Let us define

$$\delta=\delta'/\tilde{\omega}\,. \tag{18}$$

After some lengthy calculation, the coefficients in (16) are found to be

$$\left.\begin{aligned} A&=1-U_1/\delta^2\,,\\ B&=-2+(U_2/\delta^2)+(U_3/\delta^4)\,,\\ C&=1-(U_4/\delta^2)+(U_5/\delta^4)+(U_6/\delta^8)\,,\end{aligned}\right\} \tag{19}$$

where
$$U_1=T_2+2T_1\sin\theta\cos\theta-T_5\cos^2\theta-T_7\sin^2\theta\,,$$
$$U_2=T_3+2T_8\sin\theta\cos\theta+T_4\cos^2\theta+T_6\sin^2\theta\,,$$
$$U_3=E_{xx}\sin^2\theta+E_{zz}\cos^2\theta+2T_8\sin\theta\cos\theta\,,$$
$$U_4=T_6+E_{zz}\,,$$
$$U_5=T_7+T_6E_{zz}\,,$$
$$U_6=T_7E_{zz}\,,$$
$$U_7=U_5+E'_{yz}-T_8^2\,,$$
$$U_8=-U_6-E_{xx}E'_{yz}+2T_8E_{xyz}R_6R'_6\,,$$

and
$$T_1=E_{xyz}(R_6R'_6-R_7R'_8)\,,\qquad T_2=E_{xz}^2\,,$$
$$T_3=E_{xx}+E_{zz}\,,\qquad T_4=E_{yy}+E_{zz}\,,$$
$$T_5=E_{yy}E_{zz}+E'_{yz}\qquad T_6=E_{xx}+E_{yy}\,,$$
$$T_7=E_{xx}E_{yy}+E'_{xy}\,,\qquad T_8=E_{xz}\,,$$

with

$$E_{xx} = \frac{s^2\tilde{\gamma}^2\beta_{\parallel}^{02}\,W'}{\tan^2\theta\,(\xi+\tilde{\gamma}s)}\,R_8\,, \qquad E_{yy} = \frac{a_{\perp}^2\,W'}{\tilde{m}^2c^2}\,R_7\,, \qquad E_{zz} = \frac{W'}{\tilde{m}^2c^2}\,R_9\,,$$

$$E'_{xy} = -\frac{s^2\tilde{\gamma}^2a_{\perp}^2\,\beta_{\parallel}^{02}\,W'^2}{\tilde{m}^2c^2\tan^2\theta\,(\xi+\tilde{\gamma}s)^2}\,R_6^2\,, \qquad E'_{yz} = -\frac{a_{\perp}^2\,\beta_{\parallel}^{02}\,W'^2}{\tilde{m}^2c^2}\,R_6'^2\,,$$

$$E_{xz} = \frac{s\tilde{\gamma}\beta_{\parallel}^{02}\,W'}{\tan\theta\,(\xi+\tilde{\gamma}s)}\,R'_8\,, \qquad E_{xyz} = \frac{s\tilde{\gamma}a_{\perp}^2\,\beta_{\parallel}^{02}\,W'^2}{\tilde{m}^2c^2\tan\theta\,(\xi+\tilde{\gamma}s)}\,,$$

$$W' = \frac{\sigma A(c^2k_{\parallel}^2/\omega^2-1)}{\xi^2} = \frac{\sigma A}{\xi^2}\left\{\frac{(\xi+\tilde{\gamma}s)^2}{\xi^2\beta_{\parallel}^2}-1\right\}.$$

Also, in the above expressions, $\sigma =$ (density of stream)/(density of ambient plasma), $A = \omega_{\mathrm{p}}^2/\omega_H^2$, $\xi = \omega/|\omega_H|$, $\tilde{\gamma} = (1-\beta_{\parallel}^{02}-\beta_{\perp}^{02})^{\frac{1}{2}}$, $\beta_{\parallel}^0$, $\beta_{\perp}^0 = p_{\parallel}^0/\tilde{m}c$, $p_{\perp}^0/\tilde{m}c$ respectively, and

$$R_6 = R_1R_7-R_2R_8-R_3R_9\,, \qquad R'_6 = R_6-(a_{\parallel}^2/2\tilde{m}^2c^2)G_7\,,$$

$$R_7 = R_1G_4-R_2G_5-R_3G_6\,, \qquad R_8 = R_1G_1-R_2G_2-R_3G_3\,,$$

$$R'_8 = R_8-(a_{\parallel}^2/2\tilde{m}^2c^2)G_1\,, \qquad R_9 = R_4G_1-\zeta_0R_5G_2-\tfrac{1}{2}R_5G_3\,,$$

where

$$G_1 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s^2\,\zeta\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_2 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_3 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s^2\,\zeta(\zeta-\zeta_0)^2\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_4 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s'^2\,\zeta^3\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_5 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s'^2\,\zeta^3(\zeta-\zeta_0)\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_6 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s'^2\,\zeta^3(\zeta-\zeta_0)^2\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_7 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s\mathrm{J}_s'\,\zeta^2\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_8 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s\mathrm{J}_s'\,\zeta^2(\zeta-\zeta_0)\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,,$$

$$G_9 = G_0^{-1}\int_0^{\infty} \mathrm{J}_s\mathrm{J}_s'\,\zeta^2(\zeta-\zeta_0)^2\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta\,.$$

The dispersion equation for electromagnetic waves in the ambient plasma alone is given by

$$(F_{\mathrm{p}})_{\tilde{\omega}} = c^2k^2-\tilde{\omega}^2 n_j(\tilde{\omega})^2 = 0\,.$$

Expanding $(F_{\mathrm{p}})_{\omega}$ in a Taylor series about $\tilde{\omega}$, we have

$$\begin{aligned}(F_{\mathrm{p}})_{\omega} &\simeq (\partial F_{\mathrm{p}}/\partial\omega)_{\tilde{\omega}}\,\delta'+(F_{\mathrm{p}})_{\tilde{\omega}}\\ &= (\partial F_{\mathrm{p}}/\partial\omega)_{\tilde{\omega}}\,\delta'\,,\end{aligned} \tag{20}$$

where

$$\left(\frac{\partial F_{\mathrm{p}}}{\partial\omega}\right)_{\tilde{\omega}} = -2\tilde{\omega}+\omega_{\mathrm{p}}^2\left[\frac{2A}{\tilde{\omega}\xi^2}\left(B-\frac{\sin^2\theta}{2\xi^2}-D^{\frac{1}{2}}\right)-B\left\{\frac{2A}{\tilde{\omega}\xi^2}+\frac{\sin^2\theta}{\tilde{\omega}\xi^2}-\tfrac{1}{2}D^{-\frac{1}{2}}\left(-\frac{\sin^4\theta}{\tilde{\omega}\xi^4}\right.\right.\right.$$
$$\left.\left.\left.+\frac{4AB\cos^2\theta}{\tilde{\omega}\xi^4}-\frac{2B^2\cos^2\theta}{\tilde{\omega}\xi^2}\right)\right\}\right]\left(B-\frac{\sin^2\theta}{2\xi^2}-D^{\frac{1}{2}}\right)^{-2},$$

$$B = 1-A/\xi^2,$$

and

$$D = \tfrac{1}{4}y^4\sin^4\theta+(1-x)^2y^2\cos^2\theta\,.$$

Writing

$$P = \tilde{\omega}^{-1}(\partial F_{\mathrm{p}}/\partial\omega)_{\tilde{\omega}} \qquad \text{(dimensionless)}, \tag{21}$$

we can simplify equation (16) into the following form.

$$W_1\,\delta^8+W_2\,\delta^6+W_3\,\delta^5+W_4\,\delta^4+W_5\,\delta^3+W_6\,\delta^2+W_7 = 0\,, \tag{22}$$

where

$$W_1 = P^2, \qquad W_2 = P^2\,U_3\,,$$
$$W_3 = P(U_2-2U_3)\,, \qquad W_4 = U_2-U_3-U_4\,,$$
$$W_5 = PU_1\,, \qquad W_6 = U_1+U_7\,,$$
$$W_7 = U_8\,.$$

Solving (22) for complex $\delta = \delta'/\tilde{\omega}$, one can calculate the growth rate $|\mathrm{Im}(\delta')|$.

III. Discussion

Taking only terms containing $(\omega m-k_{\parallel}\,p_{\parallel}-sm_0\,\omega_H)^{-2}$ and the number 1 in the dielectric tensor components and other assumptions as stated, the dispersion equation has been derived (equation (16)). Using perturbation theory, we have expressed the dispersion equation as a polynomial in $\delta = \delta'/\tilde{\omega}$ (equation (22)), where $\delta' = \omega-\tilde{\omega}$ is complex. All that remains, therefore, is to solve equation (22).

Before attempting to solve (22), which is complicated as it stands, we consider the case of strictly longitudinal propagation, that is, $\theta = 0°$ or $180°$. Moreover, we confine ourselves to the first harmonic only, so that $s^2 = 1$. It is found that when $\sin\theta = 0$, $W_2 = W_4 = W_5 = W_6 = W_7 = 0$, while

$$W_3 = P(U_2-2U_3) \quad = \frac{pa_\perp^2 W'}{2G_0 \tilde{m}^2 c^2}\left(R_1 G_4'-R_2 G_5'-R_3 G_6'\right), \tag{23}$$

where
$$G_4' = \int_0^\infty \zeta^3 \exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta,$$

$$G_5' = \int_0^\infty \zeta^3(\zeta-\zeta_0)\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta,$$

and
$$G_6' = \int_0^\infty \zeta^3(\zeta-\zeta_0)^2\exp\{-(\zeta-\zeta_0)^2\}\,\mathrm{d}\zeta.$$

The dispersion equation now reads

$$\delta^3+\frac{a_\perp^2 W'}{2PG_0 \tilde{m}^2 c^2}\left(R_1 G_4'-R_2 G_5'-R_3 G_6'\right) = 0. \tag{24}$$

After some simple manipulation it may be seen that equation (24) agrees with Zheleznyakov's (1960*b*) result (equation (2.12)), if the following assumptions hold:

(1) $a_\perp^2 G_4'/G_0 \gg a_\parallel^2$,

(2) $G_4'/G_0 \gg |(R_1-1)G_4'/G_0-R_2 G_5'/G_0-R_3 G_6'/G_0|$,

(3) terms containing $\exp(-\xi_l'^2)$ are negligible, and

(4) terms containing $(\omega\tilde{m}-k_\parallel p_\parallel^0-s\tilde{\gamma}\omega_H \tilde{m})^{-1}$ are small in comparison with the terms containing $(\omega\tilde{m}-k_\parallel p_\parallel^0-s\tilde{\gamma}\omega_H \tilde{m})^{-2}$.

We now consider the validity of the above four approximations. If the spread in $p_\perp$ (specified by $a_\perp^2$) is of the order of the spread in $p_\parallel$ (specified by $a_\parallel^2$), we have $G_4'/G_0 \gg 1$ in cases when the spread in $p_\perp$ is not too large. More precisely, we want $\zeta_0 = p_\perp^0/a_\perp > 1$ (the least value of ζ_0 should be about 3) in order that approximation (1) above is valid. Approximations (2) and (3) are also taken in Zheleznyakov (1960*b*), and approximation (4) is the well-known assumption in the radiative instability problem of a stream–plasma system when the growth rate is small (that is, $|\omega| \gg |\delta'|$).

For another example, we consider the stream to be cold, that is, $a_\perp = a_\parallel = 0$, and we have a delta momentum distribution for the particles in the stream. In this case, where the wave-normal angle θ assumes general values, we have in equation (22) $W_4 = W_5 = W_6 = W_7 = 0$. This equation therefore reads

$$\delta^3+(W_2/W_1)\delta+(W_3/W_1) = 0. \tag{25}$$

In this particular case,

$$\frac{W_2}{W_1} = -\frac{\mathrm{J}_s^2\sigma A\beta_\parallel^{02}\cos^2\theta}{(\xi+\tilde{\gamma}s)^2}\left(\frac{(s\tilde{\gamma}+\xi)^2}{\beta_\parallel^{02}\xi^2}-1\right)$$

and

$$\frac{W_3}{W_1} = \frac{\sigma A}{\xi^2}\left(\frac{(s\tilde{\gamma}+\xi)^2}{\beta_\parallel^{02}\xi^2}-1\right)\left(\mathrm{J}_s'^2\beta_\perp^{02}+\mathrm{J}_s^2\beta_\parallel^{02}\frac{(s\tilde{\gamma}+\xi\sin^2\theta)^2}{(s\tilde{\gamma}+\xi)^2\sin^2\theta}\right)P^{-1},$$

and we see that this equation is in fact exactly the one derived by Fung (1966*a*, equation (5), article (1)). We may thus conclude that the dispersion equation derived in the present investigation agrees with that obtained by Zheleznyakov on transition from a general θ to $\theta = 0°$ or $180°$ (under the approximations stated), and, when the temperature of the stream is taken as zero, the dispersion equation (22) is simplified to the one derived by Fung.

When the coefficients of the dispersion equation (22) in the case of the v.l.f. emission in the terrestrial magnetosphere are evaluated, the results (not shown here) indicate that only the first three terms are significant, i.e. the dispersion equation can be approximated to

$$\delta^3+(W_2/W_1)\delta+(W_3/W_1) = 0\,. \tag{26}$$

This is of the same form as the dispersion equation for the case of a strictly helical beam in a cold magnetoactive plasma as derived in Fung (1966*a*). Hence, if the dispersion equation of the system considered can be approximated to the form of equation (26), we can readily obtain an exact solution for δ by Cardan's method; otherwise, equation (22) must be used to calculate complex numerical solutions.

Under different conditions, electromagnetic waves generated by normal or anomalous cyclotron radiation processes by particles in the stream may grow in the stream–magnetoactive plasma system and the power of the wave may be amplified enormously. This radiative instability may in fact happen in many natural radio emissions in radio astronomy. The study of such instability problems will help us to understand various phenomena in plasma radiation. Assuming a delta momentum distribution, the theory has been applied to terrestrial v.l.f. emissions and decametric emissions from Jupiter (Fung 1966*a*, 1966*b*). However, a perfect monoenergetic stream of charged particles is unlikely to occur in nature. As far as cyclotron radiation is concerned, the distribution function of the stream considered by Zheleznyakov (1960*b*) and in the present investigation seems to be a more realistic and important one. We note that the theory presented here is a linearized one. When the growth rate becomes large so that an appreciable amount of energy is transferred from the stream to the electromagnetic wave, the non-linearized theory has to be employed.

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