THE ENERGY DEPENDENCE OF THE EFFECTIVE INTERACTION IN SUPERCONDUCTIVITY

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Summary

The effective interaction in the BCS model of superconductivity is usually approximated by a constant. We expand the interaction in a power series in $\epsilon_k/\hbar\omega$ and treat the energy-dependent terms to first order. This introduces one more parameter in the theory. The gap, which now becomes energy dependent, is obtained by solving an integral equation by iteration. The critical field and specific heat are calculated. The value of $2\Delta(0,0)/k_B T_c$ and the jump in the electronic specific heat at the critical temperature T_c are now dependent on the parameters of the superconductor. Calculated values for the energy gap and the critical field H_c agree rather well with the experimental data.

I. Introduction

In the Bardeen–Cooper–Schrieffer (1957) theory of superconductivity (hereafter referred to as the BCS theory), the effective interaction is believed to be the result of superposing a screened, repulsive, Coulomb interaction between the electrons and a stronger, attractive, phonon-mediated electron–electron interaction. Morel and Anderson (1962) have treated the energy dependence of interaction, while Schrieffer, Scalapino, and Wilkins (1963) and Schrieffer (1964) have solved a generalized gap equation using the Debye model for the phonon spectrum. Earlier attempts, particularly by Swihart (1962, 1963), tended to favour the Eliashberg (1960) interaction. We present here simplified calculations based on the BCS approximation of a sharp cut-off of the effective interaction and show that the experimental data for the energy gap and the critical field agree extremely well with the results of the present calculations. For this purpose, we develop the interaction as a series in powers of $\epsilon_k/\hbar\omega$. We retain only the terms up to the order of $(\epsilon_k/\hbar\omega)^2$ in this expansion. The problem of the effective energy gap is then solved by a method of successive iteration.

II. The Effective Interaction

The reduced Hamiltonian of the BCS theory is

$$\mathscr{H}_{\mathrm{red}} = 2\sum_{k>k_F} \epsilon_k b_k^\dagger b_k + 2\sum_{k<k_F} |\epsilon_k| b_k b_k^\dagger - \sum_{k,k} V_{kk'} b_{k'}^\dagger b_k\,, \tag{1}$$

where $-V_{kk'}$ is the effective interaction and is the sum of the phonon part V_{ph} and

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the screened Coulomb part V_C. In the BCS theory, the effective interaction $-V_{kk'}$ is approximated by a constant $-V$ in a narrow region $-\hbar\omega < \epsilon < \hbar\omega$ about the Fermi surface and by zero outside it, i.e.

$$-V(\epsilon_k, \epsilon_{k'}) = V_{\text{ph}} + V_C \tag{2}$$

$$\left.\begin{aligned} -V(\epsilon_k, \epsilon_{k'}) &= -V & &-\hbar\omega < \epsilon < \hbar\omega \\ &= \quad 0 & &\text{outside this region.} \end{aligned}\right\} \tag{3}$$

Here $\hbar\omega$ is the average phonon frequency, usually taken as $\frac{1}{2}k_B\,\theta_D$, where k_B is the Boltzmann constant and θ_D is the Debye temperature. Quite generally, we can effect a series expansion of the effective interaction $-V(\epsilon_k, \epsilon_{k'})$ about the Fermi surface. Using the dimensionless quantities $x = \epsilon_k/\hbar\omega$ and $x' = \epsilon_{k'}/\hbar\omega$, this is

$$-V(\epsilon_k, \epsilon_{k'}) = -V - \left(x\frac{\partial V}{\partial x} + x'\frac{\partial V}{\partial x'}\right) - \tfrac{1}{2}\left(x^2\frac{\partial^2 V}{\partial x^2} + x'^2\frac{\partial^2 V}{\partial x'^2}\right) - xx'\frac{\partial^2 V}{\partial x\,\partial x'}. \tag{4}$$

Now the effective interaction $-V(\epsilon_k, \epsilon_{k'})$, occurring as it does between the electrons of energy ϵ_k and $\epsilon_{k'}$, should be symmetric in ϵ_k and $\epsilon_{k'}$. This is true of the Bardeen–Pines (1955) interaction used in the BCS theory and perhaps of many other forms of the effective interaction. Thus we restrict the form of the effective interaction so that

$$-V(\epsilon_k, \epsilon_{k'}) = -V(-\epsilon_k, -\epsilon_{k'}) \tag{5}$$

holds. The equation (5) results in the vanishing of the terms of the type $x\,\partial V/\partial x$ and $x'\,\partial V/\partial x'$. We then rewrite (4) in the form

$$-V(\epsilon_k, \epsilon_{k'}) = -V - V_1\left\{\left(\frac{\epsilon_k}{\hbar\omega}\right)^2 + \left(\frac{\epsilon_{k'}}{\hbar\omega}\right)^2\right\} - V_2\frac{\epsilon_k\,\epsilon_{k'}}{(\hbar\omega)^2}, \tag{6}$$

where $-V$ is the function used in the BCS theory. We will note in the next section that within our approximation the term containing $-V_2$ gives no contribution to the energy gap. Thus we have two parameters $-V$ and $-V_1$ to consider in the effective interaction (6). This is the interaction with which we shall be concerned in the present paper.

III. The Gap Equation

The integral equation for the energy gap is given by

$$\Delta_k = -\tfrac{1}{2}\sum_{k'} V_{kk'}\,\Delta_{k'}\,\tau_{k'}/E_{k'}, \tag{7}$$

where $$E_{k'} = (\epsilon_{k'}^2 + \Delta_{k'}^2)^{\frac{1}{2}},$$

$$\tau_{k'} = \tanh(\tfrac{1}{2}\beta E_{k'}),$$

and $$\beta = 1/k_B T.$$

The summation in equation (7) can be changed to integration and we obtain

$$\Delta(\epsilon,T) = -\tfrac{1}{2}N(0)\int_{-\hbar\omega}^{+\hbar\omega} V(\epsilon,\epsilon')\frac{\Delta(\epsilon',T)\,\tau(\epsilon',T)}{\{\epsilon'^2+\Delta^2(\epsilon',T)\}^{\frac{1}{2}}}\,\mathrm{d}\epsilon', \tag{8}$$

where $N(0)$ is the density of states (one-electron) in the normal state at the Fermi surface. Using the expression for $V(\epsilon,\epsilon')$ from equation (6), we obtain

$$\begin{aligned}\Delta(\epsilon,T) = {} & \tfrac{1}{2}N(0)\,V\,I_0(\Delta(\epsilon,T))+\tfrac{1}{2}\{N(0)\,V_1/(\hbar\omega)^2\}I_2(\Delta(\epsilon,T))\\ & +\tfrac{1}{2}\{N(0)\,V_2/(\hbar\omega)^2\}I_1(\Delta(\epsilon,T))\epsilon+\tfrac{1}{2}\{N(0)\,V_1/(\hbar\omega)^2\}I_0(\Delta(\epsilon,T))\epsilon^2,\end{aligned} \tag{9}$$

where

$$I_n(\Delta(\epsilon,T)) = \int_{-\hbar\omega}^{+\hbar\omega}\Delta(\epsilon',T)\frac{\tau(\epsilon',T)\,\epsilon'^n}{\{\epsilon'^2+\Delta^2(\epsilon',T)\}^{\frac{1}{2}}}\,\mathrm{d}\epsilon'. \tag{10}$$

We use a method of successive iteration to solve this integral equation. The zeroth-order approximation for $\Delta(\epsilon,T)$ can be obtained from equation (9) by substituting $\epsilon = 0$ in the right-hand side of (9) and solving for $\Delta(0,T)$. This gives

$$\Delta(0,T) = \tfrac{1}{2}N(0)\,V\,I_0(\Delta(0,T))+\tfrac{1}{2}\{N(0)\,V_1/(\hbar\omega)^2\}I_2(\Delta(0,T)). \tag{11}$$

Substituting the solution of this integral equation on the right-hand side of equation (9), we obtain the solution in the next approximation, namely,

$$\begin{aligned}\Delta_1(\epsilon,T) &= \tfrac{1}{2}N(0)\,V\,I_0(\Delta(0,T))+\tfrac{1}{2}\{N(0)\,V_1/(\hbar\omega)^2\}I_2(\Delta(0,T))\\ &\quad +\tfrac{1}{2}\{N(0)\,V_1/(\hbar\omega)^2\}I_0(\Delta(0,T))\,\epsilon^2\\ \Delta_1(\epsilon,T) &= \Delta(0,T)+\tfrac{1}{2}\{N(0)\,V_1/(\hbar\omega)^2\}I_0(\Delta(0,T))\,\epsilon^2.\end{aligned} \tag{12}$$

Repeating the process, the solution can be obtained to any degree of accuracy. It may be seen that in each stage of iteration the odd terms in ϵ vanish. Evaluating the integrals $I_0(\Delta(0,T))$ and $I_2(\Delta(0,T))$, we obtain

$$\begin{aligned}I_0(\Delta(0,T)) &= 2\Delta(0,T)\sinh^{-1}(\hbar\omega/\Delta(0,T))-4\Delta(0,T)\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_0(\beta m\Delta(0,T))\\ &\simeq 2\Delta(0,T)\ln(2\hbar\omega/\Delta(0,T))-4\Delta(0,T)\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_0(\beta m\Delta(0,T)),\end{aligned} \tag{13}$$

$$\begin{aligned}I_2(\Delta(0,T)) &= \Delta(0,T)(\hbar\omega)^2\{1+(\Delta(0,T)/\hbar\omega)^2\}^{\frac{1}{2}}\\ &\quad -\Delta^3(0,T)\sinh^{-1}(\hbar\omega/\Delta(0,T))+2\Delta^3(0,T)\sum_{m=1}^{\infty}(-1)^{m+1}\\ &\quad \times\{\mathrm{K}_0(\beta m\Delta(0,T))-\mathrm{K}_2(\beta m\Delta(0,T))\}\\ &\simeq (\hbar\omega)^2\Delta(0,T)\{1+(\Delta(0,T)/\hbar\omega)^2\}^{\frac{1}{2}}-\Delta^3(0,T)\ln(2\hbar\omega/\Delta(0,T))\\ &\quad +2\Delta^3(0,T)\sum_{m=1}^{\infty}(-1)^{m+1}\{\mathrm{K}_0(\beta m\Delta(0,T))-\mathrm{K}_2(\beta m\Delta(0,T))\}.\end{aligned} \tag{14}$$

The second forms of these expressions hold only for the weak coupling limit, when $\hbar\omega \gg \Delta(0,T)$. K_0 and K_2 are the modified Bessel functions of the second kind of order zero and two respectively. Equation (12) now becomes

$$\Delta_1(\epsilon,T) = \Delta(0,T)\{1+A(T)\,\epsilon^2\}, \tag{15}$$

where

$$A(T) = \{N(0)\,V_1/(\hbar\omega)^2\}\left\{\ln(2\hbar\omega/\Delta(0,T))-2\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_0(\beta m\Delta(0,T))\right\}. \tag{16}$$

As we do not propose to go to higher approximations, in what follows we shall omit the suffix 1 on Δ_1. The energy gap now depends on energy parabolically, leading to the same conclusion as that of Morel and Anderson (1962) and Swihart (1962, 1963). In this approximation then, the quasi-particle energy is given by

$$E = \{\epsilon^2+\Delta^2(\epsilon,T)\}^{\frac{1}{2}} = \{1+2A(T)\,\Delta^2(0,T)\,\epsilon^2+\Delta^2(0,T)\}^{\frac{1}{2}}, \tag{17}$$

neglecting terms of the order $A^2(T)\,\Delta^4(0,T)\,\epsilon^4$.

The sums involving the Bessel functions that occur above have simple forms in the limiting cases of T near 0 and T near T_c.

Near $T = 0°$K,

$$\left.\begin{aligned}
2\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_0(\beta m\Delta(0,T)) &\simeq (2\pi)^{\frac{1}{2}}\left\{\frac{\exp(-\beta\Delta(0,0))}{(\beta\Delta(0,0))^{\frac{1}{2}}}-\frac{\exp(-\beta\Delta(0,0))}{8(\beta\Delta(0,0))^{3/2}}\right\},\\
2\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_2(\beta m\Delta(0,T)) &\simeq (2\pi)^{\frac{1}{2}}\left\{\frac{\exp(-\beta\Delta(0,0))}{(\beta\Delta(0,0))^{\frac{1}{2}}}+\frac{15\exp(-\beta\Delta(0,0))}{8(\beta\Delta(0,0))^{3/2}}\right\},
\end{aligned}\right\} \tag{18}$$

and, for a temperature near $T = T_c$,

$$\left.\begin{aligned}
2\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_0(\beta m\Delta(0,T)) &\simeq \ln\left\{\frac{\pi}{\gamma\beta\Delta(0,T)}\right\}+\frac{7\zeta(3)}{8\pi^2}\Big(\beta\Delta(0,T)\Big)^2\\
&\quad-\frac{93\zeta(5)}{128\pi^4}\Big(\beta\Delta(0,T)\Big)^4,\\
2\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_2(\beta m\Delta(0,T)) &\simeq \frac{\frac{1}{3}\pi^2}{(\beta\Delta(0,T))^2}\left\{1-\frac{3}{2\pi^2}\Big(\beta\Delta(0,T)\Big)^2\right.\\
&\quad\left.+\frac{3q}{8\pi^2}\Big(\beta\Delta(0,T)\Big)^4\right\}.
\end{aligned}\right\} \tag{19}$$

Here

$$q = \tfrac{1}{2}\int_0^{\infty} x^{-2}\tanh^2 x\,\mathrm{d}x \simeq 0\cdot 8,$$

$\ln\gamma$ is the Euler constant, and $\zeta(s)$ is the Riemann zeta function of order s. These are identical to the expansions used by Khalatnikov and Abrikosov (1959).

(a) *Case of* T *near* 0°K

Using equations (15), (16), and (18), we obtain

$$\Delta(\epsilon,T)/\Delta(0,T) = 1+A_0(T)\,\epsilon^2, \tag{20}$$

where
$$A_0(T) = \frac{N(0)\,V_1}{(\hbar\omega)^2}\left\{\ln\left(\frac{2\hbar\omega}{\Delta(0,T)}\right)-\left(\frac{2\pi}{\beta\Delta(0,T)}\right)^{\frac{1}{2}}\exp\left(-\beta\Delta(0,T)\right)\right\} \tag{21}$$

and, from equations (11), (13), (14), and (18), we have

$$\Delta(0,T) = \Delta(0,0)-\{(2\pi/\beta)\Delta(0,0)\}^{\frac{1}{2}}\exp(-\beta\Delta(0,0)) \tag{22}$$

and
$$\Delta(0,0) = 2\hbar\omega\exp(-1/N(0)\,V)\exp(\tfrac{1}{2}V_1/V). \tag{23}$$

The quantity $\Delta(0,0)$ differs from the BCS value of the gap at $T = 0$°K by a factor $\exp(\frac{1}{2}V_1/V)$. The temperature dependence of $\Delta(0,T)$ is the same as in the BCS theory. The energy-dependent gap has an additional temperature variation through $A(T)$. From equations (20) and (23), we obtain

$$\Delta(\epsilon,0)/\Delta(0,0) = 1+(V_1/V)(1-\tfrac{1}{2}N(0)\,V_1)\{\epsilon^2/(\hbar\omega)^2\}. \tag{24}$$

This gives the parabolic variation of the energy gap with energy ϵ, at $T = 0$°K.

(b) *Case of* T *near* T_c

For this case we have to use the approximations in equations (19). We obtain, as before,

$$\Delta(\epsilon,T)/\Delta(0,T) = 1+A_c(T)\,\epsilon^2, \tag{25}$$

where

$$A_c(T) = \frac{N(0)\,V_1}{(\hbar\omega)^2}\left\{\ln\left(\frac{2\hbar\omega\gamma\beta}{\pi}\right)-\frac{7\zeta(3)}{8\pi^2}\left(\beta\Delta(0,T)\right)^2+\frac{93\zeta(5)}{128\pi^4}\left(\beta\Delta(0,T)\right)^4\right\}. \tag{26}$$

From equations (11), (13), (14), and (19), we obtain the equation for $\Delta(0,T)$ as

$$\ln\left(\frac{\gamma\beta\Delta(0,0)}{\pi}\right) = \frac{7\zeta(3)}{8\pi^2}\left(\beta\Delta(0,T)\right)^2-\frac{93\zeta(5)}{128\pi^4}\left(\beta\Delta(0,T)\right)^4+\frac{\pi^2 V_1}{6V}\left(\beta\hbar\omega\right)^{-2}$$

$$\simeq \frac{7\zeta(3)}{8\pi^2}\left(\beta\Delta(0,T)\right)^2+\frac{\pi^2 V_1}{6V}\left(\beta\hbar\omega\right)^{-2}. \tag{27}$$

At $T = T_c$, the gap becomes zero and we obtain

$$\ln(\gamma\beta_c\Delta(0,0)/\pi) = \tfrac{1}{6}\pi^2(V_1/V)(\beta_c\hbar\omega)^{-2}. \tag{28}$$

Using equation (23), this equation becomes

$$\ln\left(\frac{\pi}{\gamma}\frac{k_B T_c}{2\hbar\omega}\right) = -\frac{1}{N(0)\,V}+\frac{V_1}{2V}\left\{1-\tfrac{1}{3}\pi^2\left(\frac{k_B T_c}{\hbar\omega}\right)^2\right\}. \tag{29}$$

From equations (27) and (28), we obtain

$$\ln\left(\frac{T}{T_c}\right) = -\frac{7\zeta(3)}{8\pi^2}\left(\frac{\Delta(0,T)}{k_B T_c}\right)^2 + \frac{\pi^2 V_1}{6V}\left\{1-\left(\frac{T}{T_c}\right)^2\right\}\left(\beta_c \hbar\omega\right)^{-2}, \tag{30}$$

which can be put in the form

$$\Delta^2(0,T) = \frac{8\pi^2}{7\zeta(3)}\left(k_B T_c\right)^2\left(1-\frac{T}{T_c}\right) + \frac{4\pi^4}{21\zeta(3)}\frac{(k_B T_c)^4}{(\hbar\omega)^2}\left(1-\frac{T^2}{T_c^2}\right). \tag{31}$$

The last term on the right can be approximated by using $(1-T^2/T_c^2) \simeq 2(1-T/T_c)$. Therefore

$$\frac{\Delta^2(0,T)}{(k_B T_c)^2} = \frac{8\pi^2}{7\zeta(3)}\left(1-\frac{T}{T_c}\right)\left\{1+\frac{\pi^2 V_1}{3V}\left(\frac{k_B T_c}{\hbar\omega}\right)^2\right\}. \tag{32}$$

From equation (28), we obtain

$$\frac{\Delta(0,0)}{k_B T_c} \simeq \frac{\pi}{\gamma}\left\{1+\frac{\pi^2 V_1}{6V}\left(\frac{k_B T_c}{\hbar\omega}\right)^2\right\}. \tag{33}$$

It is clear from equation (33) that $\Delta(0,0)/k_B T_c$ is not a constant for all superconductors, as in the BCS theory, but depends on the parameters of the superconductor, as expected from the experimental data.

IV. Free Energy, Entropy, and Specific Heat

The expressions for the free energy and entropy are now obtained using equations (15) and (17) for $\Delta(\epsilon,T)$ and E, in the usual definitions (BCS theory).

$$\begin{aligned}
F_s = {} & (\hbar\omega)^2 N(0) - N(0)\,(\hbar\omega)^2\{1+2A(T)\,\Delta^2(0,T)+\Delta^2(0,T)/(\hbar\omega)^2\}^{\frac{1}{2}} \\
& - \frac{N(0)\,A(T)\,\Delta^4(0,T)}{\{1+2A(T)\,\Delta^2(0,T)\}^{3/2}}\ln\left\{\frac{2\hbar\omega}{\Delta(0,T)}\left(1+2A(T)\,\Delta^2(0,T)\right)^{\frac{1}{2}}\right\} \\
& + \frac{N(0)\,A(T)\,\Delta^2(0,T)\,(\hbar\omega)^2}{1+2A(T)\,\Delta^2(0,T)}\left\{1+2A(T)\,\Delta^2(0,T)+\Delta^2(0,T)/(\hbar\omega)^2\right\}^{\frac{1}{2}} \\
& - \frac{2N(0)\,A(T)\,\Delta^4(0,T)}{\{1+2A(T)\,\Delta^2(0,T)\}^{3/2}}\left\{\sum_{m=1}^{\infty}(-1)^{m+1}\{\mathrm{K}_2(\beta m\Delta(0,T))-\mathrm{K}_0(\beta m\Delta(0,T))\}\right\} - \tfrac{1}{2}TS_s
\end{aligned} \tag{34}$$

and

$$S_s = 4(N(0)/T)\Delta^2(0,T)\left\{\sum_{m=1}^{\infty}(-1)^{m+1}\mathrm{K}_2(\beta m\Delta(0,T))\right\}\left\{1+2A(T)\,\Delta^2(0,T)\right\}^{-\frac{1}{2}}. \tag{35}$$

As before we discuss the two limiting cases separately.

(a) Case of T *near* 0°K

Using the expansions of equations (18) in (35), we can obtain the free energy and the entropy for this range of temperatures. The results are found to be the same as in the BCS analysis, as given by Khalatnikov and Abrikosov (1959).

(b) Case of T *near* T_c

We use the expressions from (19) in (34) and (35) to find the appropriate expressions for the free energy and the entropy. Using the usual value for the electronic specific heat in the normal state, C_{en}, with γ_s the Sommerfeld constant, we obtain

$$C_{en} = \gamma_s T,$$

$$\frac{C_{es}-C_{en}}{C_{en}} = \frac{12}{7\zeta(3)}+\frac{4\pi^2}{7\zeta(3)}\frac{V_1}{V}\frac{1}{(\beta\hbar\omega)^2}+\frac{8\pi^2}{7\zeta(3)}\frac{N(0)\,V_1}{(\beta\hbar\omega)^2}$$

$$\times\left(1+\frac{\pi^2}{3}\frac{1}{(\beta\hbar\omega)^2}\frac{V_1}{V}\right)\ln\left(\frac{2\hbar\omega\gamma\beta}{\pi}\right)$$

$$+\Delta^2(0,T)\left\{\frac{3\beta^2}{2\pi^2}\left(1-\frac{7\zeta(3)}{4\pi^2}\right)-\left(1+\frac{7\zeta(3)}{4\pi^2}\right)\frac{N(0)\,V_1}{(\hbar\omega)^2}\right.$$

$$\left.\times\ln\left(\frac{2\hbar\omega\gamma\beta}{\pi}\right)-\frac{N(0)\,V_1}{(\hbar\omega)^2}\left(1+\frac{2\pi^2}{3}\frac{V_1}{V}\frac{1}{(\beta\hbar\omega)^2}\right)\right\}. \qquad (36)$$

Here C_{es} is the electronic specific heat in the superconducting state. We see that the leading term on the right is the same as in the BCS theory (Khalatnikov and Abrikosov 1959). At $T = T_c$, we obtain

$$\left.\frac{C_{es}-C_{en}}{C_{en}}\right|_{T_c} = \frac{12}{7\zeta(3)}\left\{1+\frac{2\pi^2}{3}\frac{N(0)\,V_1}{(\beta_c\hbar\omega)^2}\ln\left(\frac{2\hbar\omega\gamma\beta_c}{\pi}\right)\right\}\left\{1+\frac{\pi^2}{3}\frac{V_1}{V}\frac{1}{(\beta_c\hbar\omega)^2}\right\}. \qquad (37)$$

A new feature now appears in that this quantity is not a constant, as in the BCS theory, but depends on the parameters of the superconducting state.

V. The Critical Field

The critical field is given by the formula (BCS theory)

$$H_c^2(T)/8\pi = F_n - F_s\,. \qquad (38)$$

Substituting the expression for the free energy from equation (34) in this equation, we obtain

$$H_c^2(T)/8\pi = \tfrac{1}{2}T(S_s - S_n) - N(0)(\hbar\omega)^2 + N(0)(\hbar\omega)^2\{1 + 2A(T)\Delta^2(0,T) + \Delta^2(0,T)/(\hbar\omega)^2\}^{\frac{1}{2}}$$

$$+ \frac{N(0)A(T)\Delta^4(0,T)}{\{1+2A(T)\Delta^2(0,T)\}^{3/2}} \ln\left\{\frac{2\hbar\omega}{\Delta(0,T)}\left(1+2A(T)\Delta^2(0,T)\right)^{\frac{1}{2}}\right\}$$

$$+ \frac{2N(0)A(T)\Delta^4(0,T)}{\{1+2A(T)\Delta^2(0,T)\}^{3/2}} \left\{\sum_{m=1}^{\infty}(-1)^{m+1}\{\mathrm{K}_2(\beta m\Delta(0,T)) - \mathrm{K}_0(\beta m\Delta(0,T))\}\right\}$$

$$- \frac{N(0)A(T)\Delta^2(0,T)(\hbar\omega)^2}{1+2A(T)\Delta^2(0,T)}\left\{1+2A(T)\Delta^2(0,T)+\Delta^2(0,T)/(\hbar\omega)^2\right\}^{\frac{1}{2}}. \quad (39)$$

For the case of T near 0°K, we obtain

$$\frac{H_c^2(T)}{H_c^2(0)} = 1 - \tfrac{2}{3}\gamma^2\left(\frac{T}{T_c}\right)^2\left\{1 + B\left(1 - \frac{\pi^2}{3}\frac{k_B^2 T_c^2}{(\hbar\omega)^2}\frac{V_1}{V}\right) - \frac{\pi^2}{3}\frac{V_1}{V}\frac{k_B^2 T_c^2}{(\hbar\omega)^2}\right\}, \quad (40)$$

where

$$B = \frac{\Delta^2(0,0)}{(2\hbar\omega)^2}\left\{1 + 8N(0)V_1\ln(2\hbar\omega/\Delta(0,0)) - 8N(0)V_1(1+\tfrac{1}{2}N(0)V_1)\{\ln(2\hbar\omega/\Delta(0,0))\}^2\right\}, \quad (41)$$

$$H_c^2(0)/8\pi = -N(0)(\hbar\omega)^2 + N(0)(\hbar\omega)^2\{1+2A_0(0)\Delta^2(0,0)+\Delta^2(0,0)/(\hbar\omega)^2\}^{\frac{1}{2}}$$

$$+ \frac{N(0)A_0(0)\Delta^4(0,0)}{\{1+2A_0(0)\Delta^2(0,0)\}^{3/2}}\ln\left\{\frac{2\hbar\omega}{\Delta(0,0)}\left(1+2A_0(0)\Delta^2(0,0)\right)^{\frac{1}{2}}\right\}$$

$$- \frac{N(0)A_0(0)\Delta^2(0,0)}{1+2A_0(0)\Delta^2(0,0)}\left(\hbar\omega\right)^2\left\{1+2A_0(0)\Delta^2(0,0)+\Delta^2(0,0)/(\hbar\omega)^2\right\}^{\frac{1}{2}}. \quad (42)$$

For T near T_c, we obtain

$$H_c^2/8\pi = \tfrac{1}{8}qN(0)\beta^2\Delta^4(0,T) - \tfrac{1}{8}N(0)\Delta^4(0,T)/(\hbar\omega)^2$$

$$+ N^2(0)\Delta^4(0,T)(\hbar\omega)^{-2}V_1(1+\tfrac{1}{2}N(0)V_1)\{\ln(2\hbar\omega\gamma\beta/\pi)\}^2$$

$$- N^2(0)\Delta^4(0,T)(\hbar\omega)^{-2}V_1\ln(2\hbar\omega\gamma\beta/\pi)\left\{1 + \frac{7\zeta(3)}{8\pi^2}\left(\beta\Delta(0,T)\right)^2\left(1+N(0)V_1\right)\right\}. \quad (43)$$

TABLE 1

COMPARISON OF THE COMPUTED VALUES OF THE ENERGY GAP $\Delta(0,0)$ AND THE CRITICAL FIELD $H_c(0)$ WITH THE BCS VALUES AND EXPERIMENTAL DATA

Super-conductor	T_c* (°K)	θ_D* (°K)	γ_s* (mJ mole^{-1} degK^{-2})	$C_{es}/\gamma_s T_c$†	$2\Delta(0,0)/k_B T_c$‡		Critical Field, $H_c(0)$		
					Computed Value	Expt. Data§	BCS Value‖	Computed Value	Expt. Data*
Lead	7·18	96·3	3·0	3·65	4·23	4·20±0·08	705	818	803
Mercury (α)	4·15	69	1·91	3·18	4·32	4·5 ±0·1	361	439	412
Niobium	9·2	238	7·53	3·07	4·03	3·84±0·06	1844	1980	1944
Tin	3·73	195	1·75	2·60	3·68	3·6 ±0·1	296	318	306
Vanadium	5·1	338	9·26	2·57	3·62	3·4	1292	1325	1310

* Data taken from American Institute of Physics Handbook (1963).

† From Bardeen and Schrieffer (1961).

‡ The BCS value for $2\Delta(0,0)/k_B T_c$ is equal to 3·52 for all superconductors.

§ From Douglass and Falicov (1964, Table 5·5).

‖ Calculated using equation (3·40), p. 1187, of Bardeen, Cooper, and Schrieffer (1957).

The leading terms on the right are the same as in the BCS theory (Khalatnikov and Abrikosov 1959).

We have determined the parameters V and V_1 from equations (29) and (37), using the experimental values for $(C_{es}-C_{en})/C_{en}|_{Tc}$, T_c, and $\hbar\omega$ $(=\frac{1}{2}k_B\theta_D)$. We then computed the values of $\Delta(0,0)$ by making use of (23) and the critical field $H_c(0)$ at $T=0°K$ from the exact equation (42). The results are tabulated in Table 1. Experimental data for the energy gap $2\Delta(0,0)/k_BT_c$ and the critical field $H_c(0)$ for several metals are listed for comparison. The results for the BCS theory are also listed; the results agree quite well with the experimental data. It may be mentioned that even though no account has been taken of life-time effects (believed to be important, particularly for strong coupling superconductors lead and mercury), the results are quite close to the experimental values. In the case of niobium the experimental data for $2\Delta(0,0)/k_BT_c$ lie in the range 3·6–3·84. Leupold and Boorse (1964) found a value of 3·69, which is somewhat lower than the value of 4·03 found from our calculations. The critical field for niobium, from specific heat data of Leupold and Boorse was found to be 1994 G. McConville and Serin (1964, 1965) found $H_c(0)$ to be 1990 G. Niobium, even in the purest form, is a type II superconductor. The difference in $2\Delta(0,0)/k_BT_c$ between our value and the observed values may be possibly due to this reason. Thus on the basis of experimental data, at least for the energy gap and the critical field, a two parameter energy-dependent effective interaction within the BCS approximation seems sufficiently accurate.

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