DISSIPATION IN QUANTUM MECHANICS

By E. Harting*

[*Manuscript received April* 22, 1966]

Summary

The energy dissipation of an electromagnetic mode, weakly coupled to a solid, is derived for a simple model. The result differs in significant detail from that previously obtained by a different method. Reasons for the discrepancy are given, with reference to a recently proposed theorem in quantum optics.

I. Introduction

The energy dissipation of an electromagnetic mode in a lossy cavity has received previous attention (Senitzky 1960). In the present paper, the mode is treated as a classical driving force, and the loss mechanism to which it is weakly coupled through the electric field is quantized and assumed to be at all times in approximate thermal equilibrium. In other words, the loss mechanism consists of a quasi-continuum of energy levels, and in the uncoupled state it is adequately described by a density matrix

$$\rho^{(0)} = \frac{\exp(-\beta H^{(0)})}{\operatorname{Tr}\exp(-\beta H^{(0)})}, \tag{1}$$

where $H^{(0)}$ is the unperturbed Hamiltonian and $\beta = 1/kT$. It is only slightly perturbed by the field.

II. Energy in the Mode

The Hamiltonian of the uncoupled electromagnetic mode can be expressed as

$$H_{\text{osc}}^{(0)} = \tfrac{1}{2}\{(p^{(0)})^2+\omega^2(q^{(0)})^2\}, \tag{2}$$

where $q^{(0)}$ is defined by the equations

$$\mathbf{E} = -(1/c)\,\dot{q}^{(0)}\mathbf{A}, \qquad \mathbf{H} = q^{(0)}\,\nabla\times\mathbf{A}\,; \tag{3}$$

$p^{(0)} = \dot{q}^{(0)}$ is the variable that is canonically conjugate to $q^{(0)}$, and ω is the angular frequency of the radiation.

We assume that $p^{(0)}$ and $q^{(0)}$ are classical variables and that the coupling takes place through p. Introducing a coupling constant α, the generalized force acting on the solid is $\alpha p(t)$.

* School of Physics, University of New South Wales, Kensington, N.S.W.

Since the coupling is weak, two simplifying assumptions can be made:

(i) the signal energy density, averaged over a cycle, is equally divided between electric and magnetic energy at all times, that is,

$$u_s(t) = |p(t)|^2 = |q(t)|^2; \tag{4}$$

(ii) the frequency is not appreciably changed from ω.

Under these assumptions, the mean power abstracted from the mode is

$$\left|\frac{\alpha\, p(t)}{Z(\omega)}\right|^2 R(\omega)$$

or

$$\frac{\mathrm{d}|p(t)|^2}{\mathrm{d}t} = -|p(t)|^2 \alpha^2 G(\omega), \tag{5}$$

where $Z(\omega)$, $R(\omega)$, and $G(\omega)$ are the complex impedance, its real part, and the real part of the complex admittance, respectively, of the solid.

Bernard and Callen (1959) have shown that, with the assumptions made here about the solid, $G(\omega)$ can be expressed as

$$G(\omega) = -\pi\omega\{1-\exp(-\beta\hbar\omega)\}B(\omega), \tag{6}$$

with

$$B(\omega) = \int_{-\infty}^{\infty} \mathrm{d}E\; \eta(E)\; \eta(E+\hbar\omega)\; \rho^{(0)}(E)\; \left|\langle E|\Gamma|E+\hbar\omega\rangle\right|^2, \tag{7}$$

where $\eta(E)$ denotes the density of states function, averaged for states near E, $\rho^{(0)}(E_n)$ is the density matrix element $\rho^{(0)}_{nn}$ in the unperturbed energy representation of the solid, and the matrix elements of Γ are those of the coordinate of the solid through which the coupling takes place.

It follows that the signal energy is given by

$$u_s(t) = |p(t)|^2 = |p^{(0)}|^2 \exp\{-\alpha^2 G(\omega)\, t\}, \tag{8}$$

which describes the decay of the signal.

Apart from this coherent signal energy, the solid exhibits a random noise voltage. In thermal equilibrium (which we assume to hold approximately), the quantized version of the Nyquist theorem states that the noise energy in the mode is (Callan and Welton 1951)

$$E(\omega;\beta) = \frac{\hbar\omega}{2} + \frac{1}{\exp\beta\hbar\omega - 1}. \tag{9}$$

The total energy in the mode is therefore

$$u_t = |p^{(0)}|^2 \exp\{-\alpha^2 G(\omega)\, t\} + E(\omega;\beta). \tag{10}$$

III. Comparison with Previous Work

Senitzky (1960) considers a very similar problem; the only difference lies in the treatment of the electromagnetic mode, which in his work is quantized. Explicit expressions for the p and q operators are obtained under the same assumption of weak coupling as in Section II above. His final expression (equation (60)) for the energy in the mode is, in our notation,

$$u_t = \{|p^{(0)}|^2 - E(\omega;\beta)\}\exp\{-\alpha^2 G(\omega)t\} + E(\omega;\beta). \tag{11}$$

The difference between (10) and (11) is in the first term on the right-hand side, and it is easily shown, with reference to the derivation of (11), that this difference is *not* the result of treating the radiation quantum-mechanically. In fact, Senitzky treats the dissipation classically (cf. remarks following his equation (20)), and the whole derivation leading up to his equation (60) is independent of the commutation relation between p and q. The interpretation of the result (remarks following equation (61)), involving the zero-point energy of the radiation, cannot, therefore, be correct.

The results (10) and (11) can be reconciled if $|p^{(0)}|^2$ in (11) includes the noise energy $E(\omega;\beta)$, that is, if it denotes the *total* energy in the mode at the initial time. But, as we have seen, the noise energy is associated with the *solid* (as is evident from the fact that it depends on β); u_t of (11) thus describes the energy in a mode of the solid which is excited in some way prior to $t = 0$ and is then weakly coupled to the "remainder" of the solid. It is then understandable how a Brownian movement equation for p will result ((72) of Senitzky), this model being the analogue of a classical, harmonically bound particle. The equations of motion for the coordinates of the latter are given by Chandrasekhar (1943), and the energy calculated from these (employing assumptions (i) and (ii) of Section II) leads to the classical analogue of (11) (classically, $E(\omega;\beta) \to 1/\beta$).

As we have remarked, in Senitzky's calculation it is immaterial whether the radiation is classical or quantum-mechanical, and this is in part a result of his treatment of the dissipation in a classical approximation. As a consequence, p and Γ, the coupling coordinate of the solid, are made to commute at all times, and this is implied in the way the (combined) Hamiltonian is written (Senitzky's equation (3)). The coupling term appears as $\alpha p\Gamma$, instead of $\alpha[p,\Gamma]_+$ as required in a full quantum-mechanical treatment.

These remarks are relevant to the discussion of a theorem in quantum optics, proposed by Senitzky (1965), stating that "all sources (of radiation) on which the effect of the 'detector' is negligible may be treated as classical sources in the interaction under consideration". Since the proof of this theorem rests on the classical approximation described above, the theorem would seem to require further justification.

On the other hand, it seems reasonable to conjecture that such a theorem may well hold in the situation explicitly considered here, i.e. where the interaction is of a purely resonant nature. If one of the two interacting systems is quantized, energy can in any case only be exchanged in units of $\hbar\omega$. It is interesting to note in this connection that previous objections to the theorem (Glassgold and Holliday 1965) quote as a specific example of its shortcomings the photoelectric effect, which is clearly a more complicated interaction.

IV. Acknowledgment

The author sincerely wishes to thank Professor J. J. O'Dwyer for helpful discussions on this problem.

V. References

Bernard, W., and Callen, H. B. (1959).—*Rev. mod. Phys.* **31**, 1017.
Callen, H. B., and Welton, T. A. (1951).—*Phys. Rev.* **83**, 34.
Chandrasekhar, S. (1943).—*Rev. mod. Phys.* **15**, 1.
Glassgold, A. E., and Holliday, D. (1965).—*Phys. Rev. Lett.* **15**, 741.
Senitzky, I. R. (1960).—*Phys. Rev.* **119**, 670.
Senitzky, I. R. (1965).—*Phys. Rev. Lett.* **15**, 233.