ELECTRIC FORCES IN QUASI-STATIC SYSTEMS

By W. E. Smith†

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Summary

It is shown that the time-averaged electric forces produced in a dissipative system excited by a number of varying charge sources may be written in a form analogous to previously derived equations for magnetic forces in quasi-stationary systems. The result corresponds to a generalization of Thomson's formula of electrostatics to quasi-static fields in an energy-dissipating medium.

I. Introduction

Recently (Smith 1965*a*, 1965*b*), it was shown that average magnetic forces developed by quasi-stationary currents in energy-dissipating systems could often be expressed simply in terms of effective inductance parameters of the system. The results obtained were similar in form to the familiar equations of magnetostatics or to the previously obtained results for general energy-conserving systems (Smith 1961). For instance, the average generalized force $\overline{F}_x$ corresponding to a generalized coordinate x specifying the configuration of the source current part of the system may be written

$$\overline{F}_x = \tfrac{1}{2} I^2 \, \partial L / \partial x \,, \tag{1}$$

where, for simplicity, a single r.m.s. excitation current I of angular frequency ω has been taken and $L = L(\omega, x)$ is the effective inductance of the excitation circuit at this frequency. An immediate practical application of equation (1) relates to the estimation of electromagnetic levitation forces.

Equation (1) and generalizations of it were obtained from the relation

$$\overline{F}_x + \overline{F'_x} = \left(\frac{\partial}{\partial x}\right)_I \left\{ \mathrm{Re}\left(\int_V \mathbf{A} \cdot \mathbf{J}^* \, \mathrm{d}\tau \right) \right\}, \tag{2}$$

where $\mathbf{A}$ is the complex r.m.s. vector potential in a fixed dissipative medium arising from current sources described by the complex r.m.s. current density $\mathbf{J}$, and $\overline{F}_x$, $\overline{F'_x}$ are the average generalized forces resulting from current sources $\mathbf{J}$, $\mathbf{J}^*$. The differentiation is to be performed in a particular way which corresponds to constant current for current loops.

The similarity between expressions for average forces in conservative systems produced by either electric or magnetic stresses (Smith 1960) suggests that an electric force analogy of equation (2) should exist. It will be shown that, for electric forces

† Department of Applied Mathematics, University of New South Wales, Kensington, N.S.W.

produced in a quasi-static system as the result of a complex r.m.s. source charge distribution ρ, the replacement for equation (2) is

$$\bar{F}_x+\bar{F}'_x = -\left(\frac{\partial}{\partial x}\right)_\rho \left\{\mathrm{Re}\left(\int_V \phi\rho^* \,\mathrm{d}\tau\right)\right\}. \tag{3}$$

In this equation, ϕ is the complex r.m.s. quasi-static electric potential in a fixed dissipative medium and $\bar{F}_x$, $\bar{F}'_x$ are time-averaged forces corresponding to source charge distributions ρ, ρ^*. The medium is assumed to be both polarizable and conducting, and the forces are those resulting from a generalized displacement of the source charges. The differentiation in the right-hand side of equation (3) is to be performed in a specified manner corresponding to constant charge for the movement of point charges. Equation (3) is a generalization of an electrostatics formula due to Thomson (see Jackson 1962, p. 126) for quasi-static fields in a dissipative medium.

When the source charges are point charges, equation (3) may easily be reduced to a simple form involving the potential coefficients of the system, but, unlike the reduced form of equation (2), this does not correspond to the usual practical situation of constant voltages over conducting surfaces. Unfortunately, correspondences between constant voltage and constant charge conditions, as in electrostatics, do not apply.

Equation (3) is derived by assuming that the magnetic fields, including those produced by the flow of conduction currents, are everywhere negligible, so that the electric field is expressible in terms of a quasi-static electric potential. The steps in the derivation are analogous to those used in deriving equation (2) (Smith 1965*a*). A single excitation frequency is considered, but this is no limitation, since we consider only linear systems, in which case the forces from individual Fourier components are additive.

II. The Field Equations and their Formal Solution

All field quantities are taken as having time dependence $\exp(\mathrm{i}\omega t)$ corresponding to a single excitation frequency, and r.m.s. amplitudes are used throughout. The entire field distribution is produced by the charge density ρ as source. Assuming all magnetic fields to be negligible, the Maxwell equations reduce to the quasi-static equations (in MKS rationalized units)

$$\nabla\times\mathbf{E} = 0, \tag{4}$$

$$\nabla\cdot\mathbf{D} = \rho^{(\mathrm{t})}, \tag{5}$$

where $\mathbf{E}$ and $\mathbf{D}$ are the complex r.m.s. electric field and electric displacement vectors and $\rho^{(\mathrm{t})}$ is the total complex r.m.s. charge density. The total charge density $\rho^{(\mathrm{t})}$ is composed of the source charge density ρ together with an induced charge density $\rho^{(\mathrm{i})}$, polarization charges having already been included in $\mathbf{D}$, so

$$\rho^{(\mathrm{t})} = \rho+\rho^{(\mathrm{i})}. \tag{6}$$

The induced charge density $\rho^{(\mathrm{i})}$ is associated with conduction currents $\mathbf{J}^{(\mathrm{i})}$ flowing in the medium, and the following conservation equation is assumed:

$$\mathrm{i}\omega\rho^{(\mathrm{i})} + \nabla\cdot\mathbf{J}^{(\mathrm{i})} = 0. \tag{7}$$

That is, the source and induced charges are separately conserved, and the induced current comprises the whole of the conduction current in the medium. The requirements of charge conservation in the source system must be met other than by conduction currents in the medium. A similar condition was imposed for the magnetic force theorem (Smith 1965*a*) by requiring that the source currents **J** were conserved separately from the induced conduction currents in the medium. It should be noted that these conditions impose a restriction on the type of system for which the theory is valid.

The linear constitutive equation

$$\mathbf{D} = \kappa \mathbf{E} \qquad \kappa \text{ real} \tag{8}$$

together with Ohm's law for conduction currents,

$$\mathbf{J}^{(\mathrm{i})} = \sigma \mathbf{E} \qquad \sigma \text{ real} \tag{9}$$

are also assumed, guaranteeing the linearity of the system. From equation (4), the scalar potential ϕ may be introduced by

$$\mathbf{E} = -\nabla \phi \,. \tag{10}$$

Then, from equations (5), (6), and (8),

$$\rho + \rho^{(\mathrm{i})} = -\nabla \cdot (\kappa \nabla \phi) \,. \tag{11}$$

However, from equation (7),

$$\begin{aligned} \rho^{(\mathrm{i})} &= -(1/\mathrm{i}\omega)\, \nabla \cdot \mathbf{J}^{(\mathrm{i})} && \omega \neq 0 \\ &= -(1/\mathrm{i}\omega)\, \nabla \cdot (\sigma \mathbf{E}) && \text{using (9)} \\ &= (1/\mathrm{i}\omega)\, \nabla \cdot (\sigma \nabla \phi) && \text{from (10)}\,. \end{aligned} \tag{12}$$

Elimination of $\rho^{(\mathrm{i})}$ from equations (11) and (12) leads to

$$\nabla \cdot (\kappa + \sigma/\mathrm{i}\omega)\, \nabla \phi = -\rho \,, \tag{13}$$

which expresses the electric potential ϕ in terms of the source charges alone. Equation (13) corresponds to the introduction of the complex permittivity $\kappa + \sigma/\mathrm{i}\omega$ to account for conduction in the medium. With the imposition of a suitable boundary condition on ϕ ($\phi \to 0$ as $1/r$ or faster as $r \to \infty$), equation (13) provides a means of finding the electric field and hence the forces on the source charges.

We suppose that the solution of equation (13) is expressible in terms of a Green's function $G(\mathbf{r}, \mathbf{r}')$, for which

$$\nabla \cdot (\kappa + \sigma/\mathrm{i}\omega)\, \nabla G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \,, \tag{14}$$

where $G(\mathbf{r}, \mathbf{r}')$ satisfies the same boundary conditions as ϕ and $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function. Clearly, $G(\mathbf{r}, \mathbf{r}')$ corresponds physically to the solution of equation (14) for a single point charge source at $\mathbf{r} = \mathbf{r}'$. The solution for ϕ of equation (13) may then be written

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}')\, \rho(\mathbf{r}')\, \mathrm{d}\tau' \,. \tag{15}$$

The Green's function $G(\mathbf{r},\mathbf{r}')$ is symmetric in $\mathbf{r},\mathbf{r}'$, that is,

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r}), \tag{16}$$

which we show by considering both $G(\mathbf{r},\mathbf{r}')$ and $G(\mathbf{r},\mathbf{r}'')$, which satisfies

$$\nabla\cdot(\kappa+\sigma/\mathrm{i}\omega)\,\nabla G(\mathbf{r},\mathbf{r}'') = -\,\delta(\mathbf{r}-\mathbf{r}''). \tag{17}$$

From equations (14) and (17),

$$G(\mathbf{r},\mathbf{r}'')\,\nabla\cdot(\kappa+\sigma/\mathrm{i}\omega)\,\nabla G(\mathbf{r},\mathbf{r}') = -\,\delta(\mathbf{r}-\mathbf{r}')\,G(\mathbf{r},\mathbf{r}''), \tag{18}$$

and

$$G(\mathbf{r},\mathbf{r}')\,\nabla\cdot(\kappa+\sigma/\mathrm{i}\omega)\,\nabla G(\mathbf{r},\mathbf{r}'') = -\,\delta(\mathbf{r}-\mathbf{r}'')\,G(\mathbf{r},\mathbf{r}'). \tag{19}$$

Subtracting equation (19) from (18), and integrating over the entire volume,

$$\int_V \{G(\mathbf{r},\mathbf{r}'')\,\nabla\cdot(\kappa+\sigma/\mathrm{i}\omega)\,\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\,\nabla\cdot(\kappa+\sigma/\mathrm{i}\omega)\,\nabla G(\mathbf{r},\mathbf{r}'')\}\,\mathrm{d}\tau$$
$$= G(\mathbf{r}'',\mathbf{r}')-G(\mathbf{r}',\mathbf{r}''). \tag{20}$$

However, the following generalization of the usual scalar Green's theorem may easily be proved:

$$\int_S a(\phi\,\nabla\psi-\psi\,\nabla\phi)\cdot\mathrm{d}\mathbf{S} = \int_V (\phi\,\nabla\cdot a\,\nabla\psi-\psi\,\nabla\cdot a\,\nabla\phi)\,\mathrm{d}\tau.$$

Application of this theorem to equation (20) gives

$$\int_S (\kappa+\sigma/\mathrm{i}\omega)\{G(\mathbf{r},\mathbf{r}'')\,\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\,\nabla G(\mathbf{r},\mathbf{r}'')\}\cdot\mathrm{d}\mathbf{S} = G(\mathbf{r}'',\mathbf{r}')-G(\mathbf{r}',\mathbf{r}''),$$

and the surface integral vanishes from the boundary conditions on G, giving the symmetry expressed by equation (16) in accord with general principles of reciprocity for irreversible electromagnetic systems (Meixner 1963).

III. The Average Generalized Force

The average generalized force $\bar{F}_x$ is computed by evaluating the work $\bar{F}_x\,\delta x$ done in a displacement generated by a small change δx of the generalized coordinate x. This displacement is to be carried out sufficiently slowly for the fields at any time to be adequately described by steady-state equations. Let $\delta\mathbf{r}(\mathbf{r})$ denote the displacement of the source charges at $\mathbf{r}$ generated by δx.

Since magnetic fields have been assumed negligible, the force density operating on the source charges is $\mathrm{Re}(\rho^*\mathbf{E})$. Consequently, the work done is

$$\begin{aligned}\bar{F}_x\,\delta x &= \mathrm{Re}\Big(\int_V \rho^*\mathbf{E}\cdot\delta\mathbf{r}\,\mathrm{d}\tau\Big) && (21)\\ &= -\,\mathrm{Re}\Big(\int_V \rho^*\,\nabla\phi\cdot\delta\mathbf{r}\,\mathrm{d}\tau\Big) && \text{from (10)}\\ &= \mathrm{Re}\Big(\int_V \{\phi\,\nabla\cdot(\rho^*\,\delta\mathbf{r})-\nabla\cdot(\rho^*\phi\,\delta\mathbf{r})\}\,\mathrm{d}\tau\Big),\\ &= \mathrm{Re}\Big(\int_V \phi\,\nabla\cdot(\rho^*\,\delta\mathbf{r})\,\mathrm{d}\tau\Big)-\mathrm{Re}\Big(\int_S \rho^*\phi\,\delta\mathbf{r}\cdot\mathrm{d}\mathbf{S}\Big)\end{aligned}$$

by the Gauss theorem, where S encloses the volume V containing the source charges.

The surface integral vanishes, giving

$$\bar{F}_x \delta x = \mathrm{Re}\Big(\int_V \phi \, \nabla \cdot (\rho^* \, \delta \mathbf{r}) \, \mathrm{d}\tau \Big) \tag{22}$$

$$= \mathrm{Re}\Big(\int_V \phi(\rho^* \, \nabla \cdot \delta \mathbf{r} + \delta \mathbf{r} \cdot \nabla \rho^*) \, \mathrm{d}\tau \Big). \tag{23}$$

For the moment suppose $\delta\mathbf{r}$ to be constant, representing a simple translation of the source charges. Then $\nabla \cdot \delta\mathbf{r} = 0$, and the only contribution to the right-hand side of equation (23) is from the term involving $(\delta\mathbf{r} \cdot \nabla\rho^*)$. But $(\delta\mathbf{r} \cdot \nabla\rho^*)$ may be written $-\delta\rho^*$, where $\delta\rho$ is the change in the source charge density generated by translation of the sources *without change of the charge amplitudes*, i.e. at constant charge. Then,

$$\bar{F}_x \delta x = -\,\mathrm{Re}\Big(\int_V \phi \, \delta \rho^* \, \mathrm{d}\tau \Big). \tag{24}$$

Inclusion of the term $(\rho^* \, \nabla \cdot \delta\mathbf{r})$ in equation (23) allows for dilatation of the charge distribution with a general displacement $\delta\mathbf{r}$. We shall see that equation (24) always applies, provided constant charge is interpreted to mean that every volume moving with the displacement is always to contain the same amount of source charge. That is,

$$\delta \int_{V'} \rho \, \mathrm{d}\tau = 0 \qquad \text{all } V',$$

or

$$\int_{V'} \delta\rho \, \mathrm{d}\tau + \int_{\delta V'} \rho \, \mathrm{d}\tau = 0, \tag{25}$$

where the second integral accounts for the change in the volume of integration generated by δx. But, by geometry,

$$\int_{\delta V'} \rho \, \mathrm{d}\tau = \int_{S'} \rho \, \delta\mathbf{r} \cdot \mathrm{d}\mathbf{S}$$

$$= \int_{V'} \nabla \cdot (\rho \, \delta \mathbf{r}) \, \mathrm{d}\tau, \tag{26}$$

where S' is the surface enclosing V'. From (25) and (26),

$$\int_{V'} \{\delta\rho + \nabla \cdot (\rho \, \delta\mathbf{r})\} \, \mathrm{d}\tau = 0 \qquad \text{all } V'.$$

Thus,

$$\delta\rho = -\,\nabla \cdot (\rho \, \delta\mathbf{r}) \tag{27}$$

corresponds to constant charge displacements.

Using (27) in (22), we obtain

$$\bar{F}_x \delta x = -\,\mathrm{Re}\Big(\int_V \phi \, \delta \rho^* \, \mathrm{d}\tau \Big). \tag{28}$$

Equation (28) may be written in a more useful integral form by examining

$$\delta \int_V \phi \rho^* \, \mathrm{d}\tau = \int_V \rho^* \, \delta\phi \, \mathrm{d}\tau + \int_V \phi \, \delta\rho^* \, \mathrm{d}\tau. \tag{29}$$

But since only the source charges move under the displacement δx, we have, from equation (15),

$$\delta\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')\,\delta\rho(\mathbf{r}')\,\mathrm{d}\tau'\,;$$

therefore,

$$\int_V \rho^*\,\delta\phi\,\mathrm{d}\tau = \int_V \rho^*(\mathbf{r}) \int_V G(\mathbf{r},\mathbf{r}')\,\delta\rho(\mathbf{r}')\,\mathrm{d}\tau'\,\mathrm{d}\tau$$

$$= \int_V \phi'\,\delta\rho\,\mathrm{d}\tau\,,$$

where $\phi'(\mathbf{r})$, given by

$$\phi'(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')\,\rho^*(\mathbf{r}')\,\mathrm{d}\tau' \tag{30}$$

(using the symmetry (equation (16)) of the Green's function ϕ'), is the electric potential corresponding to the source charge distribution ρ^* rather than ρ. Therefore, equation (29) becomes

$$\delta \int_V \phi\rho^*\,\mathrm{d}\tau = \int_V \phi'\,\delta\rho\,\mathrm{d}\tau + \int_V \phi\,\delta\rho^*\,\mathrm{d}\tau\,,$$

and, using (28), we finally obtain

$$(\overline{F}_x+\overline{F'_x})\,\delta x = -\,\delta\Big\{\mathrm{Re}\Big(\int_V \phi\rho^*\,\mathrm{d}\tau\Big)\Big\} \tag{31}$$

or

$$\overline{F}_x+\overline{F'_x} = -\left(\frac{\partial}{\partial x}\right)_\rho \Big\{\mathrm{Re}\Big(\int_V \phi\rho^*\,\mathrm{d}\tau\Big)\Big\}, \tag{32}$$

where $\overline{F'_x}$ is the average generalized force corresponding to the charge distribution ρ^*.

IV. Discussion

The forces $\overline{F}_x$ and $\overline{F'_x}$ are not usually equal, since, from equation (28),

$$\delta x\,(\overline{F}_x-\overline{F'_x}) = \mathrm{Re}\Big(\int_V (\phi'\,\delta\rho - \phi^*\,\delta\rho)\,\mathrm{d}\tau\Big)$$

$$= \mathrm{Re}\Big(\int_V \delta\rho(\mathbf{r}) \int_V \{G(\mathbf{r},\mathbf{r}')-G^*(\mathbf{r},\mathbf{r}')\}\,\rho^*(\mathbf{r}')\,\mathrm{d}\tau'\,\mathrm{d}\tau\Big), \tag{33}$$

which depends on the imaginary part of the Green's function. If $\sigma = 0$, as in a loss-free system, the Green's function is real, and $\overline{F}_x = \overline{F'_x}$. Then $\int_V \phi\rho^*\,\mathrm{d}\tau$ is also real and equal to twice the average energy of the system. Thus, equation (32) corresponds to a generalization of the well-known Thomson theorem for electrostatic forces (Maxwell 1881, Section 93; Jackson 1962, p. 126). In fact, for a conservative system with $\omega \to 0$, equation (32) is just the Thomson formula

$$F_x = -(\partial W/\partial x)_\rho\,, \tag{34}$$

where W is the electrostatic energy.

For a set of point charge sources q_k at $\mathbf{r} = \mathbf{r}_k$, we have

$$\rho(\mathbf{r}) = \sum_k q_k\,\delta(\mathbf{r}-\mathbf{r}_k),$$

and, from (15),

$$\begin{aligned}\phi_k = \phi(\mathbf{r}_k) &= \sum_j G(\mathbf{r}_k, \mathbf{r}_j)\,q_j \\ &= \sum_j G_{kj}\,q_j,\end{aligned} \tag{35}$$

with

$$G_{kj} = G(\mathbf{r}_k, \mathbf{r}_j) = G_{jk} \qquad \text{from (16)}. \tag{36}$$

The G_{kj} are complex potential coefficients of the system. Then,

$$\begin{aligned}\mathrm{Re}\left(\int_V \phi\rho^*\,\mathrm{d}\tau\right) &= \mathrm{Re}\left(\sum_k q_k^*\,\phi_k\right) \\ &= \mathrm{Re}\left(\sum_{j,k} q_k^*\,G_{kj}\,q_j\right) \\ &= \sum_{j,k} q_k^*\,q_j\,\mathrm{Re}(G_{kj}) \qquad \text{from (36)}.\end{aligned} \tag{37}$$

Using equation (32),

$$\overline{F}_x + \overline{F'_x} = -\sum_{j,k} q_k^*\,q_j\,(\partial/\partial x)\{\mathrm{Re}(G_{kj})\}. \tag{38}$$

Equation (38) is the analogue of the magnetic force formula expressing magnetic forces in terms of the source currents and effective inductances (Smith 1965*a*, equation (28)). However, in contrast to the magnetic force formula, no practical applications suggest themselves, since excitation with a specified charge distribution is rather artificial. Practical situations involve excitation of conducting surfaces by voltage or current sources. The magnetic force theorem provided useful practical results because currents are the direct sources of magnetic fields, but for electric fields it is charges that are the sources. For energy-conserving systems a correspondence between constant charge and constant potential conditions may be made (e.g. Jackson 1962, p. 127), but this does not seem feasible for energy-dissipating systems.

V. References

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