ON THE ADIABATIC MOTION OF A CHARGED PARTICLE IN THE QUASI-STATIC MAGNETIC FIELD OF A SOLENOID

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Summary

Following an alternative phase–amplitude derivation of the general forms of r and θ, obtained earlier by Seymour, Leipnik, and Nicholson (SLN) for the motion of a charged particle in the time-dependent magnetic field within a long solenoid, the guiding-centre form of the SLN result for r is obtained in the adiabatic limit and is used as a guide to analysis of the charged particle motion when the solenoid magnetic field is quasi-static. This leads to identification of the associated adiabatic invariants, and the analysis then closes with a quantitative examination of the adiabatic approximation, which sheds interesting light on the required conditions for its self-consistency.

I. Introduction

In a previous paper (Seymour, Leipnik, and Nicholson 1965; hereafter referred to as SLN) the most general form of the solution for the motion of a charged particle in the time-dependent axially symmetric magnetic field within a long solenoid was obtained, together with a simple pictorial interpretation, and, for convenience of reference in the present paper, the salient features of the solution are assembled here. Essentially, for a charged particle of mass m, charge q, and velocity $\mathbf{v}$, located within the solenoid at r, θ, z, the component equations of the non-relativistic Lorentz force equation of motion

$$m\dot{\mathbf{v}} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \qquad \text{(e.m.u.)} \tag{1.1}$$

were specialized for the solenoid fields

$$\mathbf{E} = (0, -\tfrac{1}{2}r\dot{B}_z, 0) \tag{1.2}$$

and

$$\mathbf{B} = (0,0,B_z), \tag{1.3}$$

and were manipulated to obtain the results that:

(1) the z-direction motion of the charged particle is of constant velocity, and

(2) the non-trivial motion of the charged particle in any r–θ plane is represented by the equations

$$\ddot{r}+\omega_{\mathrm{L}}^2 r - C^2/r^3 = 0 \tag{1.4}$$

and

$$\theta = \int_0^t \omega_{\mathrm{L}}\,\mathrm{d}t' + C\int_0^t \frac{\mathrm{d}t'}{r^2} + \theta_0, \tag{1.5}$$

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where the time-dependent quantity

$$\omega_L(t) = -qB_z(t)/2m \tag{1.6}$$

is the Larmor angular frequency and C is a real constant.

Further transformations showed that solutions for r and θ can be obtained from equations (1.4) and (1.5) in terms of a complex time-dependent quantity $u(t) = \alpha(t)+i\beta(t)$, satisfying the equation

$$\ddot{u}+\omega_L^2 u = 0, \tag{1.7}$$

in the forms

$$r^2 = uu^* \tag{1.8}$$

and

$$\theta = \int_0^t \omega_L \,dt' - \tan^{-1}(\beta/\alpha)+\theta_0. \tag{1.9}$$

In terms of assumed real, linearly independent solutions $I_1(t)$ and $I_2(t)$ of equation (1.7), r and θ were finally obtained in the forms

$$r = \{|\mu|^2 I_1^2+|\lambda|^2 I_2^2+2|\mu||\lambda| I_1 I_2 \cos(\epsilon_1-\epsilon_2)\}^{\frac{1}{2}} \tag{1.10}$$

and

$$\theta = \int_0^t \omega_L \,dt' - \tan^{-1}\left(\frac{|\mu|\, I_1 \sin\epsilon_1 + |\lambda|\, I_2 \sin\epsilon_2}{|\mu|\, I_1 \cos\epsilon_1 + |\lambda|\, I_2 \cos\epsilon_2}\right)+\theta_0, \tag{1.11}$$

where, in terms of arbitrary real constants a_{ij}, μ and λ are complex constants given by $\mu = a_{11}+ia_{21} = |\mu| \exp i\epsilon_1$ and $\lambda = a_{12}+ia_{22} = |\lambda| \exp i\epsilon_2$, and

$$u = \mu I_1(t)+\lambda I_2(t). \tag{1.12}$$

The pictorial interpretation of the general solutions for r and θ in terms of I_1 and I_2 given by equations (1.10) and (1.11) is described in SLN at the close of Section III. Specializations of these equations for cases corresponding to the solenoid magnetic field, and therefore $\omega_L(t)$, varying in accordance with a simple power law and an exponential time dependence are given in SLN, Section IV.

In the present paper attention is focused on the case of quasi-stationary variation of the solenoid's magnetic field B_z, with a view to identifying the associated approximate integrals of the equation of motion—the adiabatic invariants.

As a preliminary step, it is a simple matter to develop an interesting alternative derivation of the results (1.10) and (1.11), more in the spirit of the phase–amplitude approach to the solution of the quasi-static case, but which is motivated by understanding gained from the solution technique presented in SLN.

II. Alternative Derivation of the General Solutions for r and θ

Noting that the induced azimuthal electric field given by equation (1.2) can be written in the vector form

$$\mathbf{E} = \tfrac{1}{2}\mathbf{r}\times\dot{\mathbf{B}}, \tag{2.1}$$

where $\mathbf{B}$ is given by (1.3), it follows that the Lorentz force equation (1.1) yields, for charged particle motion transverse to the magnetic field, the vector equation

$$\ddot{\mathbf{r}} = \tfrac{1}{2}\dot{\boldsymbol{\omega}}_{\mathrm{g}}\times\mathbf{r}+\boldsymbol{\omega}_{\mathrm{g}}\times\dot{\mathbf{r}}, \tag{2.2}$$

where

$$\boldsymbol{\omega}_{\mathrm{g}}(t) = -q\,\mathbf{B}(t)/m = 2\boldsymbol{\omega}_{\mathrm{L}} \tag{2.3}$$

(from equation (1.6)) is the charged particle gyrofrequency mentioned in SLN, Section V.

Consideration of the constant velocity motion in the z direction is omitted at this stage, as it can be combined with the r, θ solutions later to give the complete three-dimensional charged particle trajectory. Then, in Cartesian coordinates, the component equations of (2.2) are

$$\ddot{x} = -\tfrac{1}{2}\dot{\omega}_{\mathrm{g}}\,y-\omega_{\mathrm{g}}\,\dot{y} \tag{2.4}$$

and

$$\ddot{y} = \tfrac{1}{2}\dot{\omega}_{\mathrm{g}}\,x+\omega_{\mathrm{g}}\,\dot{x}, \tag{2.5}$$

where

$$\boldsymbol{\omega}_{\mathrm{g}} = \mathbf{k}\omega_{\mathrm{g}} \tag{2.6}$$

and $\mathbf{k}$ is the usual unit vector in the positive z direction.

Using a standard technique of analytical mechanics, we introduce the complex variable $R = x+\mathrm{i}y$ and recombine equations (2.4) and (2.5) to obtain the following equation in R

$$\ddot{R} = \mathrm{i}(\tfrac{1}{2}\dot{\omega}_{\mathrm{g}}\,R+\omega_{\mathrm{g}}\,\dot{R}). \tag{2.7}$$

By taking R to have the phase–amplitude form

$$R = \xi\exp\Big(\mathrm{i}\int\omega\;\mathrm{d}t\Big), \tag{2.8}$$

where $\xi(t)$ is a complex amplitude and $\omega(t)$ is, at this stage, an arbitrary time-dependent real angular frequency, we obtain by substitution in equation (2.7)

$$\ddot{\xi}+(\omega\omega_{\mathrm{g}}-\omega^2)\xi+\mathrm{i}\{2(\omega-\tfrac{1}{2}\omega_{\mathrm{g}})\dot{\xi}+(\dot{\omega}-\tfrac{1}{2}\dot{\omega}_{\mathrm{g}})\xi\} = 0, \tag{2.9}$$

after cancellation of the non-zero factor $\exp\Big(\mathrm{i}\int\omega\;\mathrm{d}t\Big)$.

Bearing in mind the relationship (2.3) and the result (1.7) (equation (3.12) in SLN), we see that equation (2.9) assumes the simple form

$$\ddot{\xi}+\omega_{\mathrm{L}}^2\,\xi = 0, \tag{2.10}$$

if a transformation to a coordinate system moving with angular velocity ω_{L} is made by choosing

$$\omega = \tfrac{1}{2}\omega_{\mathrm{g}} = \omega_{\mathrm{L}}. \tag{2.11}$$

Writing

$$\xi = |\xi|\exp\mathrm{i}\,\psi(t), \tag{2.12}$$

equation (2.8) can be written in the form

$$R = |\xi|\exp\mathrm{i}\Big(\int_0^t\omega_{\mathrm{L}}\;\mathrm{d}t' +\psi(t)+\theta_0\Big), \tag{2.13}$$

where the angle θ_0 is a constant.

Remembering that the time-dependent real and imaginary parts of ξ must satisfy equation (2.10), the Argand diagram interpretation of equation (2.13) is simply that the position of the charged particle in a plane normal to the z axis of the solenoid is referred, in terms of the real and imaginary parts of ξ, to axes moving with an angular velocity ω_L about an origin of coordinates located on the z axis. Since the cylindrical coordinates r and θ are, in terms of the Cartesian form

$$R = x+iy,$$

given by

$$r = |R|, \qquad \theta = \tan^{-1}(y/x), \tag{2.14}$$

equation (2.13) yields the angular equation

$$\theta = \int_0^t \omega_L \, dt' + \psi(t) + \theta_0. \tag{2.15}$$

Comparison of this result with equation (1.9) shows that, in terms of the SLN nomenclature,

$$\psi(t) = -\tan^{-1}(\beta/\alpha). \tag{2.16}$$

From equations (2.13) and (2.14) it is immediately evident that

$$r = |R| = |\xi|, \tag{2.17}$$

and hence, using (2.16), equation (2.12) can be written

$$\xi = r\exp\{-i\tan^{-1}(\beta/\alpha)\}. \tag{2.18}$$

Reverting to SLN, combination of their results (3.17) and (3.19), which were respectively

$$u^* = r\exp\left(iC\int_0^t \frac{dt'}{r^2}\right)$$

and

$$-\tan^{-1}(\beta/\alpha) = C\int_0^t \frac{dt'}{r^2},$$

gives

$$u^* = r\exp\{-i\tan^{-1}(\beta/\alpha)\}, \tag{2.19}$$

so that from (2.18) and (2.19)

$$\xi = u^* = \alpha(t) - i\beta(t). \tag{2.20}$$

Summarizing results for this exact phase–amplitude method of solution, we have

$$r^2 = RR^* = \xi\xi^*, \tag{2.21}$$

and

$$\theta = \int_0^t \omega_L \, dt' + \psi + \theta_0, \tag{2.22}$$

where, in terms of the SLN nomenclature,

$$\xi = u^* = \alpha(t) - i\beta(t), \qquad \text{and} \quad \psi = -\tan^{-1}(\beta/\alpha).$$

Exactly as in SLN, Section III, we express α and β as linear superpositions of assumed independent, real solutions $I_1(t)$ and $I_2(t)$, satisfying equation (2.10), to obtain the general results (1.10) and (1.11) for arbitrary time dependence of $\omega_{\mathrm{L}}(t)$, and thus of the solenoid magnetic field $B_z(t)$.

III. Guiding-centre Form of the SLN Result for r

The result (1.10) for r is not in guiding-centre form, which is the important form for charged particle motion in magnetic fields that change adiabatically in space and time. Since the adiabatic invariants become exact integrals of the motion in the limit of infinitely slow variation of the magnetic field in space and time, examination of the conversion of the SLN result (1.10) for r to guiding-centre form for the case of constant solenoid magnetic field forms an interesting tractable preliminary to discussion of the temporal adiabatic case of interest here.

We consider two-dimensional motion in a plane normal to the z axis of the solenoid, and assume that $\boldsymbol{\rho}$ is the orbit radius vector rotating with angular velocity $\omega_{\mathrm{g}} = 2\omega_{\mathrm{L}}$ about a fixed point, specified by the tip of the guiding-centre position vector $\mathbf{g}$, so that

$$\mathbf{r} = \mathbf{g}+\boldsymbol{\rho}. \tag{3.1}$$

Then

$$r^2 = \mathbf{r}.\mathbf{r} = g^2+\rho^2+2\rho g\cos(2\omega_{\mathrm{L}}t-\alpha_0), \tag{3.2}$$

where α_0 is the constant angle between the magnitude g and the x axis, where we take $t = 0$.

In the constant magnetic field case, with ω_{L} independent of time, $I_1(t)$ and $I_2(t)$ are linearly independent, real solutions of equation (1.7); $I_1(t) = \cos\omega_{\mathrm{L}}t$ and $I_2(t) = \sin\omega_{\mathrm{L}}t$, say. Then from equation (1.10)

$$r^2 = \tfrac{1}{2}(|\mu|^2+|\lambda|^2)+\tfrac{1}{2}(|\mu|^2-|\lambda|^2)\cos 2\omega_{\mathrm{L}}t+|\mu||\lambda|\cos(\epsilon_1-\epsilon_2)\sin 2\omega_{\mathrm{L}}t. \tag{3.3}$$

By expansion of the cosine term in (3.2), and equation of coefficients of $\cos 2\omega_{\mathrm{L}}t$, $\sin 2\omega_{\mathrm{L}}t$, and time-independent terms with those in (3.3), we obtain

$$\rho^2+g^2 = \tfrac{1}{2}(|\mu|^2+|\lambda|^2), \tag{3.4}$$

$$2\rho g\cos\alpha_0 = \tfrac{1}{2}(|\mu|^2-|\lambda|^2), \tag{3.5}$$

and

$$2\rho g\sin\alpha_0 = |\mu||\lambda|\cos(\epsilon_1-\epsilon_2). \tag{3.6}$$

From the last two equations we readily obtain

$$\tan\alpha_0 = \frac{2|\mu||\lambda|\cos(\epsilon_1-\epsilon_2)}{|\mu|^2-|\lambda|^2} \tag{3.7}$$

and

$$2\rho g = +\tfrac{1}{2}\{(|\mu|^2-|\lambda|^2)^2+4|\mu|^2|\lambda|^2\cos^2(\epsilon_1-\epsilon_2)\}^{\frac{1}{2}} = K \text{ say}, \tag{3.8}$$

where evidently $K > 0$.

When $t = \alpha_0/2\omega_L$, equation (3.2) yields

$$\rho+g = +\{\tfrac{1}{2}(|\mu|^2+|\lambda|^2)+K\}^{\frac{1}{2}}, \tag{3.9}$$

using the results (3.4) and (3.8); and similarly, when $t = (\alpha_0+\pi)/2\omega_L$, we have

$$g-\rho = \pm\{\tfrac{1}{2}(|\mu|^2+|\lambda|^2)-K\}^{\frac{1}{2}}. \tag{3.10}$$

From equations (3.9) and (3.10) we obtain explicit forms for ρ and g in terms of SLN quantities, namely,

$$\rho = \tfrac{1}{2}[+\{\tfrac{1}{2}(|\mu|^2+|\lambda|^2)+K\}^{\frac{1}{2}}\mp\{\tfrac{1}{2}(|\mu|^2+|\lambda|^2)-K\}^{\frac{1}{2}}] \tag{3.11}$$

and

$$g = \tfrac{1}{2}[+\{\tfrac{1}{2}(|\mu|^2+|\lambda|^2)+K\}^{\frac{1}{2}}\pm\{\tfrac{1}{2}(|\mu|^2+|\lambda|^2)-K\}^{\frac{1}{2}}]. \tag{3.12}$$

Further, use of equations (3.4) and (3.8) in equation (3.2) gives

$$r^2 = \tfrac{1}{2}(|\mu|^2+|\lambda|^2)+K\cos(2\omega_L t-\alpha_0). \tag{3.13}$$

To gain familiarity with these results and an understanding of the physical meaning of the real constant C appearing in equation (1.4), we now briefly examine two special cases of charged particle motion in a constant magnetic field.

Case 1. Motion about the Solenoid Axis

As shown by Seymour (1963), the integration constant in equation (1.4) has to be adjusted to $C = -qB_z r^2/2m = \omega_L r^2$ for circular charged particle motion concentric with the solenoid axis. From the SLN equation (3.27),

$$C = |\mu||\lambda|\sin(\epsilon_1-\epsilon_2)\, W(I_1,I_2), \tag{3.14}$$

where $W(I_1,I_2)$ is the Wronskian determinant $\begin{vmatrix} I_1 & I_2 \\ \dot{I}_1 & \dot{I}_2 \end{vmatrix}$.

Since here $I_1 = \cos\omega_L t$, $I_2 = \sin\omega_L t$, $W(I_1,I_2) = \omega_L$, and (3.14) yields

$$\rho^2 = |\mu||\lambda|\sin(\epsilon_1-\epsilon_2), \tag{3.15}$$

because in this case $r = \rho$ and $g = 0$. Further, substitution of $g = 0$ in equations (3.4) and (3.5), and combination of the results gives

$$\rho = |\mu| = |\lambda|. \tag{3.16}$$

Hence equation (3.15) reduces to $\sin(\epsilon_1-\epsilon_2) = 1$, so that

$$\epsilon_1-\epsilon_2 = \tfrac{1}{2}\pi, \tag{3.17}$$

a result that is confirmed by putting $g = 0$ into equation (3.6).

Thus the vectors of lengths $|\mu|I_1$ and $|\lambda|I_2$, with constant angle $\epsilon_1-\epsilon_2$ between them, moving about the origin of coordinates at the angular velocity $\omega_L(t)$, as described in SLN at the end of Section III, are in this case

$$\left.\begin{aligned} |\mu|I_1 &= \rho\cos\omega_L t, \\ |\lambda|I_2 &= \rho\sin\omega_L t, \\ \text{with angle}\quad \epsilon_1-\epsilon_2 &= \tfrac{1}{2}\pi \quad \text{between them.} \end{aligned}\right\} \tag{3.18}$$

Substitution of these results in equation (1.10) simply confirms that $r = \rho$, while their use in equation (1.11) gives the form

$$\theta = \int_0^t \omega_L \, dt' - (\epsilon_1 - \omega_L t) + \theta_0,$$

which, in view of the constancy of ω_L here, reduces to

$$\theta = \omega_g t - \epsilon_1 + \theta_0, \tag{3.19}$$

where we have used equation (2.3) to eliminate ω_L. Assuming for convenience the usual condition that $\theta = 0$ at $t = 0$, we see from (3.19) that $\theta_0 = \epsilon_1$, so that in this case

$$\theta = \omega_g t = 2\omega_L t. \tag{3.20}$$

The manner in which the SLN solution represents the circular charged particle motion about the solenoid axis in this case now becomes clear. First we consider a plane normal to the z axis of the solenoid, in which a complex radius vector of length ρ rotates with constant angular velocity ω_L about a point corresponding to intersection of the z axis and the plane. Then $|\mu| I_1 = \rho \cos \omega_L t$ and $|\lambda| I_2 = \rho \sin \omega_L t$ as given by (3.18) are merely the real and imaginary components respectively of the complex radius, resolved onto appropriate fixed mutually perpendicular axes. However, this representation would give $\theta = \omega_L t$, rather than the $\theta = 2\omega_L t$ required by equation (3.20), and so to complete the picture we now have to spin the plane in which the above motion is taking place, at an angular velocity of ω_L, so that the mutually perpendicular components $\rho \cos \omega_L t$ and $\rho \sin \omega_L t$ of the radius vector are referred to axes spinning with angular velocity ω_L. The representation is now complete. Although the charged particle motion in this case is most simple, the above discussion gives considerable insight into the nature of the SLN solution.

Case 2. Grazing Orbits

This case corresponds to the condition $\rho = g$, so that the circular motion does not include the solenoid axis but in fact just grazes it. From SLN, equation (2.23), we obtain by multiplying through by r

$$v_\theta = r\dot{\theta} = \omega_L r + C/r, \tag{3.21}$$

so that, as pointed out by Seymour (1963, p. 441), for v_θ to remain finite at $r = 0$, we must impose the condition

$$C = 0. \tag{3.22}$$

However, equation (3.21) of SLN states that

$$W(\alpha, \beta) = \begin{vmatrix} \alpha & \beta \\ \dot{\alpha} & \dot{\beta} \end{vmatrix} = -C, \tag{3.23}$$

and hence, using the condition $C = 0$ given by equation (3.22) above, the Wronskian of α and β vanishes. The real and imaginary parts of the SLN quantity $u = \alpha + i\beta$ (c.f. equation (2.20), Section II, present paper) are now no longer linearly independent, contrary to the assumption in SLN. Application of the SLN results to this case of grazing orbits leads to inconsistencies, as would now be expected, but this is of no concern, for from equations (1.4) and (1.5) we merely have to solve

$$\ddot{r}+\omega_{\mathrm{L}}^{2} r=0 \tag{3.24}$$

and

$$\theta=\int_{0}^{t} \omega_{\mathrm{L}}\, \mathrm{d} t'+\theta_{0} \tag{3.25}$$

for constant ω_{L} and initial conditions appropriate to the particular grazing orbit geometry being considered (Seymour 1963, pp. 441–2, covers a typical case).

IV. Analysis of the Motion when the Solenoid Magnetic Field is Quasi-static

A guiding-centre review and a trajectory approximation published earlier (Seymour 1963, pp. 436–7, 442–3) gave, in the former case, the well-known result

$$g^{2} B_{z}=\text{constant} \tag{4.1}$$

in the nomenclature of the present paper, and in the latter case the different result

$$r^{2} B_{z}=\text{constant}. \tag{4.2}$$

In this section we analyse the adiabatic motion of a charged particle in a quasi-static field $B_{z}(t)$ and from the solution obtained reconcile the results (4.1) and (4.2), which, upon multiplication through by π in each case, are clearly of a flux-conserving nature.

Recalling that in Section III, Case 1, where $B_{z}(t)$ was time independent, $I_{1}=\cos \omega_{\mathrm{L}} t$ and $I_{2}=\sin \omega_{\mathrm{L}} t$, we might guess that for a quasi-static $B_{z}(t)$

$$I_{1}=\cos\left(\int \omega_{\mathrm{L}}(t)\, \mathrm{d} t\right), \tag{4.3}$$

and

$$I_{2}=\sin\left(\int \omega_{\mathrm{L}}(t)\, \mathrm{d} t\right). \tag{4.4}$$

Equation (1.7) implies that

$$\ddot{I}_{1}+\omega_{\mathrm{L}}^{2} I_{1}=0, \tag{4.5}$$

with a similar equation for I_{2}. The formation of, say, $\ddot{I}_{1}$ from the first of these intuitively-based solutions leads to

$$\ddot{I}_{1}+\omega_{\mathrm{L}}^{2} I_{1}=-\dot{\omega}_{\mathrm{L}} \sin\left(\int \omega_{\mathrm{L}}\, \mathrm{d} t\right), \tag{4.6}$$

which merely confirms that, when ω_{L} is time independent, $I_{1}=\cos \omega_{\mathrm{L}} t$ is the exact solution of equation (4.5). With reference to equation (4.4), similar remarks apply to I_{2}. To succeed in this case, we profitably assume the phase–amplitude form

$$u=A(t) \exp \mathrm{i} \phi(t), \tag{4.7}$$

in which the time functions $A(t)$ and $\phi(t)$ are real quantities, and substitute into equation (1.7) to obtain

$$(\ddot{A}+\omega_{\mathrm{L}}^{2} A-A \dot{\phi}^{2})+\mathrm{i}(A \ddot{\phi}+2 \dot{A} \dot{\phi})=0, \tag{4.8}$$

after cancellation of the non-zero factor $\exp \mathrm{i}\phi$. Equating the real and imaginary parts to zero, we have

$$\ddot{A}+\omega_{\mathrm{L}}^2 A-A\dot{\phi}^2 = 0 \tag{4.9}$$

and

$$A\ddot{\phi}+2\dot{A}\dot{\phi} = 0. \tag{4.10}$$

Equation (4.10) integrates immediately to yield

$$\dot{\phi}^{\frac{1}{2}} A = \text{constant}. \tag{4.11}$$

If we now assume the quantity $A(t)$ to be a slowly varying function of the time, so that $\ddot{A}$ is small compared with the other terms appearing in equation (4.9), we have approximately

$$\dot{\phi} = \pm\omega_{\mathrm{L}}, \tag{4.12}$$

so that

$$\phi = \pm\int \omega_{\mathrm{L}}(t)\, \mathrm{d}t. \tag{4.13}$$

The above derivation, which is one approach to the J.W.K.B. solution technique used in quantum mechanics, enables the quasi-static solution for u to be written in the form

$$u = (C_1/\omega_{\mathrm{L}}^{\frac{1}{2}})\exp\left(\mathrm{i}\int \omega_{\mathrm{L}}\, \mathrm{d}t\right)+(C_2/\omega_{\mathrm{L}}^{\frac{1}{2}})\exp\left(-\mathrm{i}\int \omega_{\mathrm{L}}\, \mathrm{d}t\right), \tag{4.14}$$

where we have used equations (4.11), (4.12), and (4.13) in equation (4.7), and $C_1 = |C_1| \exp \mathrm{i}\nu_1$ and $C_2 = |C_2| \exp \mathrm{i}\nu_2$ are arbitrary complex constants. Then, since

$$u^* = (C_1^*/\omega_{\mathrm{L}}^{\frac{1}{2}})\exp\left(-\mathrm{i}\int \omega_{\mathrm{L}}\, \mathrm{d}t\right)+(C_2^*/\omega_{\mathrm{L}}^{\frac{1}{2}})\exp\left(\mathrm{i}\int \omega_{\mathrm{L}}\, \mathrm{d}t\right), \tag{4.15}$$

we have from equation (1.8)

$$r^2 = \frac{C_1^2+C_2^2}{\omega_{\mathrm{L}}}+2\frac{|C_1||C_2|}{\omega_{\mathrm{L}}}\cos\left(2\int \omega_{\mathrm{L}}\, \mathrm{d}t -(\nu_2-\nu_1)\right). \tag{4.16}$$

Bearing in mind the result (3.2) for the case of the static magnetic field, we can write equation (4.16) in the form

$$r^2 = g^2(t)+\rho^2(t)+2\rho(t)\, g(t) \cos\left(2\int \omega_{\mathrm{L}}\, \mathrm{d}t -\alpha_0\right), \tag{4.17}$$

where

$$g^2(t)\, \omega_{\mathrm{L}} = C_1^2, \tag{4.18}$$

$$\rho^2(t)\, \omega_{\mathrm{L}} = C_2^2, \qquad \text{say,} \tag{4.19}$$

and

$$\alpha_0 = \nu_2-\nu_1. \tag{4.20}$$

In the adiabatic limit, when the solenoid magnetic field varies infinitely slowly with the time, the result (4.17) reduces precisely to equation (3.2), in which ρ, g, and ω_{L} are all time-independent quantities. In the quasi-static case, where equation (4.17) applies, we note from equations (1.6), (4.18), and (4.19) that

$$g^2 B_z = \text{constant}, \tag{4.21}$$

and

$$\rho^2 B_z = \text{constant}. \tag{4.22}$$

Further, by time-averaging equation (4.16) over a few gyroperiods, after multiplying through by $\omega_L(t)$, we obtain

$$r^2 B_z = \text{constant}. \tag{4.23}$$

The results (4.21), (4.22), and (4.23) constitute the flux-conserving forms of the adiabatic invariants in this case. Equation (4.21) confirms the usual drift-theory result given by equation (4.1); in first-order approximation we may use the result $\rho = mv_\perp/qB_z$ (see, for example, Spitzer 1962) to transform equation (4.22) to the form

$$\mu = \tfrac{1}{2}m(v_\perp)^2/B_z = \text{constant}, \tag{4.24}$$

where μ is the magnetic moment of the spiralling charged particle, so that equation (4.22) confirms the constancy of the transverse or perpendicular adiabatic invariant μ in the temporal case; finally, equation (4.23) confirms the result (4.2), obtained earlier by Seymour (1963, p. 443) from a non-drift analysis on the assumption that r varies as $r = r_0\exp(\pm at)$, with $a \ll \omega_L$. In concluding this section, it is of interest to note that the results (4.21) and (4.22) have been obtained by Rose and Clark (1961), who used a phase–real amplitude form for the quantity R defined with a complex amplitude by equation (2.8), Section II, of the present paper, and a method of successive approximations.

V. Quantitative Examination of the Adiabatic Approximation

In the previous section the assumed form of u did not reveal the explicit forms of $I_1(t)$ and $I_2(t)$ for the adiabatic case, and so we were unable to refine the initially chosen, unsuccessful forms given by (4.3) and (4.4). However, remembering that $C_1 = |C_1|\exp i\nu_1$, $C_2 = |C_2|\exp i\nu_2$, by replacing all exponential terms that appear in equation (4.14) with their Euler expansions, the form (1.12) is readily obtained, where

$$\mu = (|C_1|\cos\nu_1 + |C_2|\cos\nu_2) + i(|C_1|\sin\nu_1 + |C_2|\sin\nu_2), \tag{5.1}$$

$$\lambda = (-|C_1|\sin\nu_1 + |C_2|\sin\nu_2) + i(|C_1|\cos\nu_1 - |C_2|\cos\nu_2), \tag{5.2}$$

$$I_1 = \omega_L^{-\frac{1}{2}}\cos\Big(\int \omega_L\,dt\Big), \tag{5.3}$$

and

$$I_2 = \omega_L^{-\frac{1}{2}}\sin\Big(\int \omega_L\,dt\Big). \tag{5.4}$$

Using, say, equation (5.3), the formation of $\ddot{I}_1$ leads now to

$$\ddot{I}_1 + \omega_L^2 I_1 = \tfrac{1}{2}\omega_L^{3/2}\left\{\frac{3}{2}\left(\frac{\dot{\omega}_L}{\omega_L^2}\right)^2 - \frac{\ddot{\omega}_L}{\omega_L^3}\right\}\cos\left(\int \omega_L\,dt\right). \tag{5.5}$$

It is thus evident that the degree to which the solution (5.3) fits equation (4.5) depends significantly on the terms $\dot{\omega}_L/\omega_L^2$ and $\ddot{\omega}_L/\omega_L^3$ that appear inside the braces on the right-hand side of equation (5.5). In the adiabatic limit, where $B_z(t)$, and hence $\omega_L(t)$, are independent of the time, these terms vanish, equation (5.5) reduces to equation (4.5), and the $\cos\omega_L t$ form then assumed by equation (5.3) becomes its exact solution.

Away from the adiabatic limit, for the particular orbits that satisfy equation (4.12) initially, and for which $\dot{A}$ is slowly varying, of course, we may specify that the form of I_1 given by equation (5.3) for an $\omega_{\mathrm{L}}(t)$ that varies slowly with time must satisfy the form

$$\ddot{I}_1+\omega_{\mathrm{L}}^2 I_1 \approx 0, \tag{5.6}$$

which, from equation (5.5), is consistent with the conditions

$$\dot{\omega}_{\mathrm{L}}/\omega_{\mathrm{L}}^2 \ll 1, \qquad \ddot{\omega}_{\mathrm{L}}/\omega_{\mathrm{L}}^3 \ll 1. \tag{5.7}$$

To see what the conditions (5.7) mean in terms of the behaviour of $B_z(t)$ in time, we consider a Taylor series expansion of $B_z(t+\delta t)$, where for convenience we choose the short characteristic time increment $\delta t \sim \omega_{\mathrm{L}}^{-1}$. Then

$$B_z(t+\delta t) = B_z+\delta t\,\dot{B}_z+(\delta^2 t/2\,!)\ddot{B}_z+\dots. \tag{5.8}$$

Thus, with neglect of terms involving the third and higher time derivatives of B_z, we have

$$\frac{\delta B_z}{B_z} = \frac{\delta\omega_{\mathrm{L}}}{\omega_{\mathrm{L}}} = \frac{\dot{\omega}_{\mathrm{L}}}{\omega_{\mathrm{L}}^2}+\frac{\omega_{\mathrm{L}}}{2\omega_{\mathrm{L}}^3}, \tag{5.9}$$

where we have made use of equation (1.6).

From the result (5.9) we see immediately that the conditions (5.7) also ensure that the coefficient $\delta B_z/B_z \ll 1$, as required for an adiabatic magnetic field change with time. In summary, the inequalities (5.7) ensure that I_1 of (5.3) is a good approximate solution of equation (4.5), and that the temporal variation of the magnetic field $B_z(t)$ is adiabatic. Since we readily find that

$$\ddot{I}_2+\omega_{\mathrm{L}}^2 I_2 = \tfrac{1}{2}\omega_{\mathrm{L}}^{3/2}\left\{\frac{3}{2}\left(\frac{\dot{\omega}_{\mathrm{L}}}{\omega_{\mathrm{L}}^2}\right)^2-\frac{\ddot{\omega}_{\mathrm{L}}}{\omega_{\mathrm{L}}^3}\right\}\sin\left(\int \omega_{\mathrm{L}}\,\mathrm{d}t\right), \tag{5.10}$$

the above remarks on I_1 apply equally to I_2, so completing this examination of the adiabatic approximation.

VI. Conclusions

A satisfactorily self-consistent temporal adiabatic analysis of the motion of a charged particle in the quasi-static magnetic field of a solenoid has been developed by suitable approximation of the exact general solution presented in SLN, and the associated adiabatic invariants have been identified. The case of harmonic time variation of the solenoid magnetic field, which, as mentioned in SLN, Section VIII, was to have been associated with this case, has been made the topic of a separate paper now in preparation.

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VIII. References

Rose, D. J., and Clark, M., Jr. (1961).—"Plasmas and Controlled Fusion." pp. 213–15. (Wiley: New York.)

Seymour, P. W. (1963).—Motions of charged particles in plasmas. *Int. J. engng Sci.* **1**, 440.

Seymour, P. W., Leipnik, R. B., and Nicholson, A. F. (1965).—Charged particle motion in a time-dependent axially symmetric magnetic field. *Aust. J. Phys.* **18**, 553–65.

Spitzer, L., Jr. (1962).—"Physics of Fully Ionized Gases." 2nd Ed. p. 2. (Interscience: New York.)