FIRST-ORDER COUPLED WAVE EQUATIONS FOR PROPAGATION IN STRATIFIED COMPRESSIBLE PLASMAS

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Summary

This paper deals with wave propagation in planar stratified, continuously varying, compressible, non-magnetized electron plasmas. The waves are coupled electromagnetic and electron acoustic waves and are described by Maxwell's equations coupled to the linearized single-fluid equations of hydrodynamics. Some first-order coupled wave equations are obtained and are transformed into convenient forms. Coupling, power flow, and approximate solutions are then discussed.

I. Introduction

In much of the work on waves in a compressible plasma, the plasma has been taken to be homogeneous, or to consist of homogeneous regions separated by sharp boundaries. In a region of homogeneous non-magnetized plasma, electromagnetic and electron acoustic waves propagate independently. Coupling between them occurs at a boundary between regions if the electric vector has a component normal to the boundary. Coupling can occur also in a continuously varying inhomogeneous region.

Approximate studies of fields in continuously stratified compressible plasmas have been made by a number of authors (Fejer 1964; Hoh 1964; Parker, Nickel, and Gould 1964; Yadavalli 1965). Musal (1965) has obtained an exact second-order wave equation.

Felsen (1966) derived some first-order and second-order coupled wave equations for propagation in planar stratified non-magnetized compressible electron plasmas. In that work, gradients in the static pressure and in the electron collision frequency were not allowed for, but the equations were otherwise rigorous. In a subsequent letter (Burman 1966) it has been pointed out that alternative first-order equations can be obtained. In the present paper, a derivation of these alternative equations is given. The equations are then transformed into convenient forms.

II. Basic Equations

The model of the plasma taken here consists of electrons neutralized by an equal number of positive ions. The motion of positive ions in the electromagnetic field is neglected. The waves are described by Maxwell's equations coupled to the single-fluid equations of hydrodynamics. The amplitude of the waves is taken to be sufficiently small that linearized equations can be used. The plasma is inhomogeneous and compressible. The effect of an imposed magnetostatic field is not considered.

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A time factor $e^{i\omega t}$ is taken, where ω is the angular frequency of the fields and t is the time. The permeability of the plasma has the free-space value μ_0. Also, ϵ_0 is the permittivity of free space and e is the charge of an electron (a negative quantity). The electrons have number density N_0+N, N_0 being the equilibrium value. In equilibrium the ordered velocity of the electrons vanishes, the medium being stationary. Maxwell's equations are

$$\nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H} \tag{1}$$

and

$$\nabla \times \mathbf{H} = i\omega\epsilon_0 \mathbf{E} + N_0 e\mathbf{v}. \tag{2}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields and $\mathbf{v}$ is the ordered velocity of electrons in the wave. $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{v}$ are time-dependent quantities. The current density term has been linearized by neglecting $N\mathbf{v}$.

The pressure in the electron fluid is denoted by p_0+p, where p_0 is the equilibrium value. In an inhomogeneous fluid in the absence of static-body forces, it follows from hydrostatics that the gradient of the equilibrium pressure vanishes (Friedlander 1958). In the present case any static forces are neglected, so that $\nabla p_0 = 0$.

The linearized equation of conservation of momentum is

$$mN_0(i\omega+\nu)\mathbf{v} = eN_0\mathbf{E} - \nabla p. \tag{3}$$

Here m is the mass of an electron and ν is the collision frequency of electrons with heavy particles. The force on the electrons due to the magnetic field of the wave disappears in the linearization. Also it can be shown that for small disturbances (Friedlander 1958)

$$i\omega p + mN_0 u_0^2 \nabla . \mathbf{v} = 0. \tag{4}$$

The acoustic speed in the electron gas at rest, u_0, is defined by $mu_0^2 = \partial p_0/\partial N_0$.

III. First-order Coupled Wave Equations

The plasma is now taken to be planar stratified. Rectangular Cartesian coordinates (x, y, z) are introduced with N_0, ν, and u_0 functions of z only. The field quantities are all taken to be independent of y.

In an incompressible isotropic planar stratified plasma, the electromagnetic field can be expressed as the sum of two partial fields. These are referred to as horizontally and vertically polarized waves in which the electric and magnetic vectors respectively are parallel to the stratifications (Budden 1961).

When compressibility is considered, a vertically polarized electromagnetic wave is coupled to the electron acoustic wave. However, a horizontally polarized electromagnetic wave propagates as an independent mode. The physical reason for this is that an electric field parallel to the stratifications cannot impart any longitudinal motion to the electrons. Thus, horizontally polarized electromagnetic waves are unaffected by the compressibility and need not be considered here.

Consider the propagation of a wave consisting of a vertically polarized electromagnetic wave and an electron acoustic wave coupled together. The magnetic field of the wave is taken to have a y component H only. The electric field has

x and z components E_x and E_z only. Similarly, the electron velocity has x and z components v_x and v_z only. Such a wave could be excited by a line source of magnetic current parallel to the y axis.

The fields are functions of x and z. Fourier transforms of the fields, taken with respect to the x coordinate, are now introduced. The field component E_x has the Fourier transform $\bar{E}_x$ given by

$$E_x(x,z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \bar{E}_x(k_0 S, z)\exp(-\mathrm{i}k_0 Sx)\,\mathrm{d}(k_0 S)\,. \tag{5}$$

Here, $k_0 = \omega(\mu_0 \epsilon_0)^{\frac{1}{2}}$ and S is independent of the coordinates. Exactly similar expressions hold for all other field components. Equation (5) shows that for all components $\partial/\partial x = -\mathrm{i}k_0 S$. Also, $\partial/\partial y = 0$ for all components, and a prime will denote differentiation with respect to z.

The symbols X and U, commonly used in magnetoionic theory (Ratcliffe 1959; Budden 1961), are defined by $X = N_0 e^2/\epsilon_0 m\omega^2$ and $U = 1-\mathrm{i}(\nu/\omega)$. The quantities $C^2 = 1-S^2$, $c = (\mu_0 \epsilon_0)^{-\frac{1}{2}}$, and $\eta_0 = (\mu_0/\epsilon_0)^{\frac{1}{2}}$ will also be required below. Equations (1)–(4) are expressed in component form and Fourier transforms are taken. This gives

$$\mathrm{i}k_0 S\bar{E}_z+\bar{E}'_x = -\mathrm{i}\omega\mu_0 \bar{H}, \tag{6}$$

$$-\bar{H}' = \mathrm{i}\omega\epsilon_0 \bar{E}_x+N_0 e\bar{v}_x, \tag{7}$$

$$-\mathrm{i}k_0 S\bar{H} = \mathrm{i}\omega\epsilon_0 \bar{E}_z+N_0 e\bar{v}_z, \tag{8}$$

$$\mathrm{i}\omega m N_0 U\bar{v}_x = eN_0 \bar{E}_x+\mathrm{i}k_0 S\bar{p}, \tag{9}$$

$$\mathrm{i}\omega m N_0 U\bar{v}_z = eN_0 \bar{E}_z-\bar{p}', \tag{10}$$

and

$$\mathrm{i}\omega\bar{p}+mN_0 u_0^2(\bar{v}'_z-\mathrm{i}k_0 S\bar{v}_x) = 0. \tag{11}$$

The quantities $\bar{E}_z$ and $\bar{v}_x$ can be eliminated. The resulting equations can be written in matrix form as

$$\mathbf{e}' = -\mathrm{i}k_0 \mathbf{A}\mathbf{e}. \tag{12}$$

Here, $\mathbf{e}$ is the column vector

$$\mathbf{e} = \{\bar{E}_x, \bar{H}, \bar{v}_z, \bar{p}\} \tag{13}$$

and $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{vmatrix} 0 & \eta_0 C^2 & \mathrm{i}S(\omega m/e)X & 0 \\ \eta_0^{-1}(1-XU^{-1}) & 0 & 0 & -\mathrm{i}S(e/\omega m)U^{-1} \\ \mathrm{i}S(e/\omega m)U^{-1} & 0 & 0 & (k_0/\omega m N_0 U)(c^2 U u_0^{-2}-S^2) \\ 0 & -\mathrm{i}S(\omega m/e)X & (\omega m/k_0)N_0(U-X) & 0 \end{vmatrix}. \tag{14}$$

Equation (12) represents four first-order coupled wave equations.

The main difference between the present formulation and the previous one (Felsen 1966) is the absence of derivatives of the plasma properties in the matrix $\mathbf{A}$ used here. Also, the present work allows the electron collision frequency to vary with z. It was mentioned by Felsen (1966) that his matrix equation is similar to one describing electromagnetic wave propagation in an incompressible planar stratified magnetoplasma (Clemmow and Heading 1954; Budden 1961, p. 389). The matrices themselves differ. In the incompressible magnetoplasma problem, the matrix in the equation corresponding to equation (12) above does not contain derivatives of the plasma properties. In this sense, the present formulation of the compressible plasma problem is mathematically closer than Felsen's to the work described by Budden (1961, Ch. 18).

Coupled wave equations are of considerable importance in the theory of electromagnetic wave propagation in cold planar stratified media (Budden 1961). This is in part due to the fact that solutions can be obtained by successive approximations (Budden 1961, Ch. 18).

IV. Transformation of the Equations

Let the column vector $\mathbf{s}^{(i)}$ be an eigenvector of $\mathbf{A}$ and let q_i be the corresponding eigenvalue, where $i = 1, 2, 3$, or 4. These quantities satisfy

$$\mathbf{A}\,\mathbf{s}^{(i)} = q_i\,\mathbf{s}^{(i)}, \tag{15}$$

where the summation convention is not used. Let $\mathbf{S}$ be a matrix the ith column of which is $\mathbf{s}^{(i)}$. The matrix $\mathbf{S}$ will diagonalize $\mathbf{A}$, that is,

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathrm{diag}\{q_1, q_2, q_3, q_4\}, \tag{16}$$

as is easily verified. The column vector $\mathbf{f}$ is introduced by writing $\mathbf{e} = \mathbf{S}\mathbf{f}$. Then equation (12) gives

$$\mathbf{f}' + \mathrm{i}k_0\,\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{f} = -\mathbf{S}^{-1}\mathbf{S}'\mathbf{f}, \tag{17}$$

provided $\mathbf{S}$ is non-singular.

The method in the above paragraph has simply followed that described by Budden (1961, p. 398). It now remains to find the eigenvectors and eigenvalues of the particular matrix used here, namely $\mathbf{A}$.

It is found that the eigenvectors of $\mathbf{A}$ can be written

$$\mathbf{s}^{(1)} = \{a_1, 1, b, 0\}\phi_1, \tag{18}$$

$$\mathbf{s}^{(2)} = \{a_2, 1, b, 0\}\phi_2, \tag{19}$$

$$\mathbf{s}^{(3)} = \{b, 0, c_3, 1\}\phi_3, \tag{20}$$

and

$$\mathbf{s}^{(4)} = \{b, 0, c_4, 1\}\phi_4. \tag{21}$$

In these expressions

$$a_1 = \eta_0 q_1/(1-XU^{-1}), \qquad a_2 = \eta_0 q_2/(1-XU^{-1}), \tag{22}$$

$$b = \mathrm{i}S\frac{e}{\omega m}\frac{\eta_0}{U-X}, \tag{23}$$

and

$$c_3 = \frac{k_0}{\omega m}\frac{q_3}{N_0(U-X)}, \qquad c_4 = \frac{k_0}{\omega m}\frac{q_4}{N_0(U-X)}. \tag{24}$$

The eigenvalues q_1 and q_2 are the two values of q^{em} given by

$$(q^{em})^2 = 1-XU^{-1}-S^2. \tag{25}$$

Similarly, q_3 and q_4 are the two values of q^{ea} given by

$$(q^{ea})^2 = (c^2/u_0^2)(U-X)-S^2. \tag{26}$$

Thus $q_2 = -q_1$ and $q_4 = -q_3$. Hence $a_1 = -a_2$ and $c_3 = -c_4$. The quantities a and c are now defined by $a_1 = -a_2 = a$ and $c_3 = -c_4 = c$ respectively.

Also ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are arbitrary functions of z. Equations (18)–(26) are easily verified by substituting them into equations (15). Equations (15) determine only the ratios of the elements of each eigenvector. This accounts for the presence of the arbitrary functions ϕ_i.

When the plasma is homogeneous, $\mathbf{S}' = 0$ and equation (17) reduces to four first-order uncoupled differential equations. Then four independent waves can exist in the plasma for a given S. These are two independent electromagnetic waves, travelling in opposite directions with respect to z, and two independent electron acoustic waves, also travelling in opposite directions with respect to z. Under various circumstances, some of these fields will be evanescent disturbances rather than propagating waves.

Consider a theoretical homogeneous plasma with properties the same as those of the actual plasma at the level considered. The four independent waves in the theoretical plasma may be referred to as the characteristic waves of the actual plasma at the level considered (Budden 1961, Ch. 18). Solving equation (17) for a homogeneous plasma gives, using equation (16),

$$f_i \propto \exp(-\mathrm{i}k_0 q_i z), \tag{27a}$$

where f_i is the ith component of $\mathbf{f}$. Suppose that f_i is the only field present, that is, $f_i \neq 0$ but $f_j = 0$ for $j \neq i$. In this case, $\mathbf{e} = \mathbf{s}^{(i)} f_i$. Then equation (27a) shows that

$$\mathbf{e} \propto \mathbf{s}^{(i)} \exp(-\mathrm{i}k_0 q_i z), \tag{27b}$$

which gives the field components of the ith characteristic wave at the level z.

Thus, the vector $\mathbf{e}$ of the ith characteristic wave is proportional to the ith eigenvector of $\mathbf{A}$. Also, the propagation constant with respect to z of the ith characteristic wave is the ith eigenvalue of $\mathbf{A}$. These considerations correspond closely to those in the incompressible magnetoplasma problem (Budden 1961, Ch. 18).

Consider the four characteristic waves. Equations (18) and (19) for a homogeneous medium represent electromagnetic waves; these waves possess all the magnetic field present, but there is no pressure deviation from the mean. The propagation constants with respect to z are the two roots of equation (25). Equations (20) and (21) for a homogeneous medium represent electron acoustic waves; these waves possess no magnetic field, but involve a pressure deviation from the mean. The propagation constants with respect to z are the two roots of equation (26).

By using equations (18)–(21), the matrix $\mathbf{S}$ can be written

$$\mathbf{S} = \begin{vmatrix} a\phi_1 & -a\phi_2 & b\phi_3 & b\phi_4 \\ \phi_1 & \phi_2 & 0 & 0 \\ b\phi_1 & b\phi_2 & c\phi_3 & -c\phi_4 \\ 0 & 0 & \phi_3 & \phi_4 \end{vmatrix}. \tag{28}$$

It is easily shown that

$$\det \mathbf{S} = 4\phi_1\phi_2\phi_3\phi_4 ac.$$

Thus, the singularities of $\mathbf{S}$ will depend on the choices made of the functions ϕ_i. If the ϕ_i are taken to be finite and independent of z, then $\mathbf{S}$ is singular where its eigenvalues become equal to zero. It is found that

$$2\mathbf{S}^{-1} = \begin{vmatrix} \dfrac{1}{a\phi_1} & \dfrac{1}{\phi_1} & 0 & \dfrac{-b}{a\phi_1} \\ \dfrac{-1}{a\phi_2} & \dfrac{1}{\phi_2} & 0 & \dfrac{b}{a\phi_2} \\ 0 & \dfrac{-b}{c\phi_3} & \dfrac{1}{c\phi_3} & \dfrac{1}{\phi_3} \\ 0 & \dfrac{b}{c\phi_4} & \dfrac{-1}{c\phi_4} & \dfrac{1}{\phi_4} \end{vmatrix} \tag{29}$$

and hence

$$2\mathbf{S}^{-1}\mathbf{S}' = \begin{vmatrix} \dfrac{2a\phi_1'+a'\phi_1}{a\phi_1} & \dfrac{-a'\phi_2}{a\phi_1} & \dfrac{b'\phi_3}{a\phi_1} & \dfrac{b'\phi_4}{a\phi_1} \\ \dfrac{-a'\phi_1}{a\phi_2} & \dfrac{2a\phi_2'+a'\phi_2}{a\phi_2} & \dfrac{-b'\phi_3}{a\phi_2} & \dfrac{-b'\phi_4}{a\phi_2} \\ \dfrac{b'\phi_1}{c\phi_3} & \dfrac{b'\phi_2}{c\phi_3} & \dfrac{2c\phi_3'+c'\phi_3}{c\phi_3} & \dfrac{-c'\phi_4}{c\phi_3} \\ \dfrac{-b'\phi_1}{c\phi_4} & \dfrac{-b'\phi_2}{c\phi_4} & \dfrac{-c'\phi_3}{c\phi_4} & \dfrac{2c\phi_4'+c'\phi_4}{c\phi_4} \end{vmatrix}. \tag{30}$$

The equation $\mathbf{e} = \mathbf{S}\mathbf{f}$ gives $\mathbf{f} = \mathbf{S}^{-1}\mathbf{e}$ provided $\mathbf{S}$ is non-singular. Hence,

$$\left.\begin{aligned} f_1 &= (\bar{E}_x+a\bar{H}-b\bar{p})/2a\phi_1, \\ f_2 &= (-\bar{E}_x+a\bar{H}+b\bar{p})/2a\phi_2, \\ f_3 &= (-b\bar{H}+\bar{v}_z+c\bar{p})/2c\phi_3, \\ \text{and} \qquad f_4 &= (b\bar{H}-\bar{v}_z+c\bar{p})/2c\phi_4. \end{aligned}\right\} \tag{31}$$

Consider waves propagating purely in the z direction in the stratified medium. Then the fields are independent of x and $S = 0$. From equation (23), $b = 0$. Hence, equations (17) and (30) show that the fields f_1 and f_2 are not coupled to the fields f_3 and f_4. Furthermore, equations (31) with $b = 0$ show that f_1 and f_2 possess a magnetic field but no pressure deviation from the mean, while the contrary holds for f_3 and f_4. Thus f_1 and f_2 here represent electromagnetic waves while f_3 and f_4 represent electron acoustic waves. For propagation purely in the z direction, there is no coupling between electromagnetic and electron acoustic waves.

Also, equation (23) shows that b vanishes if the charge on the electron is put equal to zero. Then there is no coupling between electromagnetic waves and acoustic waves in the electron gas. The latter waves then possess no electric field, as well as no magnetic field, and are purely mechanical in nature.

V. A Particular Form of the Equations

Consider the matrix equation (17). The ith component equation contains only the ith component of $\mathbf{f}$ on the left-hand side. In general it contains all components of $\mathbf{f}$ on the right-hand side. In the ith equation, the coupling terms can be defined to be the terms involving f_j for $j \neq i$. If the diagonal components of $\mathbf{S}^{-1}\mathbf{S}'$ were to vanish, the right-hand side of equation (17) would contain coupling terms only. When the coupling terms are neglected, each f_i represents an independent field. When coupling terms are allowed for, the coupling terms show how each field f_i is coupled to the remaining f_j $(j \neq i)$.

The ϕ_i are now taken to satisfy

$$\left.\begin{aligned} 2a\phi_i'+a'\phi_i &= 0 \qquad \text{for } i = 1,\ 2 \\ \text{and} \qquad 2c\phi_i'+c'\phi_i &= 0 \qquad \text{for } i = 3,\ 4. \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} \text{Hence} \qquad \phi_i &= B_i\, a^{-\frac{1}{2}} \qquad \text{for } i = 1,\ 2 \\ \text{and} \qquad \phi_i &= B_i\, c^{-\frac{1}{2}} \qquad \text{for } i = 3,\ 4, \end{aligned}\right\} \tag{33}$$

where the B_i are arbitrary constants, independent of z. Equation (30) shows that the diagonal elements of $\mathbf{S}^{-1}\mathbf{S}'$ now vanish.

For simplicity, the B_i are all taken to be unity. Then evaluating $\mathbf{S}^{-1}\mathbf{S}'$ and writing out the matrix equation (17) in full gives

$$\left.\begin{aligned} f_1'+ik_0 q_1 f_1 &= (a'/2a)f_2-\tfrac{1}{2}b'(ac)^{-\frac{1}{2}}(f_3+f_4), \\ f_2'+ik_0 q_2 f_2 &= (a'/2a)f_1+\tfrac{1}{2}b'(ac)^{-\frac{1}{2}}(f_3+f_4), \\ f_3'+ik_0 q_3 f_3 &= -\tfrac{1}{2}b'(ac)^{-\frac{1}{2}}(f_1+f_2)+(c'/2c)f_4, \\ \text{and} \qquad f_4'+ik_0 q_4 f_4 &= \tfrac{1}{2}b'(ac)^{-\frac{1}{2}}(f_1+f_2)+(c'/2c)f_3. \end{aligned}\right\} \tag{34}$$

These are coupled wave equations with the right-hand sides containing coupling terms only.

The term "coupling" refers to the process in which energy is transferred between two waves that are independent when coupling is neglected. In the region of coupling itself, the wave cannot be separated into the two independent parts. A special case of coupling is reflection. Then a wave travelling in one direction with respect to z is coupled to a wave of the same type propagating in the other direction. In the region of reflection itself, a wave cannot be separated into oppositely travelling parts. Away from a reflection region such separation can be made.

Consider coupling between the components of $\mathbf{f}$ as demonstrated by equations (34). Coupling between the first pair (f_1 and f_2) and the last pair (f_3 and f_4) occurs wherever b' is non-zero. Coupling between f_1 and f_2 occurs wherever a' is non-zero. Coupling between f_3 and f_4 occurs wherever c' is non-zero. The quantities a, b, and c depend on z through N_0, ν, and u_0.

Terms in the right-hand sides of equations (34) become large in the region where $a = 0$ or $c = 0$, that is, coupling is important in regions near zeros of q^{em} or q^{ea}. Zeros will occur, in general, for complex values of z. Their effect on the propagation will be most significant when they are on or near the real z axis.

The terms involving coupling between f_1 and f_2 become large near zeros of a. Hence a field f_1 will generate a field f_2 when passing near a zero of q^{em}. Also a field f_2 will generate a field f_1 near such a zero. Similarly, f_3 and f_4 are coupled near zeros of q^{ea}. Terms involving coupling between the first and last pairs become large near zeros of both q^{em} and q^{ea}.

The fields f_1 and f_2 are associated with the eigenvalues q_1 and q_2 $(= -q_1)$, which are roots of equation (25). When there is no coupling, these waves reduce to electromagnetic waves propagating in opposite directions with respect to z, or to evanescent electromagnetic disturbances. Similar remarks apply to the fields f_3 and f_4, which are electron acoustic in nature when coupling is neglected.

Suppose that $b'(ac)^{-\frac{1}{2}}$ is small. Then coupling between the first pair (f_1 and f_2) and the last pair (f_3 and f_4) is small. Thus f_1 and f_2 represent a quasi-electromagnetic type of wave while f_3 and f_4 represent a quasi-electron acoustic mode. Where a'/a is small, f_1 and f_2 represent oppositely propagating waves (when they are not evanescent). Reflection of a quasi-electromagnetic mode will occur when a'/a is significant. Similarly, reflection of a quasi-electron acoustic mode will occur when c'/c is significant. When $b'(ac)^{-\frac{1}{2}}$ is not negligible, the separation into quasi-electromagnetic and quasi-electron acoustic modes is not physically meaningful.

It has been mentioned by Budden (1961, p. 418) that care should be taken in the use of first-order wave equations (for the cold magnetoplasma problem) in a coupling region. Thus it will be of interest to investigate the positions of coupling regions by using second-order differential equations.

VI. Power Flow

The time-averaged power flux in a compressible plasma is given by $\mathbf{P}$, where (Seshadri 1963)

$$\mathbf{P} = \tfrac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^* + p\mathbf{v}^*). \tag{35a}$$

A particular value of the Fourier transform variable S corresponds to a particular wave in the spectrum appearing in equation (5). For any such wave, with S real,

$$\mathbf{P} = \tfrac{1}{2}\mathrm{Re}(\bar{\mathbf{E}} \times \bar{\mathbf{H}}^* + \bar{p}\bar{\mathbf{v}}^*), \tag{35b}$$

where the barred quantities are defined as in equation (5). In the present problem this vector has x and z components P_x and P_z only, and these will now be calculated.

The equation $\mathbf{e} = \mathbf{S}\mathbf{f}$, with the choice made of the ϕ_i in Section V, gives

$$\left.\begin{aligned} \bar{E}_x &= a^{\frac{1}{2}}(f_1-f_2)+bc^{-\frac{1}{2}}(f_3+f_4), \\ \bar{H} &= a^{-\frac{1}{2}}(f_1+f_2), \\ \bar{v}_z &= ba^{-\frac{1}{2}}(f_1+f_2)+c^{\frac{1}{2}}(f_3-f_4), \\ \text{and}\qquad \bar{p} &= c^{-\frac{1}{2}}(f_3+f_4). \end{aligned}\right\} \tag{36}$$

The remaining field components, $\bar{E}_z$ and $\bar{v}_x$, are given in terms of the elements of $\mathbf{e}$ by equations (8) and (9). Those equations give for the x component of $(\bar{E} \times \bar{H}^* + \bar{p}\bar{v}^*)$

$$\frac{k_0 S}{\omega\epsilon_0}\bar{H}\bar{H}^* - \frac{\mathrm{i}N_0 e}{\omega\epsilon_0}\bar{v}_z\bar{H}^* + \frac{\mathrm{i}e}{\omega m U^*}\bar{p}\bar{E}_x^* + \frac{k_0 S}{\omega m N_0 U^*}\bar{p}\bar{p}^*. \tag{37}$$

Equations (36) show that this is equal to

$$\frac{k_0 S - \mathrm{i}N_0 eb}{\omega\epsilon_0}\frac{1}{|a|}|f_1+f_2|^2 - \frac{\mathrm{i}N_0 e}{\omega\epsilon_0}\left(\frac{c}{a^*}\right)^{\frac{1}{2}}(f_1+f_2)^*(f_3-f_4)$$
$$+\frac{\mathrm{i}e}{\omega m U^*}\left(\frac{a^*}{c}\right)^{\frac{1}{2}}(f_1-f_2)^*(f_3+f_4) + \frac{k_0 S + \mathrm{i}N_0 eb^*}{\omega m N_0 U^*}\frac{1}{|c|}|f_3+f_4|^2 \tag{38}$$

and that the z component of the same vector is

$$(a/a^*)^{\frac{1}{2}}(f_1-f_2)(f_1+f_2)^* + (b+b^*)(a^*c)^{-\frac{1}{2}}(f_1+f_2)^*(f_3+f_4) + (c^*/c)^{\frac{1}{2}}(f_3+f_4)(f_3-f_4)^*. \tag{39}$$

In the remainder of this section it will be supposed that the medium is loss-free. Then a and c are real or purely imaginary, while b is purely imaginary. On using equations (22) and (24) it is found that

$$\begin{aligned} 2P_x = {} & |1-X|\left(\frac{N_0 e\beta}{k_0}+S\right)\left(\left|\frac{f_1+f_2}{q_1^{\frac{1}{2}}}\right|^2 + \left|\frac{f_3+f_4}{q_3^{\frac{1}{2}}}\right|^2\right) \\ & + X^{\frac{1}{2}}\,\mathrm{Im}[\{(q_3/q_1^*)^{\frac{1}{2}} - (q_1^*/q_3)^{\frac{1}{2}}\}(f_1^* f_3 - f_2^* f_4)] \\ & + X^{\frac{1}{2}}\,\mathrm{Im}[\{(q_3/q_1^*)^{\frac{1}{2}} + (q_1^*/q_3)^{\frac{1}{2}}\}(f_2^* f_3 - f_1^* f_4)], \end{aligned} \tag{40}$$

where $b = \mathrm{i}\beta$, β being real. Taking positive square roots, expression (39) gives

$$2P_z = \epsilon_1(|f_1|^2 - |f_2|^2) - 2\epsilon_2\,\mathrm{Im}(f_1 f_2^*) + \epsilon_3(|f_3|^2 - |f_4|^2) + 2\epsilon_4\,\mathrm{Im}(f_3 f_4^*). \tag{41}$$

Here, $\epsilon_1 = 1$ and 0 when a is real and imaginary respectively, and vice versa for ϵ_2. Similar definitions apply to ϵ_3 and ϵ_4, but with respect to c. It is interesting to note that in equation (41) there are no "cross-terms" between the first pair and the last pair of f_1, f_2, f_3, and f_4.

VII. Approximate Solutions

Equations (34) are in a very convenient form for obtaining WKB-type expressions for the fields. WKB solutions for the fields in an inhomogeneous medium are approximate solutions that neglect any coupling, so that the various characteristic modes propagate independently. When coupling terms are neglected, each of the equations (34) can be written

$$f_i' + \mathrm{i}k_0 q_i f_i \simeq 0. \tag{42}$$

Thus, the WKB solutions for the fields are

$$f_i \simeq \exp\left(-\mathrm{i}k_0 \int^z q_i \,\mathrm{d}z\right), \tag{43}$$

where the lower limit has been left unspecified. These results are very similar to those given by Budden (1961, Ch. 18) in a different problem.

Suppose that only the ith field is present. This field will propagate independently of the others to within the accuracy of the WKB method. WKB expressions for the components of $\mathbf{e}$ can be obtained by substituting equation (43) and $f_j = 0$ for $j \neq i$ into equations (36). The vector $\mathbf{e}$ is now equal to $\mathbf{s}^{(i)} f_i$, so the WKB approximation is

$$\mathbf{e} \simeq \mathbf{s}^{(i)} \exp\left(-\mathrm{i}k_0 \int^z q_i \,\mathrm{d}z\right). \tag{44}$$

Suppose that the medium is loss-free, so that the q_i are real or purely imaginary, and consider equation (41) for P_z. It is seen that, when the WKB expression for a particular f_i is used, P_z is independent of z for q_i real, while $P_z = 0$ for q_i purely imaginary; that is, the power flow in the z direction is independent of z for propagating waves and is zero for evanescent disturbances.

In the incompressible magnetoplasma problem, Budden (1961, Ch. 19) has described the use of a method of successive approximations in treating first-order equations. The method can easily be applied to the present equations, as will now be shown. The positive and negative z directions will now be referred to as the upward and downward directions. Suppose that there exists a coupling region above and below which the WKB approximations are applicable. The WKB solutions for f_1 and f_3 are taken to represent upward travelling waves. An upward travelling electromagnetic wave enters the coupling region. Thus the zero-order approximation is taken to be

$$f_1 \simeq \exp\left(-\mathrm{i}k_0 \int_0^z q_1 \,\mathrm{d}z\right), \qquad f_2 = f_3 = f_4 = 0. \tag{45}$$

This expression is substituted into the right-hand sides of equations (34). The first equation, containing f_1, is unaffected; the others become

$$\left.\begin{aligned}
f_2' + \mathrm{i}k_0 q_2 f_2 &= (a'/2a)\exp\left(-\mathrm{i}k_0 \int_0^z q_1 \,\mathrm{d}z\right),\\
f_3' + \mathrm{i}k_0 q_3 f_3 &= -\tfrac{1}{2}b'(ac)^{-\frac{1}{2}} \exp\left(-\mathrm{i}k_0 \int_0^z q_1 \,\mathrm{d}z\right),\\
\text{and} \qquad f_4' + \mathrm{i}k_0 q_4 f_4 &= \tfrac{1}{2}b'(ac)^{-\frac{1}{2}} \exp\left(-\mathrm{i}k_0 \int_0^z q_1 \,dz\right).
\end{aligned}\right\} \tag{46}$$

These equations are easily solved to give

$$\left.\begin{aligned} f_2 &\simeq \exp\left(-ik_0\int_0^z q_2\,\mathrm{d}z\right)\times\int_z^{\alpha}(a'/2a)\exp\left(ik_0\int_0^z(q_2-q_1)\,\mathrm{d}z\right)\mathrm{d}z,\\ f_3 &\simeq \exp\left(-ik_0\int_0^z q_3\,\mathrm{d}z\right)\times\int_{\beta}^{z}-\tfrac{1}{2}b'(ac)^{-\frac{1}{2}}\exp\left(ik_0\int_0^z(q_3-q_1)\,\mathrm{d}z\right)\mathrm{d}z,\\ \text{and}\quad f_4 &\simeq \exp\left(-ik_0\int_0^z q_4\,\mathrm{d}z\right)\times\int_z^{\gamma}\tfrac{1}{2}b'(ac)^{-\frac{1}{2}}\exp\left(ik_0\int_0^z(q_4-q_1)\,\mathrm{d}z\right)\mathrm{d}z. \end{aligned}\right\} \quad (47)$$

The limits α, β, and γ must be chosen to satisfy the conditions above and below the coupling region. Thus there should be no downward travelling waves above the coupling region ($f_2 = 0 = f_4$ there) and no upward travelling electron acoustic wave below that region ($f_3 = 0$ there). Hence α and γ can be taken to be anywhere above the coupling region and β anywhere below.

Equations (47) give, to a first approximation, the fields generated by the incident wave f_1. In particular, below the coupling region f_2 represents the reflected electromagnetic wave. Also, the coupling process has generated transmitted and reflected electron acoustic waves, given by f_3 above the coupling region and by f_4 below it respectively. Other results can be obtained by taking an electron acoustic wave incident on the coupling region. It is seen that the approximate methods discussed in this section correspond very closely to those used by Budden (1961, Chs. 18 and 19) in a different problem.

VIII. Concluding Remarks

The purpose of this paper has been, essentially, to apply the methods introduced by Clemmow and Heading (1954) to a problem that differs from the one considered by those authors. Both their paper and the present one concern propagation in stratified plasmas. The difference is that the work of Clemmow and Heading (1954) allowed for the effect of a magnetostatic field but not for the compressibility, while the reverse holds for the present paper.

The problem of allowing for both a magnetostatic field and compressibility forms a suitable topic for further investigation. Also, the presence of an electrostatic or gravitational field would give a non-zero gradient to the static pressure. The effect of such a gradient is another topic for future study.

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