ROTATING POLYTROPES IN THE POST-NEWTONIAN APPROXIMATION*

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It has been suggested by Fowler (1966) that rotation would make a star more stable, and Durney and Roxburgh (1967) have arrived at the same result, but so far the problem has not been treated without assuming that the star retains its shape (which it does not), and the post-Newtonian rotation terms have not been included. As a first step the equilibrium problem must be solved, and in this case the star is assumed to be a polytrope.

The field equations in the post-Newtonian approximation have been given by Chandrasekhar (1965), and for a uniformly rotating star with axial symmetry his equation (41) becomes

$$\nabla^2\Phi = -4\pi G(3p+2\rho U+2\rho\Omega^2R^2\cos^2\phi+\rho\Pi), \tag{1}‡$$

where

$$g_{00} = 1-\frac{2U}{c^2}+\frac{1}{c^4}(2U^2-2\Phi)+O(c^{-6})$$

and U is the Newtonian potential, Ω the angular velocity of the star, and ϕ the latitude. Chandrasekhar's equation (45) becomes

$$\nabla^2\psi^{(3)}-\frac{\psi^{(3)}}{R^2\cos^2\phi} = -8\pi GR\cos\phi\Omega\rho, \tag{2}$$

where

$$g_{03} = \frac{2\psi^{(3)}}{c^3}+O(c^{-5}).$$

The corresponding term in the metric is

$$\frac{4\psi^{(3)}}{c^2}R\cos\theta\,\mathrm{d}\phi\,\mathrm{d}t.$$

The post-Newtonian field equations are not, in general, soluble for any particular order of approximation in $1/c^2$. To obtain the extra equation required one must either use some of the field equations of the next order of approximation or, more simply, the conservation laws.

Chandrasekhar's equation (67) becomes

$$\begin{aligned}0 = &-P_R+\rho U_R+\rho R\cos^2\phi\Omega^2\\&+\frac{1}{c^2}\{2R^2\cos^2\phi\Omega^2\rho U_R+\rho\Phi_r+PU_R-2R\cos\phi\rho\Omega\psi_R^{(3)}\\&\quad+4\rho R\cos^2\phi\Omega^2U-2\rho\cos\phi\Omega\psi^{(3)}+R\cos^2\phi\Omega^2P\\&\quad+\rho\Pi(U_R+R\Omega^2\cos^2\phi)-P_R^{(2)}\},\end{aligned} \tag{3}$$

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‡ Φ here has twice the value of Chandrasekhar's Φ.

where the pressure $= P+P^{(2)}/c^2$, P being the Newtonian value. The terms not in $1/c^2$ give the Newtonian equation.

As the star is a polytrope

$$P = K\rho^{1+1/n}, \tag{4}$$

and we put

$$\rho = \lambda\Theta^{*n}, \qquad P = \lambda^{1+1/n}\, K\Theta^{*n+1}, \tag{5}$$

where

$$\Theta^* = \Theta(1+\mu/c^2), \tag{6}$$

Θ being the Newtonian value as in Chandrasekhar (1933) except that A_2 has a different value. Then,

$$\Pi = n\lambda\Theta^n\mu, \qquad P^{(2)} = (n+1)K\lambda^{1+1/n}\,\Theta^{n+1}\mu. \tag{7}$$

Now, using the Newtonian equation of (3) we find that

$$\frac{1}{\rho}\{\rho\Pi(U_R + R\Omega^2\cos^2\phi) - P^{(2)}_R\} = -R_0(\mu\Theta)',$$

where $R_0 = (n+1)\lambda^{1/n}\,K$ and the prime denotes differentiation with respect to R. We can integrate the relativistic part of (3) to get

$$\Phi = 2\psi^{(3)}R\Omega\cos\phi + R_0\,\Theta\left\{\mu - \frac{R_0\,\Theta}{2(n+1)}\right\} - 2R^2\cos^2\phi\Omega^2U + g(\phi). \tag{8}$$

Corresponding to (3) there is a similar conservation equation for ϕ, and its significance is that $g(\phi)$ is constant, call it c_0.

Putting $\psi^{(3)} = \Omega\,\psi(R)\cos\phi$ we obtain

$$\psi_{RR} + \frac{2\psi_R}{R} - \frac{2\psi}{R^2} = -8\pi G\rho R. \tag{9}$$

In vacuo $\psi \propto 1/R^2$ as $\psi \to 0$ as $R \to \infty$, and so

$$\psi(\xi) = \frac{8\pi GA^3}{3}\left(\xi\int_\xi^{\xi_1}\rho x\,\mathrm{d}x + \frac{1}{\xi^2}\int_0^\xi \rho x^4\,\mathrm{d}x\right), \tag{10}$$

where $R = A\xi$ and ξ_1 is the first zero of the Emden equation

$$A = \left(\frac{R_0}{4\pi G\lambda}\right)^{\frac{1}{2}}. \tag{11}$$

Since to the approximation required

$$\rho = \lambda\theta^n \text{ and } \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n,$$

these integrals can be simplified to

$$\psi(\xi) = \tfrac{8}{3}\pi GA^3\lambda f(\xi), \tag{12}$$

where

$$f(\xi) = \xi\{3\theta(\xi) - \xi_1\,\theta'(\xi_1)\} - \frac{6}{\xi^2}\int_0^\xi \xi^2\theta\,\mathrm{d}\xi; \tag{13}$$

μ can be eliminated from equations (1), (7), and (8) to give

$$\nabla_\xi^2 \Phi^* + n\Theta^{n-1}\Phi^* = \frac{2nv}{3}\xi f(\xi)\,\theta^{n-1}\cos^2\phi - \frac{5(n+2)}{2(n+1)}\Theta^{n+1} - \frac{(2n+1)}{2}v\xi^2\cos^2\phi\,\theta^n, \quad (14)$$

where $$\Phi^* = \frac{\Phi - c_0}{R_0^2}, \qquad v = \frac{\Omega^2}{2\pi G\lambda}.$$

U is the same as V in equation (32) of Chandrasekhar (1933), and the expression given there for V has been substituted for U.

Put

$$\Phi^* = \Phi^{(11)}(\xi) + v\{\Phi^{(12)}(\xi) + \Phi^{(2)}(\xi)\cos^2\phi\},$$

then

$$\Phi^{(11)}_{\xi\xi} + \frac{2}{\xi}\Phi^{(11)}_{\xi} + n\theta^{n-1}\Phi^{(11)} = -\frac{5(n+2)}{2(n+1)}\theta^{n+1}, \quad (15)$$

$$\Phi^{(12)}_{\xi\xi} + \frac{2}{\xi}\Phi^{(12)}_{\xi} + \frac{4\Phi^{(2)}}{\xi^2} + n\theta^{n-1}\Phi^{(12)} + n(n-1)\theta^{n-2}\Phi^{(11)}(\psi_0 + A_2\psi_2)$$
$$= \frac{5(n+2)}{2}\theta^n(\psi_0 + A_2\psi_3), \quad (16)$$

$$\Phi^{(2)}_{\xi\xi} + \frac{2}{\xi}\Phi^{(2)}_{\xi} + (n\theta^{n-1} - 6\xi^{-2})\Phi^{(2)} - \frac{3n(n-1)\theta^{n-2}A_2\psi_2\Phi^{(11)}}{2}$$
$$= \frac{2n\xi}{3}f(\xi)\,\theta^{n-1} + \frac{\theta^n}{2}\;\frac{15(n+2)A_2\psi_2}{2} - (2n+1)\xi^2, \quad (17)$$

where A_2, ψ_0, and ψ_2 are as in Chandrasekhar (1933).

To find the boundary conditions we note that, at $\Theta = 0$, equation (8) gives

$$\frac{\Phi^* - c_0}{R_0^2} = \tfrac{2}{3}v\xi f(\xi)\cos^2\phi. \quad (18)$$

The boundary is given by

$$\xi_0 = \xi_1 - \frac{v}{\theta'(\xi_1)}\{\psi_0(\xi_1) + A_2\psi_2(\xi_1)(1 - \tfrac{3}{2}\cos^2\phi)\}, \quad (19)$$

and so we obtain

$$\left.\begin{aligned} &\Phi^{(11)}(\xi_1) = 0, \qquad \Phi^{(12)}(\xi_1) = \frac{\Phi'^{(11)}(\xi_1)}{\theta'(\xi_1)}\{\psi_0(\xi_1) + A_2\psi_2(\xi_1)\}, \\ \text{and} \quad &\Phi^{(2)}(\xi_1) = \tfrac{2}{3}\xi_1 f(\xi_1) - 3A_2\frac{\psi_2(\xi_1)}{2\theta'(\xi_1)}\Phi'^{(11)}(\xi_1). \end{aligned}\right\} \quad (20)$$

A_2 is determined using the fact that $U + \Phi/c^2$ must be continuous over the boundary. We note that we may equate the external and internal solutions over the sphere

$\xi = \xi_1$, because their second derivatives will be continuous if the functions and their first derivatives are. This leads to

$$A_2 = \frac{-\frac{5}{6}\xi_1^2 + 2\sigma\{\Phi^{(2)}(\xi_1) - \frac{1}{3}\xi\Phi'^{(2)}(\xi_1)\}}{3\psi_2(\xi_1) + \xi_1\,\psi_2'(\xi_1)}, \tag{21}$$

where $\sigma = R_0/c^2$.

The equations for the Φ's were solved numerically. In each case there is a value to be guessed at the origin; however, owing to the linearity of the equations, once two trials have been made the correct figure can be determined exactly. The results of the numerical integrations are given in Table 1.

Table 1

METRIC COEFFICIENTS FOR A ROTATING POLYTROPE OF INDEX $n = 3$

ξ/ξ_1	$f(\xi)$	$\Phi^{(11)}$	$\Phi^{(12)}$	$\Phi^{(2)}$
0	0	−0·3933	808·1	0
0·1	0·8032	−0·5208	636·8	−0·3521
0·2	1·199	−0·6957	299·1	−2·087
0·3	1·201	−0·7085	31·03	−4·756
0·4	1·014	−0·5964	−113·4	−7·128
0·5	0·7939	−0·4501	−170·6	−8·927
0·6	0·6043	−0·3165	−179·0	−10·45
0·7	0·4600	−0·2076	−159·7	−12·06
0·8	0·3558	−0·1219	−121·9	−13·99
0·9	0·2815	−0·0542	−67·79	−16·4
1	0·228	0	3·86	−19·35

Now that the structures of rotating polytropes in general relativity have been determined it will be possible to investigate the stability of their oscillations, and this I hope to make the subject of a future paper.

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