DIFFUSE REFLECTION AND TRANSMISSION BY UNIFORM NONCOHERENTLY SCATTERING MEDIA

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[*Manuscript received January* 4, 1967]

Summary

Diffusely reflected and transmitted spectral line profiles are calculated for uniform noncoherently scattering media onto which radiation of frequency near that of a resonance line of the medium is incident. Profiles are calculated for incident light that is either collimated or uniformly diffused and that is either monochromatic, or of uniform frequency distribution, or contains an absorption line centred on the resonance line under discussion. Spectral lines generated by uniformly excited media are also calculated.

I. Introduction

Problems of diffuse reflection and transmission in the presence of noncoherent scattering arise in contexts where resonance radiation is involved, e.g. in discharge tubes and in structures in the solar atmosphere. In the present paper a method, based on the Eddington approximation, that was introduced by Jefferies and Thomas (1958) is used to investigate the following situations.

If a medium is illuminated externally by radiation whose frequency is near that of a resonance line of the medium it gives rise to diffusely reflected and transmitted spectral lines. These are evaluated below for incident light that is either collimated or uniformly diffused and that is either monochromatic, or of uniform frequency distribution, or contains an absorption line. In this last case the incident absorption line is assumed centred on the resonance line under consideration. For comparison, profiles are also given for uniformly excited media.

The approach is parametric in that the medium, assumed uniform, is characterized by given values of ϖ, the albedo for single scattering, and τ_1, its optical thickness at line centre. The ranges of values used cover those expected to be of astronomical interest and include semi-infinite media.

II. Definitions

x frequency measured from the line centre, in units of the Doppler width of the mean scattering profile of an atom within the medium;

κ_x, σ_x, a_x attenuation, scattering, and absorption coefficients respectively at frequency x ($\kappa_x = \sigma_x+a_x$);

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ϖ	albedo for single scattering $= \sigma_x/\kappa_x$;
λ	$1-\varpi = a_x/\kappa_x$;
$\psi(x)$	normalized scattering coefficient or scattering profile $= \sigma_x\left[\int \sigma_x \, \mathrm{d}x\right]^{-1}$;
$\phi(x)$	ratio of attenuation coefficient at frequency x to that at line centre;
τ	optical depth at line centre;
τ_1	optical thickness of medium at line centre;
$J(\tau, x)$	$2\pi \int_{-1}^{1} I(\tau, x, \mu) \, \mathrm{d}\mu$.

III. The Transfer Equations

We consider a uniform medium that scatters isotropically and completely noncoherently. Continuous absorption is assumed negligible, so that ϖ is constant across the line. The scattering profile is taken as Gaussian, that is,

$$\psi(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2); \qquad \phi(x) = \exp(-x^2). \tag{1}$$

Medium irradiated Externally

Let collimated monochromatic light be incident on the surface $\tau = 0$ of this medium. Following Chandrasekhar (1950) we may distinguish, at any point in the medium, between the reduced incident radiation and the diffuse radiation field. If μ_0 is the cosine of the angle of incidence, ϕ_0 the value of ϕ at the incident frequency x_0, and F_0 the flux per unit area of the incident radiation normal to the beam, then the reduced incident flux per unit area at optical depth τ is

$$F_0 \exp(-\phi_0 \tau/\mu_0). \tag{2}$$

Hence, the amount of diffuse radiation liberated into unit solid angle and unit frequency range per unit optical depth at optical depth τ is

$$\frac{\varpi F_0}{4\pi} \psi(x_0) \exp(-\phi_0 \tau/\mu_0). \tag{3}$$

By use of the Eddington approximation the equation of transfer may then be written

$$\frac{1}{3\phi_x^2} \frac{\mathrm{d}^2 J_x}{\mathrm{d}\tau^2} = J_x - \varpi \int_{-\infty}^{\infty} J_{x'} \psi(x') \, \mathrm{d}x' - \varpi F_0 \, \psi(x_0) \exp(-\phi_0 \tau/\mu_0). \tag{4}$$

Following Jefferies and Thomas we solve this equation by the method of discrete ordinates, replacing the integral by a $2n$-point Gaussian quadrature

$$\int_{-\infty}^{\infty} J_{x'} \psi(x') \, \mathrm{d}x' = \sum_{j=-n}^{n} W_j J_j. \tag{5}$$

The transfer equation then becomes the system of simultaneous differential equations

$$\frac{1}{3\phi_i^2}\frac{\mathrm{d}^2 J_i}{\mathrm{d}\tau^2} = J_i - \varpi \sum_j W_j J_j - \varpi F_0 \psi(x_0) \exp(-\phi_0 \tau/\mu_0), \tag{6}$$

where the subscripts refer to the division points of the quadrature. Extensive 15-significant-figure tabulations of division points and weights are given by Kopal (1955).

In the present problem the choice of quadrature demands considerable care, since the greater accuracy associated in principle with a finer division is offset by computing limitations when more than 8 points are used. Comparisons between the results obtained with 6- and 8-point Gauss–Hermite quadrature show significant differences in some curves. The central 8 points of the 12-point Gauss–Hermite quadrature have been chosen for the finite layers, since these best span the range of interest; the remaining 4 points make negligible contribution to the integral. For semi-infinite media the division points of the 8-point Gauss–Hermite quadrature best span the range and have therefore been used.

For a semi-infinite medium the solution of (4) that satisfies the boundary condition

$$J(\tau, x) \text{ finite as } \tau \to \infty \tag{7}$$

is

$$J(0,x) = \frac{\varpi F_0 \psi(x_0)}{2}\left\{\sum_{\alpha=1}^{n} \frac{L_\alpha}{1+k_\alpha/\sqrt{3}\phi} + \frac{\gamma}{1+\phi_0/\sqrt{3}\phi\mu_0}\right\}, \tag{8}$$

where $\{k_\alpha\}$ are the n positive roots of the equation

$$1 = 2\varpi \sum_{j=1}^{n} \frac{W_j}{1-k^2/3\phi_j^2} \tag{9}$$

and γ is defined by

$$\gamma^{-1} = 1 - 2\varpi \sum_{j=1}^{n} \frac{W_j}{1-\phi_0^2/3\phi_j^2\mu_0^2}. \tag{10}$$

The $\{L_\alpha\}$ are found from the equations

$$\sum_{\alpha=1}^{n} \frac{L_\alpha}{1-k_\alpha/\sqrt{3}\phi_i} + \frac{\gamma}{1-\phi_0/\sqrt{3}\phi_i\mu_0} = 0 \qquad (i = 1, 2, \ldots n), \tag{11}$$

which express the boundary conditions, in the form due to Krook (1955),

$$J_i = \frac{1}{\sqrt{3}\phi_i}\frac{\mathrm{d}J_i}{\mathrm{d}\tau} \quad \text{at } \tau = 0. \tag{12}$$

The solutions of (4) for a finite layer of optical thickness at line centre τ_1 are

$$J(0,x) = \frac{\varpi F_0 \psi(x_0)}{2}\left(\sum_{\alpha=-n}^{n} \frac{L_\alpha}{1+k_\alpha/\sqrt{3}\phi}[1-\exp\{-(k_\alpha+\sqrt{3}\phi)\tau_1\}] + \frac{\gamma}{1+\phi_0/\sqrt{3}\phi\mu_0}[1-\exp\{-(\phi_0/\mu_0+\sqrt{3}\phi)\tau_1\}]\right) \tag{13}$$

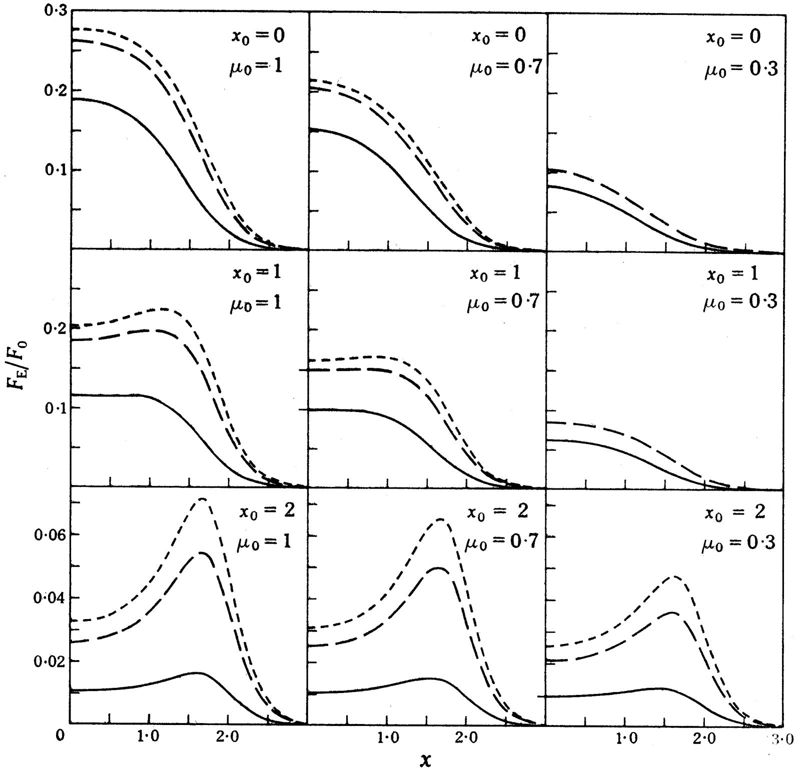

Fig. 1.—Reflected profiles for collimated monochromatic light incident on a layer of opacity at line centre 30.

——— $\lambda = 10^{-1}$; - - - - - $\lambda = 3\times10^{-2}$; -------- $\lambda = 10^{-2}$.

and

$$J(\tau_1, x) = \frac{\varpi F_0 \psi(x_0)}{2}\left(\sum_{\alpha=-n}^{n} \frac{L_\alpha}{1-k_\alpha/\sqrt{3}\phi}\{\exp(-k_\alpha\tau_1)-\exp(-\sqrt{3}\phi\tau_1)\}\right.$$
$$\left.+\frac{\gamma}{1-\phi_0/\sqrt{3}\phi\mu_0}\{\exp(-\phi_0\tau_1/\mu_0)-\exp(-\sqrt{3}\phi\tau_1)\}\right), \quad (14)$$

where $\{k_\alpha\}$ are the $2n$ roots of (9), with $k_{-\alpha} = -k_\alpha$, while γ is given by (10). The $\{L_\alpha\}$ are found from the sets of equations

$$\sum_{\alpha=-n}^{n} \frac{L_\alpha}{1-k_\alpha/\sqrt{3}\phi_i} + \frac{\gamma}{1-\phi_0/\sqrt{3}\phi_i\mu_0} = 0 \qquad (i = 1, 2, \ldots n), \quad (15)$$

which express the boundary conditions

$$J_i = \frac{1}{\sqrt{3}\phi_i}\frac{\mathrm{d}J_i}{\mathrm{d}\tau} \quad \text{at } \tau = 0, \quad (16)$$

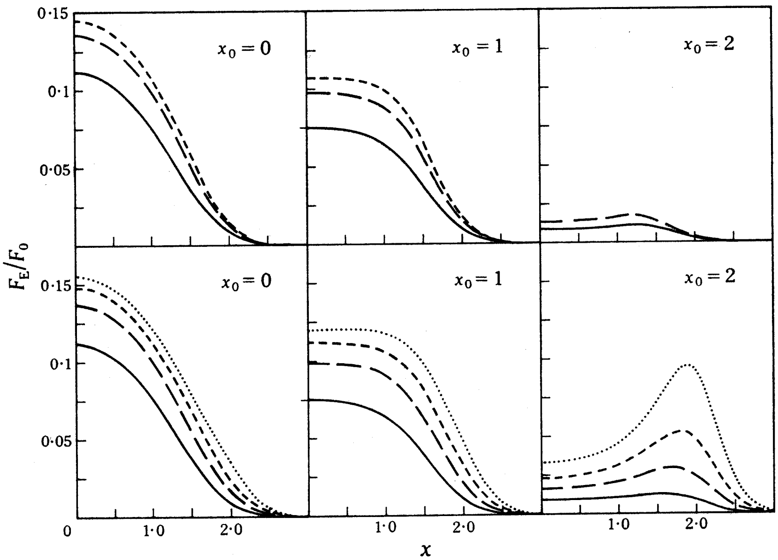

Fig. 2.—Reflected profiles for uniformly diffused monochromatic light incident on layers of opacity at line centre 10 (*upper*) and 100 (*lower*).

——— $\lambda = 10^{-1}$; – – – – – $\lambda = 3\times10^{-2}$; -------- $\lambda = 10^{-2}$; ······ $\lambda = 3\times10^{-3}$.

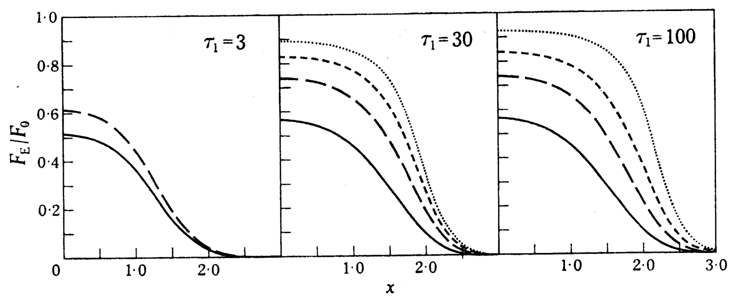

Fig. 3.—Reflected profiles for uniformly diffused incident light having a uniform continuous spectrum.

——— $\lambda = 10^{-1}$; – – – – – $\lambda = 3\times10^{-2}$; -------- $\lambda = 10^{-2}$; ······ $\lambda = 3\times10^{-3}$.

and

$$\sum_{a=-n}^{n} \frac{L_\alpha \exp(-k_\alpha \tau_1)}{1+k_\alpha/\sqrt{3\phi_i}} + \frac{\gamma \exp(-\phi_0 \tau_1/\mu_0)}{1+\phi_0/\sqrt{3\phi_i}\,\mu_0} = 0 \qquad (i = 1,\, 2,\, \dots\, n), \tag{17}$$

which express the boundary conditions

$$J_i = -\frac{1}{\sqrt{3\phi_i}}\frac{\mathrm{d}J_i}{\mathrm{d}\tau} \quad \text{at } \tau = \tau_1. \tag{18}$$

Uniformly Excited Medium

The Eddington approximation to the transfer equation for a uniformly excited medium having an emission function $B/4\pi$ (where aB is the emission from unit

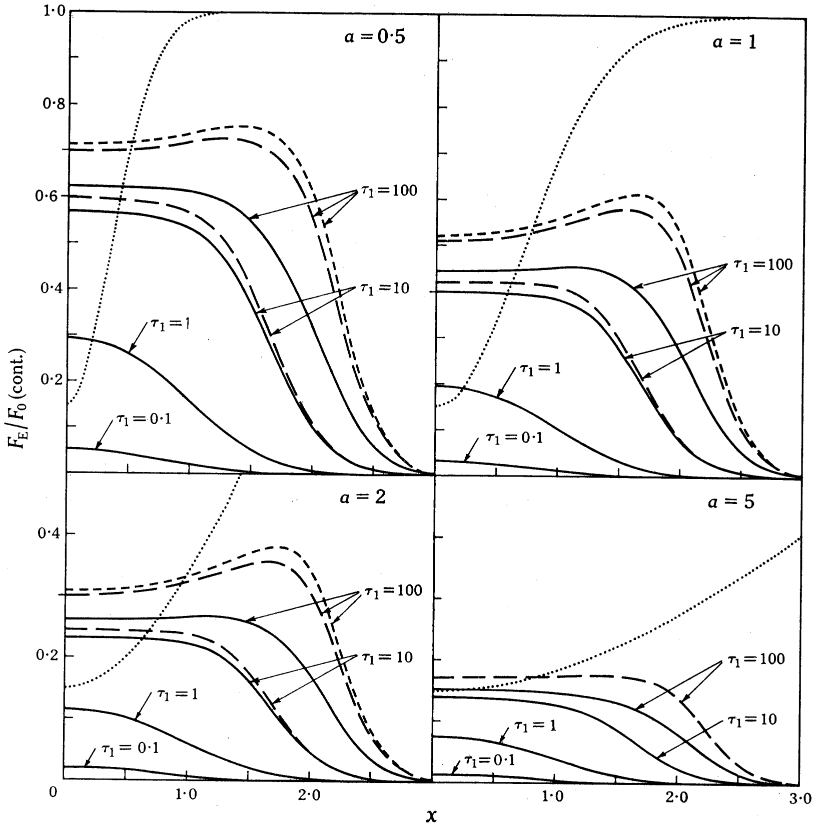

Fig. 4.—Representative reflected profiles for uniformly diffused incident light containing absorption lines of different widths.

······ incident profiles; ——— $\lambda = 10^{-2}$; – – – – – $\lambda = 10^{-3}$; -------- $\lambda = 10^{-4}$.

Profiles for values of λ less than those shown lie very close to that for the least value of λ for which a profile is drawn.

volume into unit frequency range per unit time, *excluding* scattered radiation) is

$$\frac{1}{3\phi_x^2}\frac{d^2J_x}{d\tau^2} = J_x - \varpi\int_{-\infty}^{\infty} J_{x'}\,\psi(x')\,dx' - \lambda B \tag{19}$$

and is solved in the same manner as above. The solution for a semi-infinite medium, subject to the boundary condition at infinity (7), is

$$J(0,x) = \frac{B}{2}\left(1 + \sum_{\alpha=1}^{n} \frac{L_\alpha}{1 + k_\alpha/\sqrt{3\phi}}\right), \tag{20}$$

where $\{k_\alpha\}$ are the n positive roots of (9), and the $\{L_\alpha\}$ are found from the boundary conditions (12).

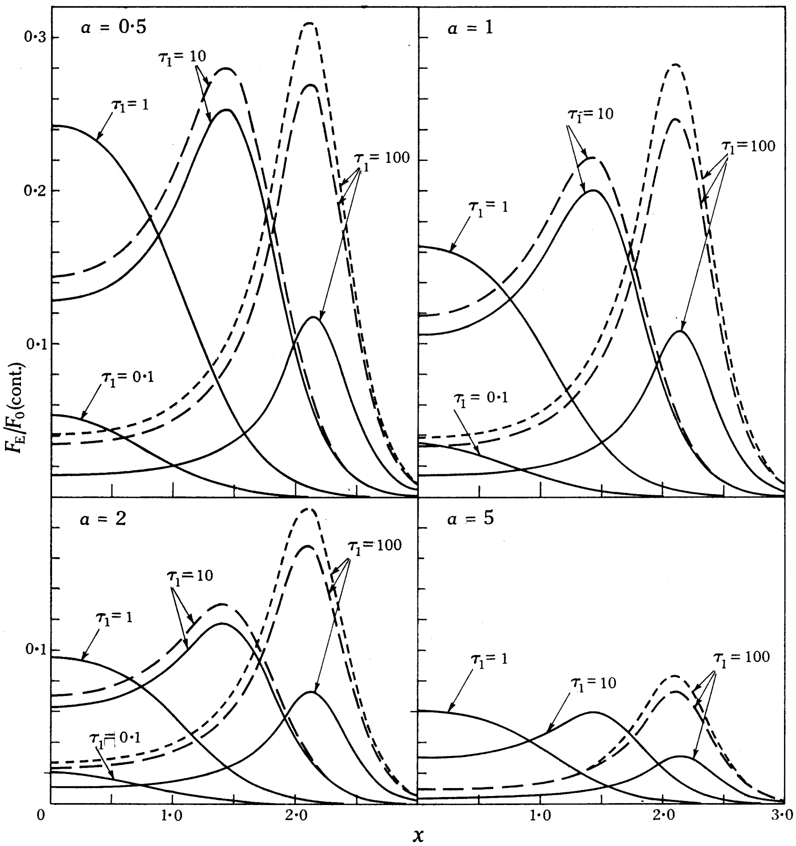

Fig. 5.—Representative diffusely transmitted profiles for uniformly diffused incident light containing absorption lines of different widths.

——— $\lambda = 10^{-2}$; - - - - - $\lambda = 10^{-3}$; -------- $\lambda = 10^{-4}$.

Profiles for values of λ less than those shown lie very close to that for the least value of λ for which a profile is drawn.

The solutions for a finite layer of optical thickness at line centre τ_1 are

$$J(0,x) = \frac{B}{2}\left(1-\exp(-\sqrt{3}\phi\tau_1) + \sum_{a=-n}^{n} \frac{L_\alpha}{1+k_\alpha/\sqrt{3}\phi}[1-\exp\{-(k_\alpha+\sqrt{3}\phi)\tau_1\}]\right) \tag{21}$$

and

$$J(\tau_1,x) = \frac{B}{2}\left(1-\exp(-\sqrt{3}\phi\tau_1) + \sum_{a=-n}^{n} \frac{L_\alpha}{1-k_\alpha/\sqrt{3}\phi}\{\exp(-k_\alpha\tau_1)-\exp(-\sqrt{3}\phi\tau_1)\}\right), \tag{22}$$

which, of course, must be equal. As before, the $\{k_\alpha\}$ are the $2n$ roots of (9), while the $\{L_\alpha\}$ are found from the boundary conditions (16) and (18).

IV. Scope of the Calculations

Emergent Flux

In each of the cases considered in Section III the emergent flux is given by the Eddington approximation, in terms of the emergent total intensity $J_E(x)$, as

$$F_E(x) = \frac{1}{\sqrt{3}} J_E(x). \tag{23}$$

In the following discussion it is profiles of emergent flux to which reference is being made.

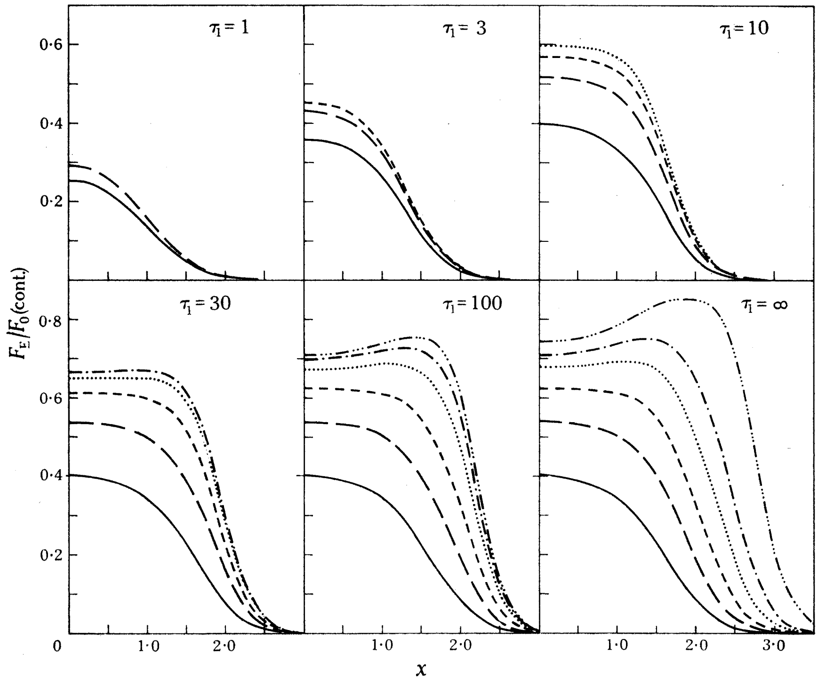

Fig. 6.—Reflected profiles as functions of λ and τ_1 for a uniformly diffused incident absorption line of Doppler width $a = 0 \cdot 5$.

——— $\lambda = 10^{-1}$; – – – – $\lambda = 3\times10^{-2}$; -------- $\lambda = 10^{-2}$; ······ $\lambda = 3\times10^{-3}$; ·-·-·-· $\lambda = 10^{-3}$; ·-·-·-·-· $\lambda = 10^{-4}$.

Except for the semi-infinite atmosphere, profiles for values of λ less than those shown lie very close to that for the least value of λ for which a profile is drawn.

Reflected profiles for a medium illuminated by collimated monochromatic light are given in Figure 1. The emergent profiles from uniformly excited media are shown in Figure 8.

Angularly Dependent Incident Radiation

Instead of collimated incident radiation we consider monochromatic radiation incident on the surface at angles to the normal in the range μ_0 to $\mu_0 + d\mu_0$. The profiles are evaluated as functions of μ_0 and then, weighted by the incident intensity,

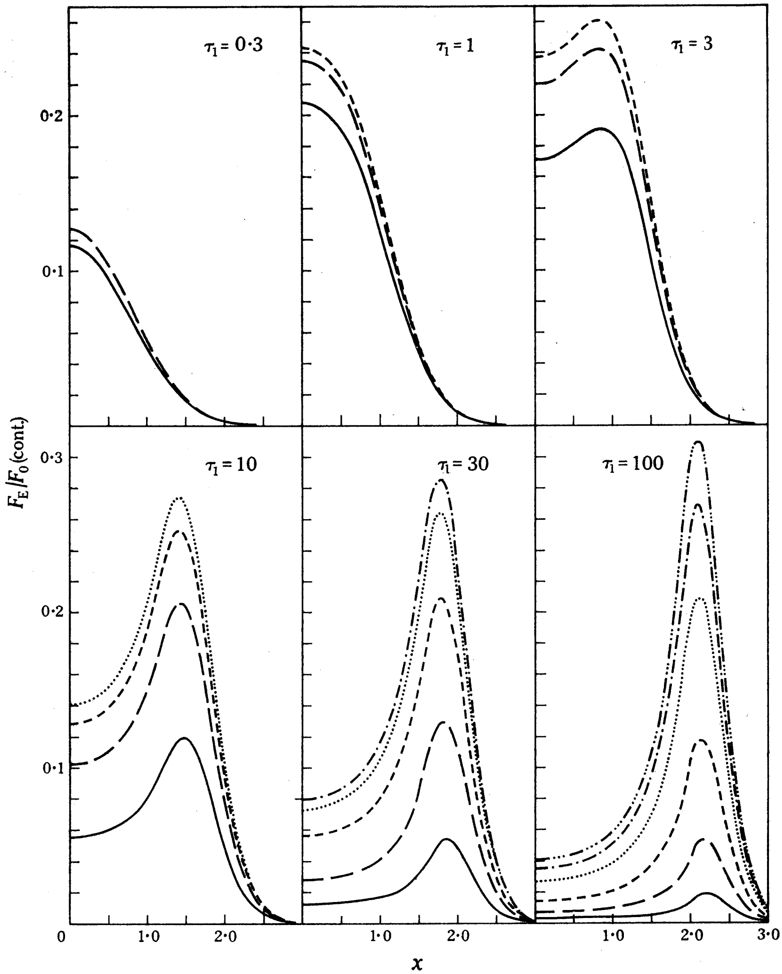

Fig. 7.—Diffusely transmitted profiles as functions of λ and τ_1 for a uniformly diffused incident absorption line of Doppler width $a = 0 \cdot 5$.

——— $\lambda = 10^{-1}$; – – – – $\lambda = 3 \times 10^{-2}$; -------- $\lambda = 10^{-2}$; ······ $\lambda = 3 \times 10^{-3}$; ·-·-·-· $\lambda = 10^{-3}$; ·-··-··-·· $\lambda = 10^{-4}$.

Profiles for values of λ less than those shown lie very close to that for the least value of λ for which a profile is drawn.

are integrated over μ_0. If $I_0(\mu_0)$ is the angular distribution of the incident intensity we thus replace F_0 by an element of incident flux $2\pi I_0(\mu_0)\,\mathrm{d}\mu_0$ in the analysis. The emergent profiles are evaluated at values of μ_0 that are the division points of the 12-point Gauss–Legendre quadrature for the range $(0, 1)$ and the required profiles obtained by numerical integration. For uniformly diffused radiation

$$I_0(\mu_0) = 1. \tag{24}$$

Reflected profiles for media illuminated by uniformly diffused monochromatic light are given in Figure 2.

Frequency-dependent Incident Radiation

Let the frequency dependence of the incident intensity be described by $I_0(x_0)$. As above, we replace F_0 by $2\pi I_0(x_0)\,\mathrm{d}\mu_0\,\mathrm{d}x_0$ if the incident radiation is uniformly diffused. The diffusely transmitted and reflected profiles are evaluated at the same values of μ_0 as above and at values of x_0 that are the division points of the 14-point Gauss–Hermite quadrature; the required profiles are then obtained by integration. For a continuous spectrum

$$I_0(x_0) = 1. \tag{25}$$

Reflected profiles for this case are given in Figure 3.

In the case of incident radiation containing an absorption line this line is taken to have a Gaussian profile and residual central intensity 0·15 of that in the continuum (this value has been selected with the solar Hα line in mind). The Doppler width being denoted by a, then

$$I_0(x_0) = 1-0\cdot 85\exp\{-(x/a)^2\}. \tag{26}$$

Reflected profiles are shown in Figures 4 and 6 for lines of a number of widths incident on a variety of media. Equivalent diffusely transmitted profiles are given in Figures 5 and 7.

V. Discussion

Let us first consider a medium in which there is a point source of radiation, whose frequency distribution follows the scattering profile. Near the source, before many scattering events have occurred, the radiation field is that of the source. Further from the source, radiation has been transferred away from the line centre, causing the characteristic central minima to develop. As the distance from the source increases the central minima become more pronounced, the peaks so formed moving further from the line centre.

Uniformly Excited Media

In this picture a uniformly excited medium may be considered as an aggregation of point sources, which everywhere augments the radiation near the line centre. From the results (Fig. 8) it is seen that for the radiation transferred to the wings to be greater than this central emission a layer of optical thickness* approximately 2·5 is required. We shall call the thickness of the layer required to produce a central minimum the *critical depth.*

Collimated Monochromatic Incident Radiation

In considering the diffuse field generated by externally incident radiation we have the added factor that the excitation decreases with depth. In the case of collimated monochromatic incident light this dependence is expressed by $\exp(-\phi_0\tau/\mu_0)$, that is, the excitation effectively occurs in the region for which $\phi_0\tau/\mu_0 \lesssim 2$ or 3. We shall call the lower limit of the region the *excitation depth* τ_e, where, for the incident radiation under discussion, $\tau_e \approx 2\mu_0/\phi_0$.

* Optical thickness, opacity, etc. refer to the line centre in this discussion.

As we have seen, the critical depth of a uniformly excited medium is about $2 \cdot 5$. Following the same line of argument it is apparent that when the excitation decreases with depth the critical depth will be somewhat greater for reflected profiles, while for diffusely transmitted radiation the critical depth will be less. The exact value of the critical depth depends on the incident radiation, since this determines the rate at which the excitation falls off with depth. Characteristic values when the incident light is collimated and monochromatic are found to be $1 \cdot 5$ for diffusely transmitted radiation and $2 \cdot 75$ for reflected radiation.

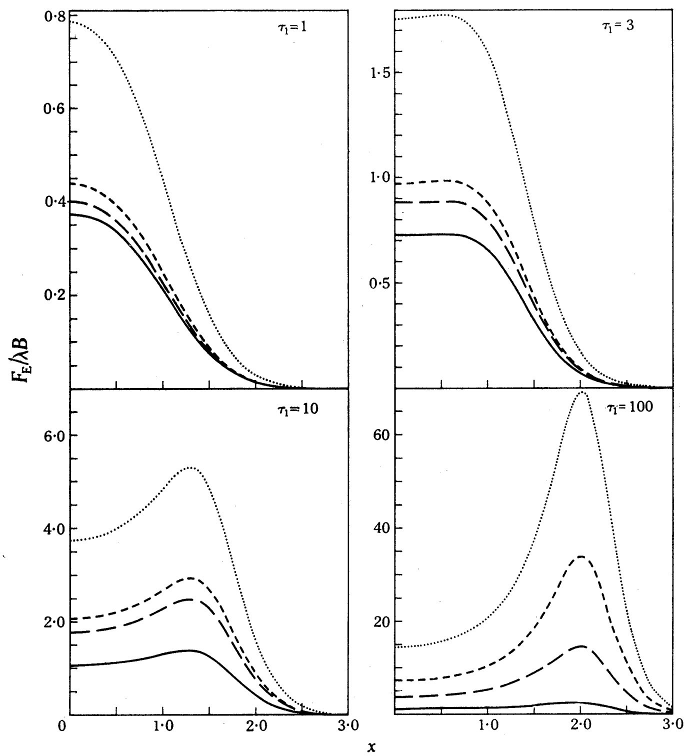

Fig. 8.—Emergent profiles for uniformly excited media.
——— $\lambda = 10^{-1}$; - - - - - $\lambda = 10^{-2}$; -------- $\lambda = 10^{-3}$; $\cdots\cdots$ $\lambda = 10^{-4}$.

The differences between the diffusely transmitted and reflected profiles are amplified by the fact that any layers beneath that in which the initial excitation occurs affect the transmitted radiation in a manner analogous to that discussed for a point source, while these layers have little effect on the reflected profile. Thus, any medium of optical thickness greater than the critical depth will generate a central minimum in the diffusely transmitted profile. On the other hand, central minima are present in the reflected profiles only if both the excitation depth and the optical thickness of the medium are greater than the critical depth.

Uniformly Diffused Monochromatic Incident Radiation

Arguments similar to those above hold when the incident light is uniformly diffused and monochromatic. In this case the depth dependence of the excitation is given by* $E_2(\phi_0 \tau)$, and hence the excitation depth is given approximately by $E_2(\phi_0 \tau_e) \approx e^{-3}$. The critical depths are found to be about $1 \cdot 5$ and 3 for diffusely transmitted and reflected radiation respectively.

Uniformly Diffused Incident Radiation having a Spectrum of Frequencies

The contribution that any one frequency makes to the excitation at any depth in the medium is proportional to both the incident intensity and the magnitude of the scattering profile at that frequency as well as to the depth factor $E_2(\phi\tau)$. This means that as the depth increases the incident frequency that contributes most to the excitation moves out from the centre of the line. However, the total excitation from all frequencies falls off rapidly with depth.

This last fact explains the insensitivity of the shape of the diffusely transmitted profiles to the width of the incident line, and also the pronounced effect that pure absorption has on them (Figs. 5 and 7).

Layers of optical thickness 3 or less give rise to reflected profiles much the same as those for uniformly diffused incident light that is monochromatic at the line centre. Light incident near the line centre also provides the major contribution to the reflected profiles from media of opacity greater than 3, but the shape of the wings depends on the amount of excitation deep in the medium and, hence, on the width of the incident line. Reflected profiles are generally flat-topped when $\tau_1 \sim 10$, and when τ_1 is 30 or more peaks may be present in the wings. Incident lines of width approximately that of the scattering profile of the medium generate more pronounced central minima in the reflected profiles than do comparatively wide or narrow lines (Fig. 4). Incident light having a continuous spectrum gives rise to reflected lines in which the intensity in the wings is augmented but not to the extent of producing peaks (Fig. 3).

Since the shape of the reflected profiles depends on the excitation deep in the medium, absorption also strongly affects the shape. High absorption inhibits the formation of central minima (Fig. 6). When $\tau_1 \gtrsim \lambda^{-1}$ the layer is equivalently semi-infinite, in that the reflected profiles have the same intensity and shape as those from a semi-infinite medium with equal value of λ.

VI. Acknowledgment

I am indebted to Dr R. G. Giovanelli for his advice and criticism at all stages of this work.

VII. References

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* $E_2(x)$ is the second exponential integral.