HIGHER MODES OF NON-RADIAL OSCILLATIONS OF STARS BY VARIATIONAL METHODS

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Summary

The Ritz method is applied to a variational principle of Chandrasekhar to determine the normal modes of non-radial oscillation of a massive star. Although good results are obtained for the f-mode and the p-modes, it is shown that the method may fail to detect the g-modes. The effect of the choice of coordinate functions is considered.

I. INTRODUCTION

Much recent work has been stimulated by Chandrasekhar's (1964) formulation as a variational principle of the complex system of differential equations representing non-radial oscillations of stars. Lebovitz (1965) has used the principle to derive what is probably the most universally valid justification yet available of Schwarzschild's well-known stability criterion. The principle has been modified to take into account rotation (Clement 1965) and magnetic fields (Kovetz 1966).

The present paper is concerned with the application of the Ritz method to Chandrasekhar's variational principle to determine higher modes of non-radial oscillation numerically. It is shown that, at least in certain circumstances, while the method gives good results for the f-mode and the p-modes, it fails even to detect the g-modes. This effect does not appear to have been noticed by others who have applied the Ritz method to the principle.

For the simpler case of purely radial oscillations, useful results for quite high modes have been obtained by Andrew (1967) using quite modest computing facilities. The calculations described in that paper used a model, constructed by Van der Borght (1964), of a star of uniform composition and mass $M = 10M_{\odot}/\mu^2$, where $M_{\odot}$ is the mass of the Sun and μ the molecular weight of the star. The same model was used for the calculations described here. For this model Wan and Van der Borght (1966) have used numerical integration to solve the full fourth-order system of equations governing non-radial oscillations and also the simpler second-order system of equations obtained by neglecting the perturbation ϕ' of the gravitational potential. Each system of equations yielded both p-modes and g-modes. These two systems, whose derivation and properties are described by Ledoux and Walraven (1958), will be designated here by IA and IB respectively.

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The calculations described here considered only the simpler case corresponding to IB. However, the essential mathematical properties of IA and IB, especially the non-linear occurrence of the eigenvalue and the two families of modes, are very similar. It seems likely that if a variational method failed to detect the g-modes of IB it could also fail with IA. In any case, in view of the current interest in the variational approach, it would seem that any apparent limitation of the method should be investigated.

II. The Variational Principle

When the approximation $\phi' = 0$ is made, Chandrasekhar's (1964) variational principle for oscillations corresponding to the spherical harmonic l may be written

$$\sigma^2 \int_0^R \rho \left(\frac{\psi^2}{r^2} + \frac{(\chi')^2}{l(l+1)} \right) \mathrm{d}r = \int_0^R \left\{ \Gamma_1 p \left(\frac{\mathrm{d}p}{\mathrm{d}r} \frac{\psi}{\Gamma_1 p} + \psi' - \chi' \right)^2 + A \frac{\mathrm{d}p}{\mathrm{d}r} \psi^2 \right\} \frac{\mathrm{d}r}{r^2}, \tag{1}$$

where the components of displacement in the radial and transverse (θ) directions are respectively

$$\psi/r^2 \qquad \text{and} \qquad \{\chi'/l(l+1)r\}\, \partial Y_l^m(\theta, \varphi)/\partial\theta,$$

R is the radius of the star, ρ and p are equilibrium values of density and total pressure at a distance r from the centre respectively, $2\pi/\sigma$ is the period,

$$A = \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r},$$

and, for a monatomic gas,

$$\Gamma_1 = \beta + 2(4-3\beta)^2/(24-21\beta),$$

where β is the ratio of gas pressure to total pressure. The Euler–Lagrange equations obtained from (1) by considering variations in ψ and χ' respectively are seen, after some simplification, to be precisely the system IB. In terms of the variables $\bar{p}$ and $\bar{t}$ used by Van der Borght (1964), equation (1) may be written

$$\begin{aligned} \lambda \int_0^1 f_4(x) \{\xi^2 + \eta^2/l(l+1)\}\, \mathrm{d}x \\ = \int_0^1 \{ f_1(x)\, \xi^2 - 2x f_2(x)\, \xi\, \mathrm{d}\xi/\mathrm{d}x + x^2 f_3(x)\, (\mathrm{d}\xi/\mathrm{d}x)^2 + f_3(x)\, \eta^2 \\ + 2 f_2(x)\, \xi\eta - 2x f_3(x)\, \eta\, \mathrm{d}\xi/\mathrm{d}x \}\, \mathrm{d}x, \end{aligned} \tag{2}$$

where

$$f_1(x) = 9x^2 \Gamma_1 \bar{p} + 6x^3 \frac{\mathrm{d}\bar{p}}{\mathrm{d}x} + x^4 \frac{\mathrm{d}\bar{p}}{\mathrm{d}x} \frac{\bar{t}}{\beta\bar{p}} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\beta\bar{p}}{\bar{t}} \right),$$

$$f_2(x) = -3x^2 \Gamma_1 \bar{p} - x^3\, \mathrm{d}\bar{p}/\mathrm{d}x, \qquad f_3(x) = x^2 \Gamma_1 \bar{p}, \qquad f_4(x) = x^4 \beta\bar{p}/\bar{t},$$

$$x = r/R, \qquad \psi = r^3 \xi, \qquad \chi' = r^2 \eta, \qquad R^3\sigma^2 = MG\omega^2,$$

and $$\lambda = GM\omega^2/\mathscr{R}, \qquad \text{where } \mathscr{R} \text{ is the gas constant.}$$

III. Numerical Calculations

The Ritz method was applied to (2) using a procedure similar to that already described (Andrew 1967) for the radial case. The interval $[0, 1]$ was divided into 25 equal subintervals. Both ξ and η were assumed to be linear in each of these subintervals and continuous throughout the entire interval. For $i = 0, \ldots, 25$,

$$\xi(x) = \xi(i/25)+(25x-i)\{\xi((i+1)/25)-\xi(i/25)\}, \tag{3}$$

for $i < 25x < i+1$. The form for η is similar. No further restrictions were imposed (boundedness of ξ and η ensures that there is no displacement at the centre) and hence ξ and η are determined by 52 parameters. Partial differentiation of (2) with respect to these parameters yields the matrix equation

$$Pu = \lambda Qu, \tag{4}$$

where in this case P and Q are each of the form (P_{ij}), $i,j = 1,2$, and each of the P_{ij} is (26×26) and tridiagonal. Matrices of this form may be triangulated without the introduction of any further non-zero elements (see Andrew 1967). This fact was utilized in the solution of (4), which was carried out on an I.B.M. 1620 computer at the Australian National University. Approximate eigenvalues of (4) were first obtained by searching for the zeros of $\det(P-\lambda Q)$. These were used to obtain accurate eigenvectors using Wilkinson's recurrence relation $(P-\lambda Q)u_r = Qu_{r-1}$. Accurate eigenvalues were then obtained by substituting these eigenvectors in (4). Each element of Pu was divided by the corresponding element of Qu and the 52 such ratios were compared. All agreed to several decimal places. The bounds thus obtained for the eigenvalues of (4) showed that errors obtained in the solution of (4) were small compared with those due to the restricted form assumed for ξ and η.

IV. Results

All 52 eigenvalues and eigenvectors were calculated in the case corresponding to $l = 2$. Of these, 26 appeared to correspond to the f-mode and the first 25 p-modes, but the remainder appeared completely spurious.

Modes obtained by numerical integration of IB show a regular pattern. The nth p-mode and the nth g-mode both contain exactly n zeros in ξ and in η. The zeros are clustered most closely near the surface of the star and ξ and η both increase sharply near the surface, these effects increasing with n. The genuine p-modes and the f-mode obtained in the present work resembled very closely the expected pattern. The differences between the computed p-modes and the expected values were similar to those noted in the radial case (Andrew 1967). The amplitudes of ξ and η were too small near the surface and the zeros spread more evenly than in the true modes. For some of the highest modes the computed values of η had one too few zeros. These differences may be explained by the fact that separations of less than 1/25 in the zeros of $\xi(x)$ and $\eta(x)$ could not be represented in the space of allowable functions.

The spurious modes were in marked contrast to the f-mode and the p-modes. The zeros of the spurious modes seemed to be almost randomly distributed and the amplitudes showed no clear pattern.

Although no restriction was put on the 52 parameters $\xi(i/25)$, $\eta(i/25)$, $i = 0, \ldots, 25$, all eigenfunctions obtained, including the spurious modes, approximately satisfied the relation $\eta(0) = 3\xi(0)$. This corresponds to the requirement that the divergence of the displacement, which is proportional to $(\psi' - \chi')/r^2$, must vanish at the centre of the star.

TABLE 1

COMPARISON OF EIGENVALUES ω^2

Mode	Variational Methods			Numerical Integration*
	Eqn. (3)	Eqn. (5), $n = 3$	Eqn. (5), $n = 6$	
f	2·6132	2·6123	2·6260	2·6120
p_1	3·9031	3·9025	3·8865	3·8847
p_2	5·2210	5·3817	5·1523	5·1481
p_3	6·5765	11·0498	6·4175	6·3911
p_4	7·987	—	7·923	7·622
p_5	9·457	—	10·567	8·891

* Values obtained by Wan and Van der Borght (1966).

TABLE 2

POSITIONS OF ZEROS OF ξ AND η

Mode	Variational Methods		Numerical Integration*	Variational Methods		Numerical Integration*
	Eqn. (3)	Eqn. (5)		Eqn. (3)	Eqn. (5)	
	First zero of ξ			First zero of η		
p_1	0·646	0·648	0·648	0·806	0·809	0·808
p_2	0·524	0·532	0·529	0·686	0·690	0·691
p_3	0·443	0·455	0·452	0·599	0·608	0·610
p_4	0·380	0·402	0·395	0·531	0·510	0·548
p_5	0·332	0·310	0·350	0·476	0·435	0·496
	Second zero of ξ			Second zero of η		
p_1	—	—	—	—	—	—
p_2	0·824	0·838	0·830	0·896	0·901	0·901
p_3	0·727	0·741	0·737	0·807	0·812	0·816
p_4	0·649	0·606	0·665	0·730	0·738	0·746
p_5	0·583	0·533	0·606	0·664	0·618	0·686

* Values obtained by Wan and Van der Borght (1966).

The eigenvalues obtained by three variational methods for the f-mode and the first five p-modes are compared in Table 1 with those obtained by Wan and Van der Borght (1966) using the more accurate (but much slower) method of numerical integration of the differential equations. Table 2 compares the corresponding forms of ξ and η by showing the positions of the first two zeros. The values of ω^2 obtained for p-modes higher than p_5 are:

10·96, 12·4, 13·8, 15·4, 17·2, 19·3, 21·6, 24·1, 26·7, 29·7,
32·9, 36·5, 40·5, 45·1, 50·3, 56·4, 63·6, 72·2, 82·8, 105·4.

The eigenvalues increase regularly, as would be expected. For the spurious modes the values of ω^2 were irregular, as shown by the following list, in which the numbers in brackets after each spurious ω^2 represent the number of zeros in ξ and in η respectively:

0·187	(3, 4),	0·257	(4, 5),	0·263	(3, 4),	0·321	(5, 6),	0·388	(6, 7),
0·460	(5, 8),	0·533	(7, 8),	0·598	(8, 9),	0·691	(9,10),	0·777	(2, 5),
0·822	(7,11),	0·969	(9,12),	1·15	(9,13),	1·37	(11,14),	1·64	(11,15),
1·97	(11,16),	2·39	(9,17),	2·93	(8,14),	3·64	(10,14),	4·61	(9,15),
5·98	(13,17),	8·05	(9,15),	11·53	(13,15),	18·76	(13,16),	41·9	(19,21),
				154·6	(24,25).				

A complete listing of the f-mode, the first 20 p-modes, and selected spurious modes is given in the author's unpublished thesis (Andrew 1966), where considerable discussion of the result is also given. The very clear-cut division of the modes into genuine and spurious should be evident from these values.

V. Use of Other Coordinate Functions

More recently Robe and Brandt (1966) reported that they had obtained both p-modes and g-modes for a number of polytropic models, using Chandrasekhar's variational principle. The form they assumed for ψ and χ was different from that used in the calculations just described. Following Chandrasekhar and Lebovitz (1964), they assumed

$$\psi = \sum_{i=0}^{n} c_i r^{l+2i+1}, \qquad \chi = c_0 r^{l+1} + \sum_{i=1}^{n} d_i r^{l+2i+1}.$$

In the case $l = 2$, this corresponds to

$$\xi = \sum_{i=0}^{n} a_i x^{2i}, \qquad \eta = 3a_0 + \sum_{i=1}^{n} b_i x^{2i}. \tag{5}$$

Like Chandrasekhar and Lebovitz (who considered only the case $n = 1$ and found only the f-mode), they made the approximation $\Gamma_1 =$ constant, which can be strictly true only for a polytrope of index 3. This approximation was not made here (in the present model β varies from 0·794 at the centre to 0·904 at the surface). Robe and Brandt used $n = 0, 1, 2, 3$ and considered both the case $\phi' = 0$ and the general case. They reported that for $l = 2$ the method gave p-modes and g-modes in both cases. However, for stable models, the eigenvalues they obtained for the g-modes were not particularly accurate and their paper makes no mention of the eigenfunctions. Since it is mainly consideration of the eigenfunctions that makes clear the division between true and spurious modes in the present work, it is perhaps conceivable that their g-modes could be illusory.

Most of the standard theory on the convergence of the Ritz method deals with operators that are positive definite (Mikhlin 1964). The operator associated with radial oscillations has this property, but that associated with non-radial oscillations does not. Hence it is conceivable that the different results obtained by Robe and Brandt (1966) could be due to the choice of coordinate functions. In order to test this hypothesis, the Ritz method was again applied to equation (2)

for Van der Borght's model, this time assuming the form (5) for ξ and η. Separate calculations were made using $n = 3, 4, 5$, and 6. In each case the method failed to detect the g-modes, yielding only the f-mode, p-modes, and some clearly spurious modes.

Since this time P and Q in (4) did not have a special simple form, solution of (4) was carried out by standard methods (hence the smaller number of parameters). Calculations were carried out on the C.D.C. 3200 computer at CSIRO, Melbourne, and the programme incorporated the standard CSIRO subroutines EVAL and EVECT. Again the computed eigenvectors were substituted in (4) and the ratio of each element of Pu to the corresponding element of Qu was computed. Comparison of the ratios showed that errors in the solution of (4), while slightly larger than when (3) was used, were in general much smaller than the error caused by the restricted form assumed for ξ and η.

One similarity of the present results with those of Robe and Brandt (1966) is that the accuracy of the f-mode did not always increase with n. A difference is that the middle eigenvalue did not represent the f-mode. (For $n = 3, 4, 5, 6$, the f-mode and the first n p-modes were obtained. The lowest $n-1$ eigenvalues and the second highest eigenvalue were spurious. This may not be true for all n.) The eigenvalues ω^2 obtained using equation (5) with $n = 3$ and $n = 6$ are shown in Table 1 and the positions of the first two zeros of ξ and η in the case $n = 6$ are shown in Table 2.

Theoretical questions raised by non-occurrence of the g-modes will be considered in a later paper.

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