EFFECTS OF AIR COMPRESSION BETWEEN PARALLEL PLATES IN INTERFEROMETRIC AND CAPACITIVE DISPLACEMENT TRANSDUCERS

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Summary

When one of two adjacent surfaces of a displacement transducer, such as a parallel plate capacitor or an optical interferometer, is moved towards the other the compression of air between the surfaces alters the dielectric constant and the refractive index of the air during the motion. In making a precise measurement of a transient displacement with one or other transducer not *in vacuo*, a correction to the measured value of capacitance or path difference may be necessary. An analysis of the effects of air compression is given, and the magnitude of the correction is assessed for a representative case. The results indicate that the required corrections may be significant. For example, in the particular case considered the surfaces are circular and of radius $2 \cdot 54$ cm, initially separated by $0 \cdot 254$ cm, and an acceleration of $0 \cdot 1g$ is suddenly applied to one surface, maintained for 50 msec, and then suddenly reversed to a value of $-0 \cdot 1g$. For this case the maximum correction to the dielectric constant amounts to 10% and that for the refractive index 5%. These are also the corrections to be applied to the capacitance change and path difference change respectively.

I. Introduction

In order to measure a transient displacement in terms of a light wavelength, a transducer has been devised that consists of a multiplebeam interferometer of the Fizeau type in which the silvered surfaces of the optical flats function also as capacitor plates, as shown in Figure 1. Both of the active electrodes (1 and 2) are surrounded by a guard electrode (3). The transient motion to be measured is imparted to one surface of the transducer and the time rate of change of capacitance between electrodes 1 and 2 is measured by means of a transformer bridge (Thompson 1958). The capacitor is calibrated by interference measurements.

For practical reasons it is convenient to operate the transducer in air rather than in a vacuum. In accordance with the Lorentz–Lorenz law (Born and Wolf 1959), the dielectric constant (ϵ) and refractive index (n) will change with compression, causing the changes in capacitance and path difference associated with motion of one of the plates to differ from their values at normal temperature and pressure. Because

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there is a definite correspondence between capacitance change and path difference change through Maxwell's relation $\epsilon = n^2$, if the variation of one is known that of the other can be directly calculated.

For reasons connected with the velocity of the interference fringes and the bandwidth consequently necessary in a photoelectric detector to resolve the detail in the fringe signal in the presence of noise (Goldberg 1963), the recording of the transient motion directly in terms of interference fringe movement is not an attractive proposition. It is therefore considered preferable to calibrate the capacitor by static measurements with the interferometer and to then apply a calculated correction to the measured capacitance to allow for the change in the dielectric constant of air that occurs with the pressure change.

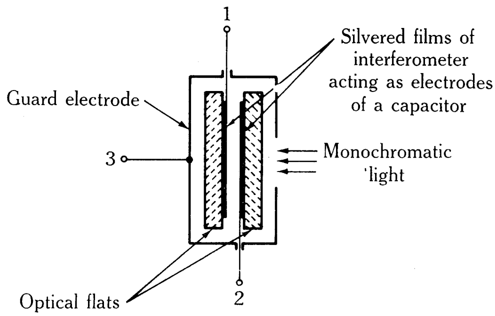

Fig. 1.—A transducer that combines an interferometer with a capacitor.

The correction to be applied is calculated by the Lorentz–Lorenz law after determining the pressure in the air gap when one surface undergoes the transient motion.

II. Physical Conditions in the Air Gap

The excess pressure in the air gap at any instant is determined by two components. One is the compression of the enclosed volume of air as one surface moves towards the other; the second is the relief of this pressure as air flows through the boundary of the gap. It will be shown that the interaction of these two components makes the behaviour of the system nonlinear. Furthermore, in the second component two conditions of air flow can be distinguished: in one the flow occurs with a boundary layer at the surfaces while the main body of air behaves as a compressible inviscid fluid; in the other the entire body of air in the gap flows under viscous retardation.

The first condition corresponds to moderate separations of the transducer surfaces, the second to separations that are extremely small compared with the surface dimensions. One would expect the pressure loss to be greater in the first case, and an evaluation of this loss might be expected to set an upper bound to it. The present paper is concerned with the first condition, since this is the one involved in the particular application.

It is convenient to make the surfaces of the transducer circular; hence the system has radial symmetry. The coordinate system and nomenclature are shown in Figure 2.

The boundary layer flow can be taken as laminar if the surfaces are smooth and the air velocity is small. It can be shown (Goldstein 1938) that in laminar flow the magnitude of the total change of pressure throughout the boundary layer along a normal to the surface is of the second order of the thickness of the layer. The pressure may be therefore taken as constant along a normal to the surface and equal to its value in the main stream just outside the layer. If it is assumed further that the ratio of the diameter $2a$ to the gap thickness d (Fig. 2) is always large, then the pressure variation in the z direction in the main air stream may be neglected. The problem is now reduced to one in which the pressure dependence is radial only.

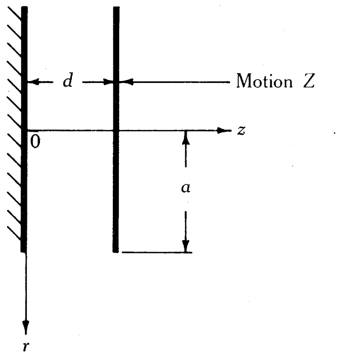

Fig. 2.—Diagram illustrating coordinate system.

III. Nomenclature

The following nomenclature is used

ϵ	dielectric constant of air
n	refractive index of air
$\mathbf{u}$	vector particle velocity of air flow
$\mathbf{r}$	vector position measured from the axis of the cylindrical gap
p_0	atmospheric air pressure
ρ_0	atmospheric air density
p	excess pressure
ρ	excess density
γ	ratio of specific heats C_p/C_v
K	bulk modulus of air, equal to γp_0
s	condensation, equal to $(\rho-\rho_0)/\rho_0$
p_c	excess pressure component due to compression of air volume
p_l	pressure loss due to air flow
d	separation of transducer surfaces
Z	displacement of a surface of the transducer tending to reduce the separation
a	radius of transducer surface
$J_0(x)$	ordinary Bessel function of order zero and argument x
$\bar{H}(\zeta_i) \equiv \bar{H}$	finite Hankel transform of p_l

Other symbols are defined as required.

IV. Differential Equation for Pressure Distribution and its Solution

Let $\mathbf{u}$ be the vector particle velocity at the vector position $\mathbf{r}$ measured from the axis of the cylindrical gap when air flow is occurring in the gap due to the motion of one surface.

Let p_0 and ρ_0 be the atmospheric values of pressure and density, and p and ρ be the excess pressure and density when the medium is disturbed. It is assumed that the changes in these quantities occur adiabatically. It is shown in textbooks on fluid mechanics that the equation of conservation of mass is

$$\partial(\rho+\rho_0)/\partial t+\operatorname{div}(\rho+\rho_0)\mathbf{u} = 0 \tag{1}$$

and that the momentum equation is

$$(\rho_0+\rho)\partial\mathbf{u}/\partial t+\operatorname{grad}(p_0+p) = 0\,. \tag{2}$$

On introducing the quantity $(\rho-\rho_0)/\rho_0 = s$, the condensation, the excess pressure p required to produce this condensation can be written as

$$p = Ks\,, \tag{3}$$

where K is the adiabatic bulk modulus and is equal to γp_0 (γ being the ratio of the specific heats C_p/C_v in the usual notation). If p_0 is constant and ρ_0 is independent of t, we obtain from equations (1), (2), and (3)

$$(\rho_0/K)\partial^2 p/\partial t^2-\operatorname{div}\operatorname{grad}p = 0\,. \tag{4}$$

The excess pressure p is determined by two components: p_c, the pressure due to the compression of the air volume and p_l, the pressure loss due to air flow out of the air gap. Thus

$$p = p_c-p_l \qquad (r \leqslant a)\,. \tag{5}$$

If the initial separation of the air gap is d and an inward movement Z of one of the transducer surfaces occurs, then

$$p_c = p_0\{d/(d-Z)\}^\gamma-p_0\,. \tag{6}$$

Substituting expressions (5) and (6) into equation (4) we obtain

$$-\nabla^2 p_l = \frac{\rho_0}{d}\left\{\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}\left(1-\frac{Z}{d}\right)^{-\gamma-1}+\frac{1}{d}\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2\left(\gamma+1\right)\left(1-\frac{Z}{d}\right)^{-\gamma-2}\right\}-\frac{\rho_0}{K}\frac{\partial^2 p_l}{\partial t^2}\,. \tag{7}$$

Because of the assumed radial symmetry and because the pressure is independent of the z coordinate, equation (7) can be written in terms of cylindrical coordinates as

$$-\left(\frac{\partial^2 p_l}{\partial r^2}+\frac{1}{r}\frac{\partial p_l}{\partial r}\right) = \frac{\rho_0}{d}\left\{\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}\left(1-\frac{Z}{d}\right)^{-\gamma-1}+\frac{1}{d}\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2\left(\gamma+1\right)\left(1-\frac{Z}{d}\right)^{-\gamma-2}\right\}-\frac{\rho_0}{K}\frac{\partial^2 p_l}{\partial t^2}\,. \tag{8}$$

Equation (8) is a nonlinear partial differential equation that has to be solved for an assigned variation of the displacement Z with time and subject to the boundary conditions

$$p_l = 0 \quad \text{at} \quad r = a \qquad \text{and} \qquad p_l = 0 \quad \text{at} \quad t = 0\,.$$

At first sight it may appear that equation (8) could be solved only by purely numerical methods. However, we shall seek a solution in closed form by a combination of analytical and numerical methods.

Equation (8) can be reduced to an ordinary nonlinear type by removing the dependence of p_l on the radial coordinate r. The finite Hankel transform method (Sneddon 1946) devised from the properties of Fourier–Bessel series provides a systematic procedure for this reduction.

We multiply equation (8) by $r\mathrm{J}_0(r\zeta_i)$ and integrate between 0 and a, where the ζ_i are all the positive roots of $\mathrm{J}_0(a\zeta_i) = 0$. Thus

$$-\int_0^a \left(r.\frac{\partial^2 p_l}{\partial r^2}+\frac{\partial p_l}{\partial r}\right).\mathrm{J}_0(r\zeta_i)\,\mathrm{d}r$$
$$=\int_0^a \frac{\rho_0}{d}\left\{\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}\left(1-\frac{Z}{d}\right)^{-\gamma-1}+\frac{1}{d}\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2\left(\gamma+1\right)\left(1-\frac{Z}{d}\right)^{-\gamma-2}\right\}r.\mathrm{J}_0(r\zeta_i)\,\mathrm{d}r$$
$$-\frac{\rho_0}{K}\int_0^a \frac{\partial^2 p_l}{\partial t^2}.r\mathrm{J}_0(r\zeta_i)\,\mathrm{d}r.$$

Carrying out the integrations we obtain

$$a\zeta_i\,p_l(a)\,\mathrm{J}_0'(a\zeta_i)+\zeta_i^2\,\overline{H}(\zeta_i)$$
$$=\frac{\rho_0}{d}\left\{\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}\left(1-\frac{Z}{d}\right)^{-\gamma-1}+\frac{1}{d}\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2\left(\gamma+1\right)\left(1-\frac{Z}{d}\right)^{-\gamma-2}\right\}.\frac{a\mathrm{J}_1(a\zeta_i)}{\zeta_i}$$
$$-\frac{\rho_0}{K}\frac{\partial^2\overline{H}(\zeta_i)}{\partial t^2},$$

where $\overline{H}(\zeta_i)$ is defined as the Hankel transform of p_l. Because $p_l(a) = 0$, the boundary condition for $\overline{H}(\zeta_i)$ is $\overline{H}(\zeta_i) = 0$ at $r = a$ and $t = 0$.

The reduced differential equation for $\overline{H}(\zeta_i)$ is therefore

$$\frac{\partial^2\overline{H}}{\partial t^2}+\frac{K}{\rho_0}\zeta_i^2\,\overline{H}=\frac{K}{d}\left\{\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}\left(1-\frac{Z}{d}\right)^{-\gamma-1}+\frac{1}{d}\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2\left(\gamma+1\right)\left(1-\frac{Z}{d}\right)^{-\gamma-2}\right\}.\frac{a\mathrm{J}_1(a\zeta_i)}{\zeta_i}. \tag{9}$$

A solution of this reduced differential equation now has to be obtained in closed form so that it may be inverted to obtain the pressure p_l. The expression in braces is a highly nonlinear function of time t. A series expansion of this expression by the binomial theorem is found to converge too slowly for computational purposes. However, if a polynomial function of time t with a finite number of terms could be fitted to it, the equation could be integrated by the well-known D-operator methods (Ince 1949) and the resulting solution for the Hankel transform of the pressure p_l could be readily inverted.

Suppose then that the nonlinear expression can be fitted to a polynomial of the form

$$a_0+a_1t^2+a_2t^4+a_3t^6 \qquad (0 \leqslant t \leqslant t_1),$$

where the coefficients a_0, a_1, a_2, and a_3 are found by regression analysis.* Thus, in the interval $0 \leqslant t \leqslant t_1$, equation (9) can be put in the form

$$(\mathrm{D}^2+m^2)\bar{H} = \nu_i(a_0+a_1t^2+a_2t^4+a_3t^6)\,, \tag{10}$$

where

$$\nu_i = (K/d)a\mathrm{J}_1(a\zeta_i)/\zeta_i\,, \qquad m = (K\zeta_i^2/\rho_0)^{\frac{1}{2}}\,, \qquad \text{and} \quad \mathrm{D} \equiv \partial/\partial t\,.$$

The solution of equation (10) consists of the sum of a complementary function that is a solution of $(\mathrm{D}^2+m^2)\bar{H} = 0$ and of particular integrals of the terms of the right-hand side.

Using the result

$$\frac{1}{\mathrm{D}^2+m^2}a_rt^n = \frac{a_r}{m^2}\left(t^n - \frac{n(n-1)}{m^2}t^{n-2} + \frac{n(n-1)(n-2)}{m^4}t^{n-4} - \ldots\right),$$

the various particular integrals are:

$$r = 0, \qquad \bar{H}_1 = \frac{\nu_i a_0}{m^2}\,;$$

$$r = 1, \qquad \bar{H}_2 = \frac{\nu_i a_1}{m^2}\left(t^2 - \frac{2}{m^2}\right);$$

$$r = 2, \qquad \bar{H}_3 = \frac{\nu_i a_2}{m^2}\left(t^4 - \frac{12}{m^2}t^2 + \frac{24}{m^4}\right);$$

$$r = 3, \qquad \bar{H}_4 = \frac{\nu_i a_3}{m^2}\left(t^6 - \frac{30}{m^2}t^4 + \frac{120}{m^4}t^2 - \frac{360}{m^6}\right).$$

The complete solution is then

$$\begin{aligned}\bar{H} = A\cos mt + B\sin mt + \frac{\nu_i}{m^2}\Bigg\{&\left(a_0 - \frac{2a_1}{m^2} + \frac{24a_2}{m^4} - \frac{360a_3}{m^6}\right)\\ &+\left(a_1 - \frac{12a_2}{m^2} + \frac{120a_3}{m^4}\right)t^2\\ &+\left(a_2 - \frac{30a_3}{m^2}\right)t^4 + a_3t^6\Bigg\}.\end{aligned} \tag{11}$$

The least value of ζ_i is $2\cdot4048/a$ (Watson 1944), and therefore with $K = 1\cdot414\times 1\cdot01\times10^6$ dyn/cm^2 and $\rho = 1\cdot29\times10^{-3}$ g/cm^3 for air we find that the least value of m^2 is about 10^9. With the values of a_0, a_1, a_2, and a_3 encountered in the particular application, we need retain only the leading term in each coefficient of t^0, t^2, t^4, and t^6. This is demonstrated in a numerical example given in Section V. With this simplification the solution for $\bar{H}$ reduces to

$$\bar{H} = A\cos mt + B\sin mt + (\nu_i/m^2)(a_0+a_1t^2+a_2t^4+a_3t^6)\,. \tag{12}$$

* Computer programmes for rapidly carrying out this analysis are available for the Control Data 3600 computer. The programmes originated from the Michigan State University, U.S.A. The author wishes to acknowledge the help of Mr. C. H. Gray of the Computing Research Section, CSIRO, in this aspect of the work.

Now $\overline{H} = 0$ at $t = 0$ and therefore $A = -\nu_i a_0/m^2$. The final expression for $\overline{H}$ is

$$\overline{H} = (\nu_i a_0/m^2)(1-\cos mt)+(\nu_i/m^2)(a_1 t^2+a_2 t^4+a_3 t^6). \tag{13}$$

The Hankel inversion theorem then gives for the pressure loss

$$p_l = \frac{2}{a^2}\sum_i \overline{H}.\mathrm{J}_0(r\zeta_i)/\{\mathrm{J}_1(a\zeta_i)\}^2, \tag{14}$$

where the summation is taken over all positive roots of $\mathrm{J}_0(a\zeta_i) = 0$. It is not always possible to approximate, with one polynomial, the nonlinear term over its entire range. If three ranges are required for the approximation, we write the polynomials thus:

$$\left.\begin{array}{ll} a_{01}+a_{11}t^2+a_{21}t^4+a_{31}t^6, & (0 \leqslant t \leqslant t_1); \\ a_{02}+a_{12}t^2+a_{22}t^4+a_{32}t^6, & (t_1 \leqslant t \leqslant t_2); \\ a_{03}+a_{13}t^2+a_{23}t^4+a_{33}t^6, & (t_2 \leqslant t \leqslant t_3). \end{array}\right\} \tag{15}$$

The general solution for $\overline{H}$ will be of the same form within each range. However, it is necessary to make the relevant solutions agree at the transition times t_1 and t_2. A criterion of agreement which can be used is that $\overline{H}$ and its derivative be continuous at these values of t.

For example, for the range $t_1 \leqslant t \leqslant t_2$

$$\overline{H} = A_1\cos mt+B_1\sin mt+(\nu_i/m^2)(a_{02}+a_{12}t^2+a_{22}t^4+a_{32}t^6),$$

where A_1 and B_1 are determined by making $\overline{H}$ and $\partial\overline{H}/\partial t$ agree at $t = t_1$ with the solution for $\overline{H}$ in the interval $0 \leqslant t \leqslant t_1$. This latter solution has already been derived (equation (13)). On carrying out these steps we obtain

$$\begin{aligned} A_1 = {} & -(\nu_i/m^2)a_{01}+(\nu_i/m^2)\{(a_{01}-a_{02})+(a_{11}-a_{12})t^2+(a_{21}-a_{22})t^4+(a_{31}-a_{32})t^6\}\cos mt_1 \\ & -(\nu_i/m^3)\{2(a_{11}-a_{12})t_1+4(a_{21}-a_{22})t_1^3+6(a_{31}-a_{32})t_1^5\}\sin mt_1, \\ B_1 = {} & (\nu_i/m^2)\{(a_{01}-a_{02})+(a_{11}-a_{12})t_1^2+(a_{21}-a_{22})t_1^4+(a_{31}-a_{32})t_1^6\}\sin mt_1 \\ & +(\nu_i/m^3)\{2(a_{11}-a_{12})t_1+4(a_{21}-a_{22})t_1^3+6(a_{31}-a_{32})t_1^5\}\cos mt_1. \end{aligned}$$

In the third range $t_2 \leqslant t \leqslant t_3$

$$\overline{H} = A_2\cos mt+B_2\sin mt+(\nu_i/m^2)(a_{02}+a_{12}t^2+a_{22}t^4+a_{32}t^6),$$

where A_2 and B_2 are determined by making $\overline{H}$ and $\partial\overline{H}/\partial t$ agree with the previous solution at time $t = t_2$. The values are

$$\begin{aligned} A_2 = {} & A_1+(\nu_i/m^2)\{(a_{02}-a_{03})+(a_{12}-a_{13})t_2^2+(a_{22}-a_{23})t_2^4+(a_{32}-a_{33})t_2^6\}\cos mt_2 \\ & +(\nu_i/m^3)\{2(a_{13}-a_{12})t_2+4(a_{23}-a_{22})t_2^3+6(a_{33}-a_{32})t_2^5\}\sin mt_2, \\ B_2 = {} & B_1+(\nu_i/m^2)\{(a_{02}-a_{03})+(a_{12}-a_{13})t_2^2+(a_{22}-a_{23})t_2^4+(a_{32}-a_{33})t_2^6\}\sin mt_2 \\ & -(\nu_i/m^3)\{2(a_{13}-a_{12})t_2+4(a_{23}-a_{22})t_2^3+6(a_{33}-a_{32})t_2^5\}\cos mt_2. \end{aligned}$$

For a larger number of intervals we can proceed by induction to find the coefficients A and B.

V. Numerical Example

We now consider a realistic numerical example in order to assign magnitudes to the pressure components p_c and p_l. The following data are applicable

$$\text{radius of surface} \qquad a = 2\cdot 54 \text{ cm},$$

$$\text{initial separation} \qquad d = 0\cdot 254 \text{ cm}.$$

In the analysis, the form of the displacement Z of the moving surface has not been defined. Suppose then that at time $t = 0$ a constant acceleration f is applied to one

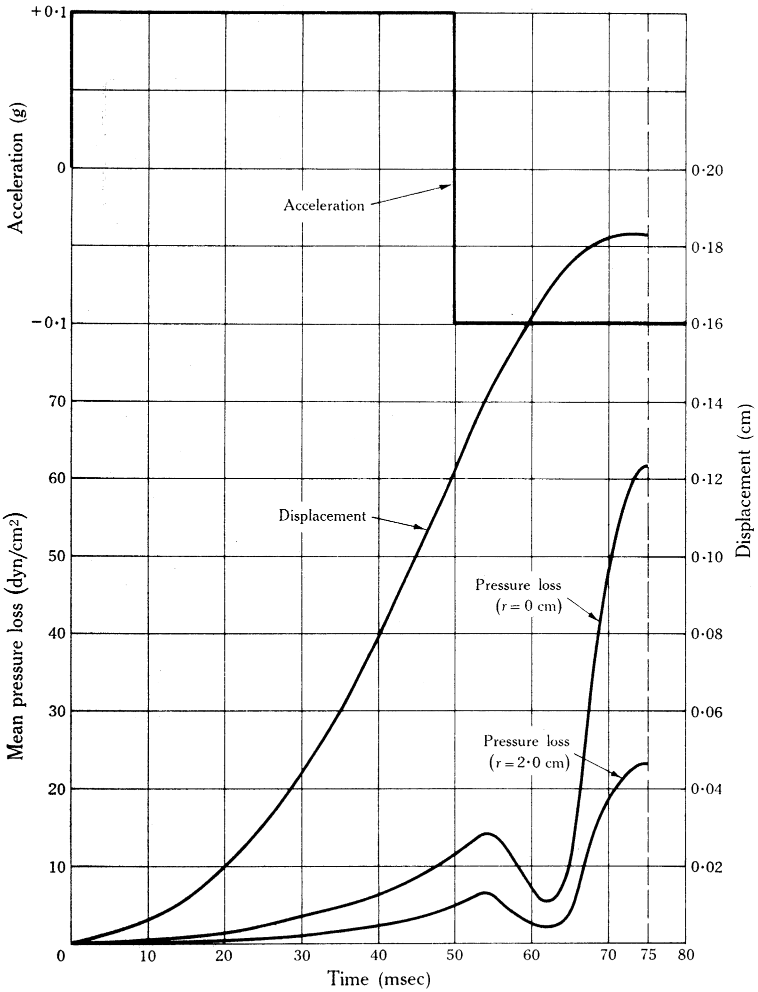

Fig. 3.—Displacement, acceleration, and pressure variation with time in the air gap.

of the surfaces, reducing the initial separation d by the amount Z. At time t the value of Z is $\frac{1}{2}ft^2$ and if at a later time $t = T_0$ the acceleration is reversed to a value $-f$, the subsequent displacement is $Z = fT_0 t - ft^2$. This means that the maximum displacement occurs at a time $t = \frac{3}{2}T_0$ and has the value $Z_{\max} = \frac{3}{4}fT_0^2$.

In equation (9), the nonlinear term is evaluated over the range $0 \leqslant t \leqslant \frac{3}{2}T_0$ using the formulae

$$Z = \tfrac{1}{2}ft^2, \qquad t \leqslant T_0;$$

$$Z = (fT_0t - ft^2) + \tfrac{1}{2}ft^2, \qquad t \geqslant T_0;$$

where $f = 0 \cdot 1g$ and $T_0 = 50 \times 10^{-3}$ sec.

Regression analysis is then applied to fit the nonlinear term in three parts to the polynomials (15). The polynomial coefficients a_{ij} are

$a_{01} = 97 \cdot 98$, $\quad a_{02} = 185 \cdot 34$, $\quad a_{03} = -84649 \cdot 27$,

$a_{11} = 13 \cdot 78 \times 10^4$, $\quad a_{12} = 1 \cdot 70 \times 10^5$, $\quad a_{13} = 800 \cdot 157 \times 10^5$,

$a_{21} = 0 \cdot 679 \times 10^8$, $\quad a_{22} = 0 \cdot 109 \times 10^9$, $\quad a_{23} = -24 \cdot 269 \times 10^9$,

$a_{31} = 0 \cdot 046 \times 10^{12}$, $\quad a_{32} = 0 \cdot 001 \times 10^{13}$, $\quad a_{33} = 0 \cdot 24 \times 10^{13}$.

The justification for the neglect of the terms involving a_1/m^2, a_2/m^2, . . . in equation (11) is obvious by inspection.

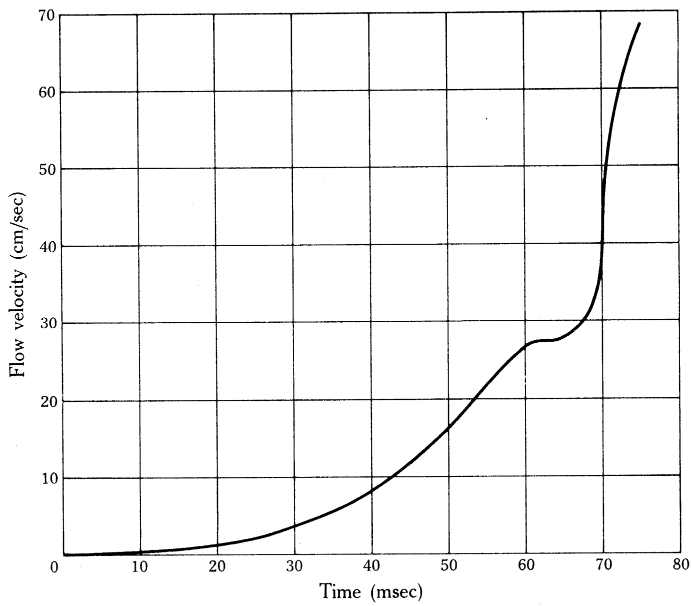

Fig. 4.—Velocity of air flow at a position 1 cm from the central axis of the gap.

The pressure loss is then calculated by summing the series on the right-hand side of equation (14) to 40 terms; this number was found to be necessary to ensure convergence of the series to an acceptable accuracy. The expression for $\overline{H}$ is, of course, the relevant one in a particular range of the time t. The mean pressure loss has been computed on the Control Data 3600 computer against time t at two radial positions in the gap: at the centre and at a distance $2 \cdot 0$ cm from the centre. The two pressure–time curves are shown in Figure 3, together with the displacement and acceleration of the moving surface. The maximum pressure loss occurs at the centre of the gap and amounts to $61\cdot5$ dyn/cm^2 after an elapsed time of 75 msec. It would serve no purpose to compute the pressure loss beyond the time corresponding to the minimum gap, since on physical grounds the loss would begin to decrease as the gap started to increase.

The inflexions in the pressure curves appear to be related to the change in velocity of the moving surface when the reverse acceleration is applied to it. The mean velocity of air flow can be found in an approximate way at a distance $r = 1$ cm from the axis of the gap. The momentum equation (2) may be written

$$\left(\frac{\partial u}{\partial t}\right)_{r=1\,\mathrm{cm}} \simeq -\frac{1}{\rho_0}\frac{p_2-p_0}{r_2-r_0},$$

where p_2, p_0 are the pressures, at time t, at radial positions r_2, r_0. In this case $r_2 = 2$ cm and $r_0 = 0$. Using the values of p at time t obtained from Figure 3, graphical integration then gives the velocity at time t at $r = 1$ cm. As shown in Figure 4, the variation of velocity indicates a slowing of the air motion when the reverse acceleration is suddenly applied to the moving surface.

VI. Calculation of the Correction

From equation (6), the excess pressure component is

$$p_c = p_0\{d/(d-Z)\}^{\gamma}-p_0 .$$

Evaluating this expression for the maximum displacement of the surface $Z_{max} = 0\cdot183$ cm, it is found that $p_c = 6\cdot055p_0$.

It is now evident from the calculation made in Section V that the maximum pressure loss is about 10^{-5} of the maximum excess pressure computed above for the effect of volume change. This is so despite the fact that compression involves 70% reduction of the initial gap. The pressure loss during the transient compression can thus be ignored and the pressure change can be estimated by knowing only the initial and final values of the surface separation.

In order to relate this pressure change to the dielectric constant of the air, use is made of the Lorentz–Lorenz law, which states that the molar refractivity A is given by

$$A = \frac{R\theta}{p}\frac{n^2-1}{n^2+2}, \tag{16}$$

where θ is the absolute temperature, R is the universal gas constant, and p (the pressure) is a constant for gases over a very wide range of pressures. Born and Wolf (1959) quote the value of A as $2\cdot17$, which will be used in the calculation that follows.

The absolute temperature θ at the end of the compression is calculated from the formula

$$\theta = \theta_0\{d/(d-Z)\}^{\gamma-1},$$

where θ_0 is taken as 293°K (normal room temperature). In this example θ at the end of compression is found to be 494°K. Taking the value of the gas constant R as $8\cdot31\times10^{7}$ erg mole^{-1} degK^{-1}, it is found that

$$n^2 = \epsilon = 1\cdot10016\,,$$

whereas for air at normal temperature and pressure

$$n^2 = \epsilon = 1\cdot00000\,.$$

Thus, the dielectric constant ϵ changes by about 10% in this example, and this is the correction to be applied to the measured value of capacitance change. The correction to the refractive index is found to be 5%.

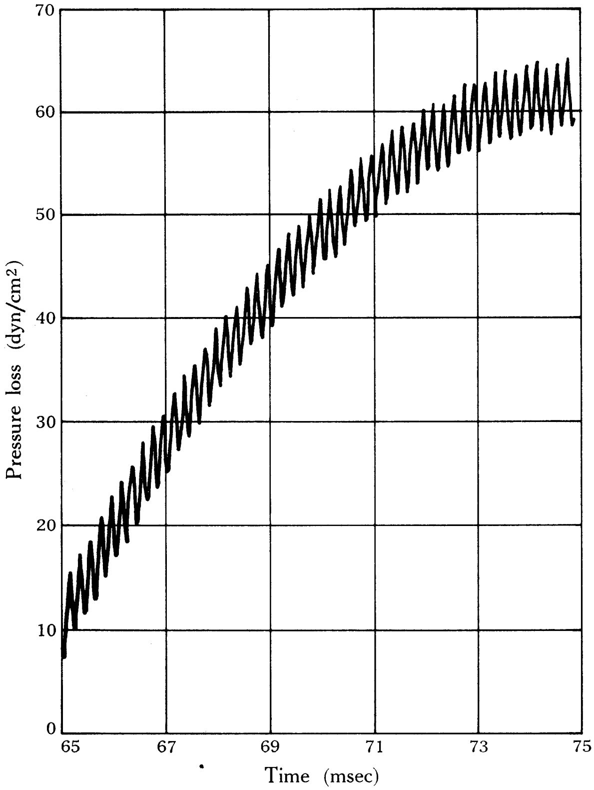

Fig. 5.—Illustrating the periodic pressure fluctuation that is probably related to a vibration mode of the air within the cylindrical gap.

VII. Other Phenomena that may occur in the Air Gap

The physical significance of the periodic terms in equations (12) and (14) is of interest. If the instantaneous pressure is calculated as a function of time, it is found that a periodic fluctuation in pressure is superposed on the mean value. For example, a part of the graph obtained on the computer is shown in Figure 5; the graph shows a fundamental fluctuation frequency of about 5 kHz. This pressure fluctuation could be associated with a mode of vibration of the cylindrical volume of air contained between the surfaces. The mode frequency would depend on the radius a and, in fact, it is found that if the radius is doubled, for example, the calculated frequency is halved.

An experiment to detect this pressure oscillation is at present being considered: as the velocity of air flow at the boundary also contains a fluctuating component at the same frequency, this could give rise to an acoustically radiated component of pressure at the boundary which might possibly be measurable with a probe microphone. The experimental detection of this phenomenon, while of academic interest only, may provide some confirmation of the theory.

Quite independently of the above phenomena, there is also the possibility of a weak acoustical wave arising in the air gap owing to the disturbance of the air by the transient motion, since equation (7) is a form of wave equation for the pressure loss. However, the form of solution obtained here for this pressure does not exhibit a wave disturbance explicitly. The physical effect of such a disturbance would in any case be extremely small in this application.

VIII. Conclusions

The pressure changes that occur in the air gap of a parallel plate transducer when a transient motion is applied to one surface have been examined and the influence of these pressure changes on the refractive index and dielectric constant of the enclosed air has been considered.

In a numerical example for a typical system it was shown that the corrections required for the refractive index and dielectric constant may be significant and can be worked out on the basis of volume change of air in the gap because the pressure loss due to air flowing out at the system boundary is negligible during the period up to the maximum compression.

Although it would be unwise to generalize from a particular case, it would seem that in other cases in which the ratio of the radius to the plate separation is large throughout the motion, a similar result might reasonably be expected.

IX. References

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