# STRUCTURE OF POLYTROPES IN MAGNETOSTATIC EQUILIBRIUM

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*Summary*

This paper investigates the structure of a polytrope in which a magnetic field, having both poloidal and toroidal components, is present. The influence of the toroidal component on the oblateness of the configuration is also considered.

## I. Introduction

In a recent series of papers, Monaghan (1965, 1966*a*) has investigated the structure of a poloidal magnetic field in polytropes and its effect on their structure. Using a pseudo-polytropic approximation he was then able, with the help of a perturbation method, to investigate the structure of such fields in upper main sequence stars (Monaghan 1966*b*).

The problem of the structure of a purely toroidal field was considered previously by Roxburgh (1963*a*, 1963*b*) and again recently (Roxburgh 1966). In this last paper Roxburgh investigates the structure of a magnetic field in polytropes when the field has both poloidal and toroidal components. This form of the magnetic field was considered earlier by Woltjer (1960) and Wentzel (1961) in the case of simple density distributions.

In the present investigation we study the effect of such a field on the structure of a polytrope and the dependence of the oblateness of the configuration on the strength of the toroidal component.

## II. Basic Equations

In the absence of rotation and if no circulation is present the equation of hydrostatic equilibrium can be written

$$\rho^{-1}\,\mathrm{grad}\,p = -\,\mathrm{grad}\,\Phi + (c\rho)^{-1}\,\mathbf{j}\times\mathbf{H}\,, \tag{1}$$

where $p$ is the pressure, $\rho$ the density, $\Phi$ the gravitational potential, $\mathbf{j}$ the current density, and $\mathbf{H}$ the magnetic field intensity.

In addition the following equations must be satisfied: the Poisson equation,

$$\nabla^2\Phi = 4\pi G\rho\,, \tag{2}$$

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the Maxwell equations,

$$\operatorname{curl} \mathbf{H} = (4\pi/c)\mathbf{j} \tag{3}$$

and

$$\operatorname{div} \mathbf{H} = 0\,, \tag{4}$$

and the equation of state,

$$p = K\rho^{1+n^{-1}}\,, \tag{5}$$

where $K$ is a constant and $n$ is the polytropic index. Following Roxburgh (1966), we shall consider a magnetic field of the form

$$\mathbf{H} = (H_r, H_\theta, H_\phi) = \left(\frac{2A\cos\theta}{r^2}, \frac{-A'\sin\theta}{r}, \frac{CA\sin\theta}{r}\right), \tag{6}$$

where $C$ is a constant, $A(r)$ is a function of $r$ to be determined, and a prime denotes differentiation with respect to $r$. It is easily seen that in this case equation (4), that is,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 H_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, H_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(H_\phi\right) = 0\,,$$

is satisfied.

Eliminating $\mathbf{j}$ between equations (3) and (1), we have

$$\rho^{-1}\operatorname{grad} p = -\operatorname{grad}\Phi + (4\pi\rho)^{-1}\operatorname{curl}\mathbf{H}\times\mathbf{H} \tag{7}$$

and we are therefore interested in the components of the vector

$$\mathscr{H} = \operatorname{curl}\mathbf{H}\times\mathbf{H}$$

for the magnetic field defined in (6).

It can be shown that

$$\left.\begin{aligned} \mathscr{H}_r &= -\frac{A'\sin^2\theta}{r^2}\left(A'' - \frac{2A}{r^2} + C^2 A\right), \\ \mathscr{H}_\theta &= -\frac{2A\sin\theta\cos\theta}{r^3}\left(A'' - \frac{2A}{r^2} + C^2 A\right), \\ \mathscr{H}_\phi &= 0\,. \end{aligned}\right\} \tag{8}$$

The $r$ and $\theta$ components of equation (7) can then be written

$$\frac{\partial p}{\partial r} = -\rho\frac{\partial\Phi}{\partial r} - \frac{1}{4\pi}\frac{A'\sin^2\theta}{r^2}F(A)\,, \qquad \frac{\partial p}{\partial\theta} = -\rho\frac{\partial\Phi}{\partial\theta} - \frac{1}{4\pi}\frac{2A\sin\theta\cos\theta}{r^2}F(A)\,, \tag{9}$$

where

$$F(A) = A'' - 2Ar^{-2} + C^2 A\,. \tag{10}$$

We now consider the case of a magnetic field of small intensity and we write

$$A(r) = \lambda \bar{A}(r),$$

where $\lambda$ is a small parameter and the corresponding perturbations $p_d$, $\rho_d$, and $\Phi_d$ of the pressure, density, and gravitational potential can then be written

$$p = p_0 + \lambda^2 p_d, \qquad \rho = \rho_0 + \lambda^2 \rho_d, \qquad \Phi = \Phi_0 + \lambda^2 \Phi_d, \tag{11}$$

$p_0$, $\rho_0$, and $\Phi_0$ being the equilibrium values of these variables in the absence of a magnetic field.

Substituting these equations in (9) and equating coefficients of $\lambda^2$, we obtain

$$\left.\begin{aligned} \frac{\partial p_d}{\partial r} &= -\rho_0 \frac{\partial \Phi_d}{\partial r} - \rho_d \frac{\partial \Phi_0}{\partial r} - \frac{1}{4\pi}\frac{\bar{A}'}{r^2}\sin^2\theta\, F(\bar{A}), \\ \frac{\partial p_d}{\partial \theta} &= -\rho_0 \frac{\partial \Phi_d}{\partial \theta} - \frac{1}{4\pi}\frac{\bar{A}\, F(\bar{A})}{r^2} 2\sin\theta\cos\theta. \end{aligned}\right\} \tag{12}$$

In these equations $\rho_d$, $p_d$, and $\Phi_d$ are functions of $r$ and $\theta$, and we now assume a $\theta$ dependence of the form

$$\rho_d = \rho_1 + \rho_2 \sin^2\theta, \qquad p_d = p_1 + p_2 \sin^2\theta, \qquad \Phi_d = \Phi_1 + \Phi_2 \sin^2\theta, \tag{13}$$

where $\rho_1$, $p_1$, $\Phi_1$, $\rho_2$, $p_2$, $\Phi_2$ are functions of $r$ only. Substituting these expressions in (12), we have

$$p_1' = -\rho_0 \Phi_1' - \rho_1 \Phi_0', \tag{14}$$

$$p_2' = -\rho_0 \Phi_2' - \rho_2 \Phi_0' - \frac{1}{4\pi}\frac{\bar{A}'\, F(\bar{A})}{r^2}, \tag{15}$$

$$p_2 = -\rho_0 \Phi_2 - \frac{1}{4\pi}\frac{\bar{A}\, F(\bar{A})}{r^2}. \tag{16}$$

It can be shown, using the above variables, that the Poisson equation (2) leads to the following equations

$$\left.\begin{aligned} \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi_1}{\mathrm{d}r}\right) + \frac{4\Phi_2}{r^2} &= 4\pi G \rho_1, \\ \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi_2}{\mathrm{d}r}\right) - \frac{6\Phi_2}{r^2} &= 4\pi G \rho_2, \end{aligned}\right\} \tag{17}$$

whereas the equation of state (5) leads to the equations

$$p_1 = K(1+n^{-1})\rho_0^{n^{-1}} \rho_1, \qquad p_2 = K(1+n^{-1})\rho_0^{n^{-1}} \rho_2. \tag{18}$$

In the case of a polytropic equation of state, or in fact for any barotropic law, the above equations can be considerably simplified. This follows from the fact that, in this case, we can derive from equation (1) the simple result

$$\operatorname{curl}\{\rho^{-1}(\mathbf{H}\times\operatorname{curl}\mathbf{H})\} = 0\,. \tag{19}$$

Differentiating (16) with respect to $r$ and eliminating $p_2'$ between this equation and (15), we obtain

$$\frac{p_2}{\rho_0^2}\rho_0' = -\frac{\rho_2}{\rho_0}\Phi_0' + \frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\bar{A}\,F(\bar{A})}{\rho_0 r^2}\right) - \frac{1}{4\pi}\left(\frac{\bar{A}'\,F(\bar{A})}{\rho_0 r^2}\right). \tag{20}$$

Substituting $p_2$ from (18),

$$\frac{\rho_2}{\rho_0}\left(\frac{K(1+n^{-1})\rho_0^{n^{-1}}\rho_0'}{\rho_0} + \Phi_0'\right) = \frac{1}{4\pi}\bar{A}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{F(\bar{A})}{\rho_0 r^2}\right).$$

Since

$$p_0 = K\rho_0^{1+n^{-1}},$$

we have

$$p_0' = K(1+n^{-1})\rho_0^{n^{-1}}\rho_0'$$

and the above equation can be written

$$\frac{\rho_2}{\rho_0}\left(\frac{p_0'}{\rho_0} + \Phi_0'\right) = \frac{1}{4\pi}\bar{A}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{F(\bar{A})}{\rho_0 r^2}\right).$$

Since $p_0'/\rho_0' = -\Phi_0'$ from the equation of hydrostatic equilibrium, in the absence of a magnetic field, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{F(\bar{A})}{\rho_0 r^2}\right) = 0 \qquad \text{or} \qquad F(\bar{A}) = -D\,\rho_0 r^2, \tag{21}$$

where $D$ is a constant. We can therefore use equation (21) instead of (15) and we finally obtain the following system of differential equations

$$\frac{\mathrm{d}p_1}{\mathrm{d}r} = -\rho_0 f_1 - \frac{p_1}{K(1+n^{-1})\rho_0^{n^{-1}}}\Phi_0', \tag{22}$$

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}r} = f_1, \tag{23}$$

$$\frac{\mathrm{d}\Phi_2}{\mathrm{d}r} = f_2, \tag{24}$$

$$\frac{\mathrm{d}f_1}{\mathrm{d}r} = -\frac{2}{r}f_1 - \frac{4}{r^2}\Phi_2 + \frac{4\pi G p_1}{K(1+n^{-1})\rho_0^{n^{-1}}}, \tag{25}$$

$$\frac{\mathrm{d}f_2}{\mathrm{d}r} = -\frac{2}{r}f_2 + \frac{6}{r^2}\Phi_2 + \frac{4\pi G p_2}{K(1+n^{-1})\rho_0^{n^{-1}}}, \tag{26}$$

where

$$p_2 = -\rho_0 \Phi_2 - \frac{1}{4\pi}\frac{\bar{A}\,F(\bar{A})}{r^2}, \tag{27}$$

$$\bar{A}'' - \frac{2\bar{A}}{r^2} + C^2\bar{A} = -D\,\rho_0 r^2. \tag{28}$$

## III. Integration of the Equations

We now introduce a dimensionless system of variables $\xi$, $\gamma$, $\bar{f}_1$, $\bar{f}_2$, $\bar{\Phi}_1$, $\bar{\Phi}_2$, $\bar{p}_1$, and $\bar{p}_2$, which are defined by the following equations

$$\left.\begin{aligned}
&\bar{A} = D\,\rho_c a^4\gamma, && && a = \left(\frac{K(1+n)\rho_c^{\,n^{-1}-1}}{4\pi G}\right)^{\frac{1}{2}},\\
&Ca = \alpha, && r = a\xi, && \rho_0 = \rho_c\theta_0^n,\\
&f_2 = \rho_c\frac{D^2a^3}{4\pi}\bar{f}_2, && \Phi_2 = \rho_c\frac{D^2a^4}{4\pi}\bar{\Phi}_2, && p_2 = \rho_c^2\frac{D^2a^4}{4\pi}\bar{p}_2,\\
&f_1 = \rho_c\frac{D^2a^3}{4\pi}\bar{f}_1, && \Phi_1 = \rho_c\frac{D^2a^4}{4\pi}\bar{\Phi}_1, && p_1 = \rho_c^2\frac{D^2a^4}{4\pi}\bar{p}_1,
\end{aligned}\right\} \tag{29}$$

where $\theta_0$ satisfies the Emden equation.

Using these new variables we obtain the following system of equations

$$\gamma'' - \frac{2\gamma}{\xi^2} + \alpha^2\gamma = -\theta_0^n\,\xi^2, \tag{30}$$

$$\bar{f}_2' = -\frac{2}{\xi}\bar{f}_2 + \frac{6}{\xi^2}\bar{\Phi}_2 + n\theta_0^{n-1}(-\bar{\Phi}_2+\gamma), \tag{31}$$

$$\bar{f}_1' = -\frac{2}{\xi}\bar{f}_1 - \frac{4}{\xi^2}\bar{\Phi}_2 + \frac{n}{\theta_0}\bar{p}_1, \tag{32}$$

$$\bar{p}_1' = -\theta_0^n\bar{f}_1 + \frac{n}{\theta_0}\bar{p}_1\frac{d\theta_0}{d\xi}, \tag{33}$$

$$\bar{\Phi}_1' = \bar{f}_1, \tag{34}$$

$$\bar{\Phi}_2' = \bar{f}_2, \tag{35}$$

$$\bar{p}_2 = (-\bar{\Phi}_2+\gamma)\theta_0^n. \tag{36}$$

The above equations were integrated, using a Runge–Kutta method, together with the Emden equation

$$\frac{d^2\theta_0}{d\xi^2} + \frac{2}{\xi}\frac{d\theta_0}{d\xi} + \theta_0^n = 0.$$

The following expansions at the centre were used

$$\left.\begin{aligned} \gamma = \gamma_0\,\xi^2, \qquad \bar{\Phi}_2 = \bar{\Phi}_{02}\,\xi^2, \qquad \bar{\Phi}_1 = -\tfrac{2}{3}\bar{\Phi}_{02}\,\xi^2, \\ \bar{p}_1 = \tfrac{2}{3}\bar{\Phi}_{02}\,\xi^2, \qquad \theta_0 = 1-\tfrac{1}{6}\xi^2. \end{aligned}\right\} \tag{37}$$

Equation (30) is identical to the one used by Roxburgh (1966) and can be integrated independently of the other equations.

TABLE 1

STRUCTURE OF MAGNETIC FIELD AND CORRESPONDING PERTURBATIONS IN PRESSURE AND GRAVITATIONAL POTENTIAL AS A FUNCTION OF RADIUS FOR A POLYTROPE OF INDEX 3

| $x = \xi/\xi_1$ | $\gamma/\gamma_0\,\xi^2$ | $\bar{\Phi}_2/\bar{\Phi}_{02}\,\xi^2$ | $\bar{\Phi}_1/-\frac{2}{3}\bar{\Phi}_{02}\,\xi^2$ | $\bar{p}_2/(-\bar{\Phi}_{02}+\gamma_0)\xi^2$ | $\bar{p}_1/\frac{2}{3}\bar{\Phi}_{02}\,\xi^2$ |
|---|---|---|---|---|---|
| 0 | 1·0000 | 1·0000 | 1·0000 | 1·0000 | 1·0000 |
| 0·1 | 0·7609 | 0·7876 | 0·8706 | 0·6131 | 0·6916 |
| 0·2 | 0·3905 | 0·4191 | 0·6012 | 0·1722 | 0·2573 |
| 0·3 | 0·1587 | 0·1803 | 0·3680 | 0·0303 | 0·0665 |
| 0·4 | 0·0576 | 0·0728 | 0·2183 | 0·0043 | 0·0147 |
| 0·5 | 0·0196 | 0·0300 | 0·1323 | 0·0005 | 0·0030 |
| 0·6 | 0·0061 | 0·0132 | 0·0837 | 0·0001 | 0·0006 |
| 0·7 | 0·0016 | 0·0063 | 0·0558 | 0·0000 | 0·0001 |
| 0·8 | 0·0003 | 0·0033 | 0·0391 | 0·0000 | 0·0000 |
| 0·9 | 0·0000 | 0·0018 | 0·0286 | 0·0000 | 0·0000 |
| 1·0 | 0·0000 | 0·0011 | 0·0218 | 0·0000 | 0·0000 |

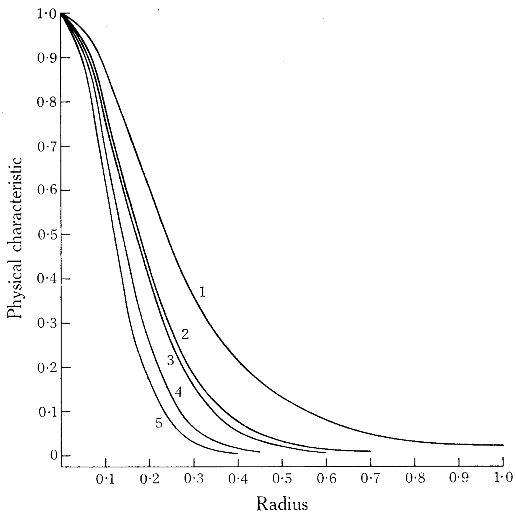


Fig. 1.—Variation of the physical characteristics as a function of radius for a polytrope of index 3. Curves 1 and 2 illustrate the perturbation in gravitational potential, curve 3 the structure of the magnetic field, and curves 4 and 5 the perturbation of the pressure:

1, $\bar{\Phi}_1/-\frac{2}{3}\bar{\Phi}_{02}\,\xi^2$
2, $\bar{\Phi}_2/\Phi_{02}\,\xi^2$
3, $\gamma/\gamma_0\,\xi^2$
4, $\bar{p}_1/\frac{2}{3}\,\bar{\Phi}_{02}\,\xi^2$
5, $\bar{p}_2/(-\bar{\Phi}_{02}+\gamma_0)\xi^2$

The expansion for $\gamma$ in the outer layers of the polytrope can be written

$$\gamma = -\frac{\theta_{01}^n\,\xi_1^2}{(n+2)(n+1)}\left(\xi_1-\xi\right)^{n+2}, \tag{38}$$

where $\theta_{01} = -(\mathrm{d}\theta_0/\mathrm{d}\xi)_{\xi=\xi_1}$ and $\xi_1$ are given in tables of Emden functions.

The integrations were started from the centre of the configuration, the values of the parameters $\gamma_0$ and the eigenvalue $\alpha$ being determined to ensure continuity of $\gamma$ and its first derivative with the expansion given in formula (38). Once these two parameters have been found, for a given polytropic index it is possible to integrate equations (31) and (35). The integration can also be started at the centre, selecting a value of $\bar{\Phi}_{02}$ such that $\Phi_2 = B/\xi^3$ in the outer layers of the star, i.e. in such a way that the boundary condition

$$\left(\xi\frac{\mathrm{d}\bar{\Phi}_2}{\mathrm{d}\xi}+3\bar{\Phi}_2\right)_{\xi=\xi_1} = 0$$

is satisfied.

In Table 1 we give the results of such an integration for the fundamental mode corresponding to the polytropic index $n = 3$. The structure of the magnetic field and the corresponding perturbation values in pressure and gravitational potential are illustrated in Figure 1.

## IV. Oblateness of the Configuration

From the previous analysis

$$\Phi = \Phi_0+\lambda^2\rho_c(D^2a^4/4\pi)(\bar{\Phi}_1+\bar{\Phi}_2\sin^2\theta)\,. \tag{39}$$

If the outer boundary is defined by the condition

$$\Phi(\xi_0) = 0\,, \tag{40}$$

and keeping in mind that

$$\Phi_0(\xi_0) = -(\mathrm{d}\Phi_0/\mathrm{d}\xi)_{\xi_1}(\xi_0-\xi_1)$$

and

$$\Phi_0 = -K(n+1)\rho_c^{n^{-1}}\theta_0\,,$$

it follows that condition (40) can be written

$$0 = -K(n+1)\rho_c^{n^{-1}}(\mathrm{d}\theta_0/\mathrm{d}\xi)_{\xi_1}(\xi_0-\xi_1)+(\mu^2\rho_c a^4/4\pi)\{\bar{\Phi}_1(\xi_1)+\bar{\Phi}_2(\xi_1)\sin^2\theta\}$$

or

$$\xi_0 = \xi_1+\frac{\mu^2\rho_c a^4}{4\pi K(n+1)\rho_c^{n^{-1}}(\mathrm{d}\theta_0/\mathrm{d}\xi)_{\xi_1}}\{\bar{\Phi}_1(\xi_1)+\bar{\Phi}_2(\xi_1)\sin^2\theta\}\,, \tag{41}$$

where $\mu = \lambda D$.

The oblateness of the configuration is then given by

$$\frac{\bar{\Phi}_2(\xi_1)}{\xi_1}\frac{\mu^2\rho_c^{1-n^{-1}}a^4}{4\pi K(n+1)(\mathrm{d}\theta_0/\mathrm{d}\xi)_{\xi_1}}\,. \tag{42}$$

Values of $\bar{\Phi}_2/\bar{\Phi}_{02}\,\xi_1^2$ are given in Table 2 for various modes, corresponding to increasing importance of the toroidal component of the configuration, in the case of a polytrope

TABLE 2

DEPENDENCE OF SHAPE OF CONFIGURATION ON MODE

| Mode | $\bar{\Phi}_2(\xi_1)/\bar{\Phi}_{02}\,\xi_1^2$ | $\bar{\Phi}_1(\xi_1)/-\frac{2}{3}\bar{\Phi}_{02}\,\xi_1^2$ | $\bar{\Phi}_{02}$ |
|---|---|---|---|
| 1 | $1\cdot075\times10^{-3}$ | $2\cdot178\times10^{-2}$ | $1\cdot766\times10^{-2}$ |
| 2 | $1\cdot084\times10^{-3}$ | $2\cdot199\times10^{-2}$ | $1\cdot524\times10^{-2}$ |
| 3 | $1\cdot091\times10^{-3}$ | $2\cdot215\times10^{-2}$ | $1\cdot320\times10^{-2}$ |
| 4 | $1\cdot097\times10^{-3}$ | $2\cdot228\times10^{-2}$ | $1\cdot146\times10^{-2}$ |
| 5 | $1\cdot102\times10^{-3}$ | $2\cdot238\times10^{-2}$ | $1\cdot006\times10^{-2}$ |

of index 3. It can be seen that the oblateness increases with the mode, but not to any marked degree. The table also shows contraction of the star for a polytropic index of 3, as found by Monaghan (1966*a*) in the case of a poloidal field. This contraction increases with the mode, but only slightly.

## V. REFERENCES

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