# ANGULAR VELOCITY DISTRIBUTION IN ROTATING MASSIVE STARS

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*Summary*

The angular velocity distribution in rotating massive stars with uniform composition and opacity due to electron scattering is calculated on the assumption that meridional circulation is neglible. The effects of radiation pressure are taken into account in the determination of the differential rotation and the angular velocity is assumed to be independent of latitude.

## I. Introduction

The equilibrium configurations of rotating upper main sequence stars have been determined previously by Schwarzschild (1947) and Roxburgh (1964) under the following assumptions:

(1) The star has reached a steady state in which there is no meridional circulation.

(2) The star is of uniform composition and all energy generation takes place in the convective core.

(3) In the convective core turbulent viscosity is isotropic and the core therefore rotates as a solid body.

(4) Radiation pressure is negligible.

A further property of massive stars is that it is possible to describe their structure in terms of the mass-composition factor

$$\mathscr{M} = \mu_e^2 M/M_\odot ,$$

where $M$ is the mass of the star and $\mu_e$ the molecular weight of stellar material in the outer layers of the star. In terms of the factor $\mathscr{M}$ it is possible to determine when the assumption (4) above becomes invalid (Van der Borght and Meggitt 1963). For pure hydrogen this corresponds to a mass of the order of $40M_\odot$ ($\mathscr{M} \simeq 10$) while for pure helium stars it is of the order of $9M_\odot$.

The effects of radiation pressure are noticed in the increase of central density, the decrease in luminosity, and the increase in size of the convective core, and it is to be expected that there should be some corresponding change in the angular velocity distribution. The aim of this paper is to derive the equations governing the structure of a rotating massive star with $\mathscr{M} > 10$ and the method used follows closely that of Schwarzschild and of Roxburgh in that we consider the case of small rotations treated as a perturbation about an equilibrium state in which there is no rotation.

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## II. Structure Equations

Taking radiation pressure into account, the structure equations of Roxburgh (1964) must be modified and can be written in the form

$$\nabla P/\rho = -\nabla\Phi + \Omega^2\boldsymbol{\omega}\,, \tag{1}$$

$$\nabla^2\Phi = 4\pi\mathscr{G}\rho\,, \tag{2}$$

$$P = \mathscr{R}\rho T/\mu + \tfrac{1}{3}aT^4\,. \tag{3}$$

where $\Omega$, $\boldsymbol{\omega}$, $\mathscr{G}$, $a$, and $\mathscr{R}$ denote the angular velocity, the vectorial distance from the rotation axis, and the gravitational, radiation, and gas constants respectively. The radiative equations become

$$\mathbf{F} = -(4ac/3)(T^3/\kappa\rho)\nabla T\,, \tag{4}$$

$$\nabla.\mathbf{F} = 0\,. \tag{5}$$

The convective equation becomes

$$\mathrm{d}(\ln P)/\mathrm{d}(\ln T) = \Gamma_1/(\Gamma_3 - 1)\,, \tag{6}$$

where

$$\Gamma_1 = \beta + (4-3\beta)^2(\gamma-1)/\{\beta+12(\gamma-1)(1-\beta)\}$$

and

$$\Gamma_3 = 1 + (\Gamma_1 - \beta)/(4-3\beta)$$

are the generalized adiabatic indices (see Ledoux and Walraven 1958), $\beta = P_g/P$ is the ratio of gas pressure $P_g$ to total pressure $P$, $\mu$ is the molecular weight of stellar material, $\Phi$ is the gravitational potential, and $\mathbf{F}$ is the radiative flux.

In the present case we assume that the stellar material can be treated as a monatomic gas and that consequently we may take the ratio of specific heats $\gamma = 5/3$. The opacity $\kappa$ for pure electron scattering is given by

$$\kappa = 0\cdot 2004\,(1+X)\,,$$

where $X$ is the abundance by weight of hydrogen.

Since the energy generation takes place entirely within the core, the energy equation

$$L = \int_{\text{volume}} \rho\epsilon\,\mathrm{d}v \tag{7}$$

can be detached from the main structure equations and evaluated where necessary as a separate quadrature.

Two other properties of massive stars enable the problem to be dealt with more simply and with slightly greater generality. Firstly, it is convenient to eliminate the unknown radius by using as variable $RT$ instead of $T$, and, secondly, in the case of uniform composition and electron scattering opacity it is possible to eliminate explicit reference to $\mu$ so that the results obtained can be applied to all massive stars with uniform composition.

As in previous work (Schwarzschild 1947; Roxburgh 1964; Roxburgh and Strittmatter 1966), we make no attempt to solve the structure equations explicitly but look for a solution in the case of small rotations that is a perturbation of an equilibrium state with no rotation. This is obtained in terms of the expansion parameter

$$\lambda = \Omega_c^2 R^3/\mathscr{G}M\,, \tag{8}$$

where $\Omega_c$ is the angular velocity of the uniformly rotating core. In order to obtain a system of equations in variables compatible with those of Van der Borght (1964), we make a change of variable and expand as:

$$\left.\begin{aligned}
(R^4\mu_e^4/M_\odot^2)P &= \bar{p} = \bar{p}_u + \lambda(\bar{p}_u/\mathscr{R}\bar{t}_u)\bar{p}_d\,,\\
(R\mu_e/M_\odot)T &= \bar{t} = \bar{t}_u + \lambda\bar{t}_d\,,\\
(4\pi r^2 F_r/L) &= \bar{l} = 1 + \lambda(x\bar{t}_u^4/\bar{c}\beta_u\bar{p}_u)(\mathscr{G}/\mathscr{R})\bar{l}_d\,,\\
(\Phi R/\mathscr{G}M) &= \phi = \phi_u + \lambda\phi_d\,,\\
\beta &= \beta_u + \lambda\beta_d\,,
\end{aligned}\right\} \tag{9}$$

where

$$x = r/R\,, \quad \bar{c} = 3l_e/16\pi acM_\odot^3\mathscr{R}\,, \quad \text{and} \quad l_e = \kappa_e\mu_e^2 L\,.$$

In the above equations $L$, $R$, and $M$ are the luminosity, radius, and mass of the star respectively and the subscript e is used to denote the value of the associated quantity in the surface layers of the star. All other symbols have their usual meanings. We eliminate explicit reference to the mass and composition of the star by introducing the mass–composition factor

$$\mathscr{M} = \mu_e^2 M/M_\odot \tag{10}$$

and use the variable

$$\sigma = \Omega^2/\Omega_c^2 \tag{11}$$

to describe the angular velocity distribution throughout the star.

## III. Basic Model

Equating terms of first order in the expansions of the preceding section we find

$$\frac{\mathscr{R}}{\mathscr{G}\mathscr{M}}\frac{\bar{t}_u}{\beta_u\bar{p}_u}\frac{d\bar{p}_u}{dx} = -\frac{d\phi_u}{dx}\,, \tag{12}$$

$$\frac{d^2\phi_u}{dx^2} + \frac{2}{x}\frac{d\phi_u}{dx} = \frac{4\pi}{\mathscr{R}\mathscr{M}}\frac{\beta_u\bar{p}_u}{\bar{t}_u}\,; \tag{13}$$

for the radiative equation

$$\frac{\mathrm{d}\bar{t}_{\mathrm{u}}}{\mathrm{d}x} = -\bar{c}\frac{\beta_{\mathrm{u}}\bar{p}_{\mathrm{u}}}{x^2\bar{t}_{\mathrm{u}}^4}; \tag{14}$$

for the convective equation

$$\frac{\mathrm{d}\beta_{\mathrm{u}}}{\mathrm{d}\bar{t}_{\mathrm{u}}} = -\frac{3}{2}\frac{(1-\beta_{\mathrm{u}})\beta_{\mathrm{u}}^2}{(4-3\beta_{\mathrm{u}})\bar{t}_{\mathrm{u}}}, \tag{15}$$

together with

$$\beta_{\mathrm{u}} = 1-\tfrac{1}{3}aM_{\odot}^2\,\bar{t}_{\mathrm{u}}^4/\bar{p}_{\mathrm{u}}. \tag{16}$$

These are equivalent to the structure equations of Van der Borght (1964) and have been solved by him subject to the boundary conditions given in Section IV.

TABLE 1

VALUES OF $\bar{p}_{\mathrm{u}}$, $\bar{t}_{\mathrm{u}}$, AND $\beta_{\mathrm{u}}$ FOR A TYPICAL STELLAR MODEL

$\mathscr{M} = 10$, $x_{\mathrm{c}} = 0\cdot3916$

| $x$ | $\bar{p}_{\mathrm{u}}$ | $\bar{t}_{\mathrm{u}}$ | $\beta_{\mathrm{u}}$ |
|---|---|---|---|
| 0·0 | $2\cdot3300\times10^{-5}$ | $4\cdot6920\times10^{-15}$ | 0·7942 |
| 0·1 | $2\cdot0872\times10^{-5}$ | $4\cdot5060\times10^{-15}$ | 0·7981 |
| 0·2 | $1\cdot3563\times10^{-5}$ | $3\cdot9975\times10^{-15}$ | 0·8133 |
| 0·3 | $7\cdot0005\times10^{-6}$ | $3\cdot3553\times10^{-15}$ | 0·8338 |
| 0·4 | $2\cdot7988\times10^{-6}$ | $2\cdot4711\times10^{-15}$ | 0·8679 |
| 0·5 | $8\cdot5605\times10^{-7}$ | $1\cdot7563\times10^{-15}$ | 0·8898 |
| 0·6 | $2\cdot0540\times10^{-7}$ | $1\cdot2030\times10^{-15}$ | 0·8988 |
| 0·7 | $3\cdot7763\times10^{-8}$ | $7\cdot8101\times10^{-16}$ | 0·9023 |
| 0·8 | $4\cdot4700\times10^{-9}$ | $4\cdot8687\times10^{-16}$ | 0·9033 |
| 0·9 | $1\cdot7514\times10^{-10}$ | $2\cdot0316\times10^{-16}$ | 0·9035 |
| 0·99 | $1\cdot1964\times10^{-14}$ | $1\cdot8469\times10^{-17}$ | 0·9035 |

The values of $\bar{p}_{\mathrm{u}}$, $\bar{t}_{\mathrm{u}}$, and $\beta_{\mathrm{u}}$ for a typical model ($\mathscr{M} = 10$) are given in Table 1. The present work is based on four models with uniform composition and $\mathscr{M} = 10$, 20, 40, and 60.

## IV. Perturbation Equations

The equations relating to the perturbation variables are obtained as the coefficients of $\lambda$ in the expansions of Section II, and following Roxburgh (1964) we assume that the solutions in the envelope can be expressed in terms of a series of Legendre polynomials up to second order. Expanding all variables in the form

$$Q_d(x,\mu) = Q_0(x)+Q_2(x)P_2(\mu), \tag{17}$$

we find that the variables $x, \mu$ can be separated and that we can replace the original set of partial differential equations by the following two sets of ordinary differential equations.

*Equations (A)*

$$\frac{1}{\mathscr{G}\mathscr{M}\beta_u}\left\{x\frac{d\bar{p}_0}{dx}+\frac{V}{n+1}\bar{p}_0-\left(\bar{t}_0-\frac{\bar{t}_u}{\beta_u}\beta_0\right)V\mathscr{R}\right\}=-x\frac{d\phi_0}{dx}+\tfrac{2}{3}\sigma x^2, \tag{18}$$

$$x\frac{dq_0}{dx}+q_0=\frac{4\pi}{\mathscr{R}\mathscr{M}}\beta_u\left(\frac{\bar{p}_0}{\mathscr{R}}-\bar{t}_0+\frac{\beta_0\bar{t}_u}{\beta_u}\right)\overline{UV}; \tag{19}$$

the radiative equations

$$x\frac{d\bar{t}_0}{dx}+\frac{\mathscr{G}}{\mathscr{R}}\bar{l}_0-\frac{V}{n+1}\left(4\bar{t}_0-\beta_0\frac{\bar{t}_u}{\beta_u}-\frac{\bar{p}_0}{\mathscr{R}}\right)=0, \tag{20}$$

$$x\frac{d\bar{l}_0}{dx}+\bar{l}_0\left(1-\frac{4V}{n+1}+V-\frac{x}{\beta_u}\frac{d\beta_u}{dx}\right)=0, \tag{21}$$

$$x\,d\phi_0/dx=q_0; \tag{22}$$

the convective equation

$$\bar{t}_0=-\frac{2}{3}\frac{4-3\beta_u}{(1-\beta_u)\beta_u^2}\bar{t}_u\beta_0+\alpha_0\bar{t}_u, \tag{23}$$

where $\alpha_0$ is a constant of integration; and the auxiliary equation

$$\beta_0=\frac{1-\beta_u}{\bar{t}_u}\left(\frac{\bar{p}_0}{\mathscr{R}}-4\bar{t}_0\right). \tag{24}$$

*Equations (B)*

$$\frac{1}{\mathscr{G}\mathscr{M}\beta_u}\left\{x\frac{d\bar{p}_2}{dx}+\frac{V}{n+1}\bar{p}_2-\left(\bar{t}_2-\frac{\bar{t}_u}{\beta_u}\beta_2\right)V\mathscr{R}\right\}=-q_2-\tfrac{2}{3}\sigma x^2, \tag{25}$$

$$\frac{1}{\mathscr{G}\mathscr{M}\beta_u}\bar{p}_2=-\phi_2-\tfrac{1}{3}\sigma x^2, \tag{26}$$

$$x\frac{dq_2}{dx}+q_2-6\phi_2=\frac{4\pi}{\mathscr{R}\mathscr{M}}\beta_u\left(\frac{\bar{p}_2}{\mathscr{R}}\bar{t}_2+\beta_2\frac{\bar{t}_u}{\beta_u}\right)\overline{UV}; \tag{27}$$

the radiative equations

$$x\frac{d\bar{t}_2}{dx}+\frac{\mathscr{G}}{\mathscr{R}}\bar{l}_2-\frac{V}{n+1}\left(4\bar{t}_2-\beta_2\frac{\bar{t}_u}{\beta_u}-\frac{\bar{p}_2}{\mathscr{R}}\right)=0, \tag{28}$$

$$x\frac{d\bar{l}_2}{dx}+\bar{l}_2\left(1-\frac{4V}{n+1}+V-\frac{x}{\beta_u}\frac{d\beta_u}{dx}\right)-\frac{6\mathscr{R}}{\mathscr{G}}\bar{t}_2=0, \tag{29}$$

$$x\,d\phi_2/dx=q_2; \tag{30}$$

the convective equation

$$\bar{t}_2 = -\frac{2}{3}\frac{4-3\beta_u}{(1-\beta_u)\beta_u^2}\bar{t}_u\beta_2; \tag{31}$$

and the auxiliary equation

$$\beta_2 = \frac{1-\beta_u}{\bar{t}_u}\left(\frac{\bar{p}_2}{\mathscr{R}} - 4\bar{t}_2\right). \tag{32}$$

In these equations we have introduced the quantities

$$\overline{UV} = \frac{\bar{p}_u}{\bar{t}_u}x^2, \qquad V = -\frac{x}{\bar{p}_u}\frac{d\bar{p}_u}{dx}, \qquad n+1 = \frac{d(\ln\bar{p}_u)}{d(\ln\bar{t}_u)}, \tag{33}$$

which are related to the homology invariants of Schwarzschild (1958).

These equations must be solved subject to the boundary conditions of the problem and these are found to be identical in every way to the boundary conditions given by Roxburgh (1964).

For a complete solution we must include the energy equation (7), which has so far been omitted. This leads to boundary conditions on $\bar{l}_0$ only, and to obtain this condition we note that

$$L = \int_{\text{volume}} \epsilon\rho\, d\tau = \int_{\text{surface}} \mathbf{F}.d\mathbf{S}$$

$$= L_u\{1+\lambda(x\,\bar{l}_0\,\bar{t}_u^4/\bar{c}\,\beta_u\,\bar{p}_u)\}_{\text{surface}}. \tag{34}$$

If we now assume a power-law energy generation within the core of the form

$$\epsilon = \epsilon_0\,\rho T^{17},$$

then

$$L_u = \int_{\text{core}} \epsilon_0\,\rho_u^2\,T_u^{17}\,d\tau$$

$$= \mathscr{L}\int_{x=0}^{x_c} \bar{p}_u^2\beta_u^2\bar{t}_u^{15}x^2\,dx, \tag{35}$$

where $x_c$ denotes the position of the interface in the unperturbed model, and the constant factor $\mathscr{L}$ depends on the mass, the radius, and the constant $\epsilon_0$. Apart from this factor the integral in (35) can be determined explicitly for any given stellar model.

The change in $L$ (to first order) is then given by

$$L_d = L_0 = \mathscr{L}\int_{x=0}^{x_c} \bar{p}_u^2\beta_u^2\bar{t}_u^{15}\left(\frac{2\bar{p}_0}{\bar{t}_u\mathscr{R}} + \frac{2\beta_0}{\beta_u} + \frac{15\bar{t}_0}{\bar{t}_u}\right)x^2\,dx,$$

since the terms in $P_2(\cos\theta)$ vanish on integration. Substituting for $\beta_0$ in the above we find finally

$$L_d = \mathscr{L}\int_{x=0}^{x_c} \bar{p}_u^2\beta_u\bar{t}_u^{14}\{2\bar{p}_0/\mathscr{R} + (23\beta_u-8)\bar{t}_0\}x^2\,dx. \tag{36}$$

Equating coefficients of $\lambda$ we obtain

$$l_0\,|_{x=x_c} = (\bar{c}\beta_u\,\bar{p}_u/x\bar{t}_u^4)|_{x=x_c}\int_{x=0}^{x_c}\bar{p}_u^2\,\beta_u\,\bar{t}_u^{14}\{2\bar{p}_0/\mathscr{R}+(23\beta_u-8)\bar{t}_0\}x^2\,\mathrm{d}x$$

$$\div\int_{x=0}^{x_c}\bar{p}_u^2\,\beta_u^2\,\bar{t}_u^{15}\,x^2\,\mathrm{d}x\,. \qquad (37)$$

It should be noted that if, for example, an exponential type of energy generation were to be used it would no longer be possible to eliminate the constant $\mathscr{L}$. In this case the above integration could only be performed for a particular model, whereas here we have derived a result that is applicable to any massive star in which a power-law energy generation is assumed.

There is no boundary condition on $\bar{l}_2$ at the interface. We note, however, that any discontinuity in $\mathrm{d}\bar{t}_2/\mathrm{d}x$ (which may be reasonably expected from consideration of the different modes of energy transport in the core and envelope) may be compensated for by a non-zero value of $\bar{l}_2$ at the interface. This will ensure continuity of energy transport across the interface.

## V. Numerical Solutions

The equations (A) involving the radial variations cannot be solved until the function $\sigma$ is known, and we therefore start by solving the set (B) for the angular variations. In the envelope, eliminating $\beta_2$ through (32) and eliminating $\sigma$ between (25) and (26), there remain five linear homogeneous first-order differential equations for the five variables $\bar{p}_2$, $\bar{t}_2$, $\phi_2$, $q_2$, and $\bar{l}_2$. There are two boundary conditions at the surface and hence the envelope solution can be obtained in terms of three parameters, which we choose to be $\bar{p}_{2e}$, $\phi_{2e}$, and $\bar{l}_{2e}$.

In the core $\sigma = 1$ and it is possible although lengthy to show that (25) and (26) together imply (31). Elimination of $\beta_2$ then leads to two linear first-order differential equations in $q_2$ and $\phi_2$. The boundary conditions at the centre are satisfied by taking $\phi_2 = \phi_{2c}\,x^2$ and the core solution can be obtained in terms of the parameter $\phi_{2c}$.

The four conditions of continuity at the interface then serve to determine uniquely the four parameters $\phi_{2c}$, $\phi_{2e}$, $\bar{p}_{2e}$, and $\bar{l}_{2e}$. Once a solution has been determined the angular velocity distribution $\sigma$ can be found from (26).

With the introduction of radiation pressure it is obvious that a polytropic solution with constant polytropic index in the core is no longer possible. Near the surface there are rapid variations in $\bar{p}_u$ and $\bar{t}_u$ and we therefore make use of the expansions of Van der Borght (1964) in ascending powers of $y = 1-x$. Thus

$$\beta_u = \beta_e\,, \qquad (38)$$

$$t_u = \beta_e\left(\frac{\mathscr{G}\mathscr{M}}{4\mathscr{R}}\right)\left(\frac{y}{1-y}\right), \qquad (39)$$

and also

$$V \simeq 4/y, \qquad n+1 \simeq 4,$$

and

$$\overline{UV} \simeq \bar{c}^{-1}\beta_e^2(\mathscr{G}\mathscr{M}/4\mathscr{R})^3 y^2. \tag{40}$$

Introducing these expansions into (A) and (B) we obtain, on elimination of $\beta_2$ and $\sigma$, the following set.

*Equations* (*A*′)

$$\left.\begin{aligned}
(1-y)\frac{\mathrm{d}\bar{p}_0}{\mathrm{d}y} &= \mathscr{G}\mathscr{M}\beta_e q_0 - \tfrac{2}{3}\mathscr{G}\mathscr{M}\beta_e(1-y)^2\sigma + \frac{4-3\beta_e}{\beta_e}\left(\bar{p}_0 - 4\mathscr{R}\bar{t}_0\right)\frac{1}{y},\\
(1-y)\frac{\mathrm{d}q_0}{\mathrm{d}y} &= q_0 - \frac{4\pi}{\bar{c}}\beta_e^2\left(\frac{\mathscr{G}\mathscr{M}}{4\mathscr{R}}\right)^3 y^2\left(\bar{p}_0 - \mathscr{R}(4-3\beta_e)\bar{t}\right),\\
(1-y)\frac{\mathrm{d}\phi_0}{\mathrm{d}y} &= -q_0,\\
(1-y)\frac{\mathrm{d}\bar{t}_0}{\mathrm{d}y} &= -\frac{4}{y}\frac{\bar{t}_0}{\beta_e} + \frac{1}{y}\frac{\bar{p}_0}{\mathscr{R}\beta_e} + \frac{\mathscr{G}}{\mathscr{R}}\bar{l}_0,\\
(1-y)\frac{\mathrm{d}l_0}{\mathrm{d}y} &= l_0.
\end{aligned}\right\} \tag{41}$$

*Equations* (*B*′)

$$\left.\begin{aligned}
(1-y)\frac{\mathrm{d}\bar{p}_2}{\mathrm{d}y} &= \bar{p}_2\left(\frac{4-3\beta_e}{y\beta_e} - 2\right) - 4(4-3\beta_e)\frac{\mathscr{R}\bar{t}_2}{y} + \mathscr{G}\mathscr{M}\beta_e(q_2 - 2\phi_2),\\
(1-y)\frac{\mathrm{d}q_2}{\mathrm{d}y} &= q_2 - 6\phi_2 - \frac{4\pi}{\mathscr{R}\mathscr{M}\bar{c}}\beta_e^2\left(\frac{\mathscr{G}\mathscr{M}}{4\mathscr{R}}\right)^3 y^2\left(\bar{p} - (4-3\beta_e)\bar{t}_2\right)\\
(1-y)\frac{\mathrm{d}\bar{t}_2}{\mathrm{d}y} &= \frac{\mathscr{G}}{\mathscr{R}}\bar{l}_2 - \frac{1}{\beta_e}\left(4\bar{t}_2 - \frac{\bar{p}_2}{\mathscr{R}}\right)\frac{1}{y},\\
(1-y)\frac{\mathrm{d}\bar{l}_2}{\mathrm{d}y} &= \bar{l}_2 + 6\bar{t}_2\frac{\mathscr{R}}{\mathscr{G}},\\
(1-y)\frac{\mathrm{d}\phi_2}{\mathrm{d}y} &= -q_2.
\end{aligned}\right\} \tag{42}$$

If we now expand all variables in (B′) in ascending powers of $y$ in the form

$$v_2 = v_{2e} + v_{21}y + v_{22}y^2 + \ldots,$$

then it is possible to obtain expressions for all coefficients in terms of the parameters $\bar{p}_{2e}$, $\phi_{2e}$, and $\bar{l}_{2e}$. No attempt has been made to obtain explicit independent solutions satisfying the boundary conditions.

At the centre of the star we look for a series solution in ascending powers of $x$ and, eliminating $\bar{t}_2$ and $\bar{p}_2$ in (B′), we find the following set of equations in $q_2$ and $\phi_2$.

$$\left.\begin{aligned} x\frac{\mathrm{d}q_2}{\mathrm{d}x}+q_2+\phi_2\left(\frac{4\pi\overline{UV}}{\mathscr{R}^2}\frac{3\beta_u^2(8-7\beta_u)\mathscr{G}}{32-24\beta_u-3\beta_u^2}-6\right)+\frac{4\pi\overline{UV}}{\mathscr{R}^2}\frac{\beta_u^2(8-7\beta_u)}{32-24\beta_u-3\beta_u^2}x^2 = 0\,, \\ q_2 = x\,\mathrm{d}\phi_2/\mathrm{d}x\,. \end{aligned}\right\} \tag{43}$$

In the core

$$\overline{UV} = \frac{aM_\odot^2}{3}\frac{\bar{t}_0^2}{1-\beta_u}\left(\frac{\beta_c\exp(-4/\beta_c)}{1-\beta_c}\frac{1-\beta_u}{\beta_u\exp(-4/\beta_u)}\right)^{4/3}x^2\,,$$

where the subscript c indicates the central value of the associated variable. Expanding

$$\phi_2 = \phi_{2c}x^2+\phi_{23}x^3+\phi_{24}x^4+O(x^5)\,,$$

$$q_2 = 2\phi_{2c}x^2+O(x^3)\,,$$

we find

$$\phi_2 = \phi_{2c}x^2-\frac{2\pi\mathscr{G}\bar{p}_c}{7\mathscr{R}^2\bar{t}_c^2}\frac{\beta_c^2(8-7\beta_c)}{32-24\beta_c-3\beta_c^2}(1+3\phi_{2c})x^4+O(x^6)\,. \tag{44}$$

The solution was started at the surface by choosing a set of parameters and using the series expansion to give starting values for a fourth-order Runge–Kutta integration at $x = 0{\cdot}96$. At the interface the final values of $\phi_2$ and $q_2$ were used as initial values for numerical integration in the core, and hence the continuity of $q_2$ and $\phi_2$ was assured. The final values $\bar{p}_2^{(e)}$, $\bar{t}_2^{(e)}$ of $\bar{p}_2$ and $\bar{t}_2$ were then compared with the values $\bar{p}_2^{(c)}$, $\bar{t}_2^{(c)}$ obtained from substitution of $\phi_2$ in (26) with $\sigma = 1$ and subsequent substitution in (31).

The integration was continued until $x = 0{\cdot}04$ at which point an estimate of $\phi_{2c}$ was obtained by solving (44) for $\phi_{2c}$ in terms of $\phi_2(x)$ and $x$. This estimate was then substituted in the expansion for $q_2$ to give a value $q_2^{(\mathrm{th})}$ at $x = 0{\cdot}04$, which was compared with the value $q_2^{(\mathrm{i})}$ obtained as the end point of the numerical integration. Putting

$$T_p = (\bar{p}_2^{(e)}-p_2^{(c)})/\bar{p}_2^{(e)}\,,$$

$$T_t = (\bar{t}_2^{(e)}-\bar{t}_2^{(c)})/\bar{t}_2^{(e)}\,,$$

and

$$T_q = (q_2^{(\mathrm{i})}-q_2^{(\mathrm{th})})/q_2^{(\mathrm{i})}\,,$$

and regarding

$$(T_p,T_q,T_t) = \mathbf{f}(\phi_{2e},\bar{p}_{2e},\bar{t}_{2e})\,,$$

where $\mathbf{f}$ is some vector "function" of the input parameters, an iterative procedure was then used to find the set $(\phi_{2e},\bar{p}_{2e},\bar{t}_{2e})$ giving

$$\mathbf{f}(\phi_{2e},\bar{p}_{2e},\bar{t}_{2e}) = (0,0,0)\,.$$

The resulting angular velocity distributions were calculated for the four stellar models ($\mathscr{M} = 10$, 20, 40, and 60) and are given in Table 2. The surface value of angular velocity at the equator is illustrated as a function of $\mathscr{M}$ in Figure 1.

TABLE 2

ANGULAR VELOCITY DISTRIBUTIONS FOR FOUR STELLAR MODELS

| $x$ | $\sigma = \Omega^2/\Omega_c^2$ for: $\mathscr{M} = 10$ | $\mathscr{M} = 20$ | $\mathscr{M} = 40$ | $\mathscr{M} = 60$ |
|---|---|---|---|---|
| 1·0 | 0·715 | 0·799 | 0·875 | 0·899 |
| 0·9 | 0·735 | 0·822 | 0·898 | 0·924 |
| 0·8 | 0·764 | 0·854 | 0·930 | 0·954 |
| 0·7 | 0·801 | 0·900 | 0·969 | 0·988 |
| 0·6 | 0·872 | 0·959 | 0·997 | 0·998 |
| 0·582 | | | | 1·00 |
| 0·542 | | | 1·00 | |
| 0·5 | 0·953 | 0·991 | 1·00 | 1·00 |
| 0·467 | | 1·00 | | |
| 0·4 | 0·995 | 1·00 | 1·00 | 1·00 |
| 0·392 | 1·00 | | | |
| 0·3 | 1·00 | 1·00 | 1·00 | 1·00 |
| 0·2 | 1·00 | 1·00 | 1·00 | 1·00 |
| 0·1 | 1·00 | 1·00 | 1·00 | 1·00 |
| 0·0 | 1·00 | 1·00 | 1·00 | 1·00 |

Having obtained the angular velocity distribution function $\sigma$, it is then possible to obtain the complete solution for the "radial" variations (variables with subscript 0). The method of solution is similar to that described above for the "angular" variations with the exception that in this case it is possible to eliminate the series expansion at

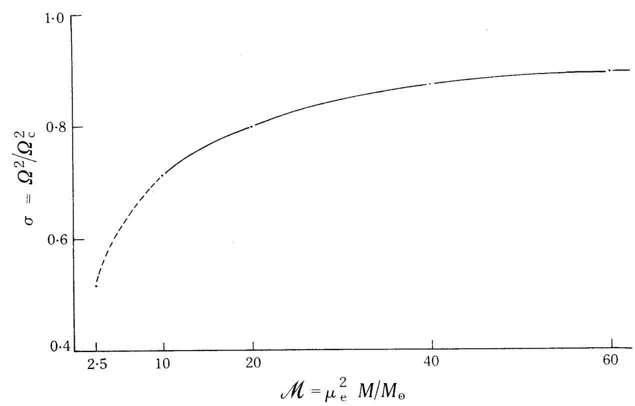


Fig. 1.—Equatorial surface value of $\sigma$ as a function of $\mathscr{M}$.

the surface and to integrate from the centre outwards. This is brought about by the fact that in the expansions so derived the coefficients of powers of $y$ up to the second order in both $\bar{t}_0$ and $\bar{p}_0$ are independent of the parameter $\bar{p}_{0e}$ and are determined explicitly in terms of the angular velocity distribution near the surface. The parameter $\bar{p}_{0e}$ enters the coefficients of only third and higher order powers of $y$, and it is then sufficiently accurate to use the boundary condition $\bar{p}_{0e} = 4\mathscr{R}t_{0e}$ as a test of goodness of fit of the solution. The variables $\phi_0$ and $q_0$ at the surface behave as $ky^4$ and $ky^3$ respectively, and as the other tests of goodness of fit we can use the boundary conditions $\phi_0 = 0 = q_0$.

Table 3

VALUES OF PERTURBATIONS AT THE SURFACE FOR FIVE STELLAR MODELS

| $\mathscr{M}$ | $\phi_2$ | $q_2$ | $\bar{x}_0$* | $\bar{x}_2$* | $\bar{l}_0$* | $\bar{l}_2$* |
|---|---|---|---|---|---|---|
| 2·5† | 0·0138 | −0·0416 | 0·5408 | −0·1868 | −0·1768 | 0·3022 |
| 10 | 0·0181 | −0·0542 | 0·1517 | −0·2564 | −0·6599 | 1·831 |
| 20 | 0·0197 | −0·0592 | 0·1546 | −0·2861 | −1·155 | 2·363 |
| 40 | 0·0198 | −0·0596 | 0·2427 | −0·3116 | −1·892 | 3·906 |
| 60 | 0·0192 | −0·0577 | 0·2573 | −0·3192 | −2·485 | 5·051 |

* Note that $x_d = (1/\beta_e)\bar{x}_d$ and $l_d = \bar{l}_d/\mathscr{M}$, where $x_d$ and $l_d$ are the corresponding variables of Roxburgh and Strittmatter (1966).

† Equivalent to the $R_3$ model of Roxburgh and Strittmatter.

The surface values of the perturbations are given in Table 3 together with those of Roxburgh and Strittmatter (1966) for the $R_3$ model, which is equivalent to $\mathscr{M} = 2·5$. (The variable $\bar{l}_d$ is related to $l_d$ of Roxburgh and Strittmatter by $l_d = \bar{l}_d/\mathscr{M}$.)

## VI. Discussion

The results of the present work can be seen to agree fairly well with those of Roxburgh and Strittmatter (1966) for their $R_3$ model, which is essentially a limiting case of the present work as $\beta \to 1$. The variation of surface angular velocity with the mass–composition factor shows that, as may be expected, the more massive the star the faster will be its observed rate of rotation. The effects on the observable properties, namely luminosity and effective temperature, have not been considered, since their determination requires a calculation of the actual temperature and radius of the star. The main purpose of the present paper has been to derive results applicable to any massive star with energy generation assumed to be of a power-law type.

It is perhaps of importance to note that the perturbed models so obtained are very sensitive to small changes in the equilibrium models. For example, a change of 0·01% in the structure variables of an equilibrium model can cause a change of up to 10% in the angular velocity at the surface. For this reason, discussion of stability of the models obtained will be left until a detailed treatment of the effects of non-radial oscillations has been carried out.

The effect of varying composition and evolution on the structure will be considered in a later paper in which three- and four-zone equilibrium models will be taken as bases for the perturbation analysis.

## VII. Acknowledgment

The author would like to thank Professor R. Van der Borght for suggestion of this problem and for his many helpful comments.

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