ELECTROMAGNETIC DIFFRACTION BY A PLANE SLIT APERTURE

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Summary

An asymptotic solution of the integral equations for diffraction by an E-polarized slit aperture is given. The results for the diffracted field are in excellent agreement with the exact solution and compare favourably with the results given by other approximate solutions.

I. Introduction

The problem of diffraction by a slit aperture in a plane screen has been the subject of many investigations in the past. It is well known that a rigorous solution can be found in terms of eigenfunction series of Mathieu functions (Morse and Rubenstein 1938; Hsu 1959), but its usefulness is limited to $ka \leqslant 10$ (where k is the wave number and a the half-aperture width in wavelengths) because of the difficulty of tabulating the Mathieu functions and the poor convergence of the series. Solutions as series of increasing powers in ka, also restricted to small slit widths, have been given by Groschwitz and Honl (1952), Honl and Zimmer (1953), Muller and Westpfahl (1953), and Bouwkamp (1954). For large slit widths, Clemmow (1956) used the concept of edge currents, while Millar (1958) presented an asymptotic solution of the integral equations by successive iteration. Grinberg (1958), using the concept of shadow currents, showed how the integral equations for the shadow currents may also be solved iteratively. A Wiener–Hopf treatment of the integral-equation approach was given by Levine (1959). More recent papers on this topic include those of Kleinman and Timman (1961), Stockel (1962), Khaskind and Vainshteyn (1964), Kieburtz (1964), and Popov (1965). Much of this previous work has been concerned with evaluating the transmission coefficient and the far-field patterns, and relatively little attention has been given to the near field. Only a few measurements have been reported (Hadlock 1958; Hsu 1959).

It is the purpose of the present paper to give an asymptotic solution of the integral equations for the screen currents, using a new approach. The near field is calculated and compared with the rigorous solution, and also with the Karp and Russek (1956) and second Rayleigh integral methods.

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II. Solution of Integral Equations

With an E-polarized plane wave incident on the slit aperture of width $2a$, the integral equations for the induced currents are (see Fig. 1)

$$j_{\mathrm{A}}(r)=j_{\mathrm{A0}}(r)+\int_0^\infty j_{\mathrm{B}}(\rho)\left(-\frac{\exp\{-ik(r+\rho+2a)\}}{\pi r^{\frac{1}{2}}(r+\rho+2a)}(\rho+2a)^{\frac{1}{2}}\right)\mathrm{d}\rho, \tag{1}$$

$$j_{\mathrm{B}}(r)=j_{\mathrm{B0}}(r)+\int_0^\infty j_{\mathrm{A}}(\rho)\left(-\frac{\exp\{-ik(r+\rho+2a)\}}{\pi r^{\frac{1}{2}}(r+\rho+2a)}(\rho+2a)^{\frac{1}{2}}\right)\mathrm{d}\rho, \tag{2}$$

where j_{A0} and j_{B0} are the half-plane currents on screens A and B respectively. These equations have been derived by Baker and Copson (1950) and Millar (1958) using Fox's formula, and also by Skal'skaya (1963) and Tan (1966) using integral transforms. The usual procedure is to solve (1) and (2) by a straightforward iterative method,

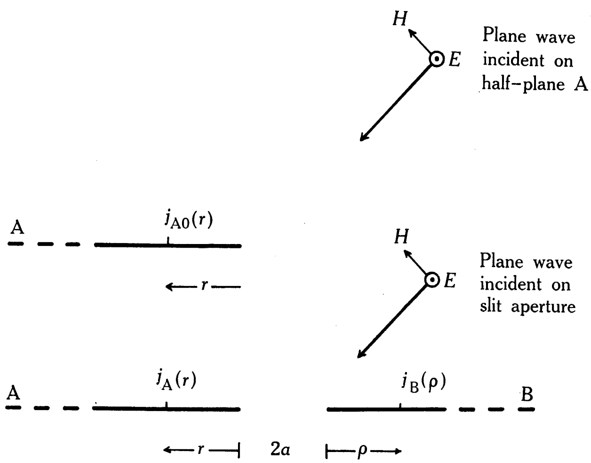

Fig. 1.—Schematic diagrams for diffraction by a slit aperture.

but this is cumbersome because of the difficulty of evaluating the integrals in each iteration. The present treatment shows how this iteration scheme can be carried out more conveniently using Karp and Zitron's (1964) theorem.

The form of equations (1) and (2) shows that the interactions between the two half-planes are represented by the integral terms. The kernel in each integral $G(r,\rho)$ is exactly the current induced at r on one half-plane due to the presence of a unit line current at ρ on the other half-plane. This interpretation of $G(r,\rho)$ then makes the form of equations (1) and (2) intuitively obvious, i.e. by superposition, the total induced current j_{A} is the sum of the unperturbed half-plane current j_{A0} and the currents induced in A by the presence of j_{B}. Each of the line currents j_{B} radiates a cylindrical wave that can be developed in an expansion of plane waves and derivatives of plane waves with respect to their angles of incidence (Karp and Zitron 1964). The currents induced on half-plane A by j_{B} are therefore equivalent to those induced by a series of plane waves and their derivatives. These induced currents can then be calculated from the half-plane solution and from the fact that the current induced by a derivative of a plane wave is the derivative of the current induced by a plane wave.

Writing j_{A0} and $G(r,\rho)$ as

$$j_{A0}(r,\gamma) = -\frac{c}{\pi k}\sum_{n=0}^{\infty}\frac{i}{2r}(2n+1)\,i^{n+\frac{1}{2}}\,J_{n+\frac{1}{2}}(kr)\sin\{(n+\tfrac{1}{2})\gamma\}, \tag{3}$$

$$G(r,\rho) = \sum_{n=0}^{\infty}\frac{i}{2r}(2n+1)\,J_{n+\frac{1}{2}}(kr)\,H^{(2)}_{n+\frac{1}{2}}(k(\rho+2a))\sin\{(n+\tfrac{1}{2})\gamma\} \qquad r<2a, \tag{4}$$

and using the Karp and Zitron theorem, we obtain

$$G(r,\rho) = H_{op}(\rho,d/d\gamma)\,j_{A0}(r,\gamma) \qquad r<2a, \tag{5}$$

where the operator $H_{op}(\rho, d/d\gamma)$ contains the series of plane waves and their derivatives. The expression for H_{op} follows from the expansion of the Hankel function in the following way

$$H^{(2)}_{n+\frac{1}{2}}(kb) = \left(\frac{2}{\pi kb}\right)^{\frac{1}{2}}\exp\left(-ikb+\tfrac{1}{4}i\pi+\tfrac{1}{2}i(n+\tfrac{1}{2})\pi\right)$$
$$\times\left(\sum_{m=0}^{\infty}\frac{\{1+4\,d^2/d\gamma^2\}\{9+4\,d^2/d\gamma^2\}\dots\{(2m-1)+4\,d^2/d\gamma^2\}}{m!\,(-8ikb)^m}\right)g(\gamma), \tag{6}$$

where $g(\gamma)$ represents a plane wave with angle of incidence γ. Hence

$$H_{op}(\rho,d/d\gamma) = \left(\frac{2}{\pi k(\rho+2a)}\right)^{\frac{1}{2}}\exp\left(-ik(\rho+2a)+\tfrac{1}{4}i\pi\right)\left(1+i\frac{1+4\,d^2/d\gamma^2}{8k(\rho+2a)}-\dots\right). \tag{7}$$

Equation (1) then becomes

$$j_A(r) = j_{A0}(r) + \left(\int_0^{\infty} j_B(\rho)\,H_{op}(\rho,d/d\gamma)\,d\rho\right)j_{A0}(r,\gamma), \tag{8}$$

where j_{A0} has been taken outside the integral because γ is independent of ρ. Equation (2) takes the same form. A solution by successive iteration is now easily carried out because each iteration yields the same type of integral. For example, if the incident plane wave makes an angle θ with the half-plane A, the third iteration gives

$$j_A(r,\theta) = j_{A0}(r,\theta) + \{C_B(\pi-\theta,d/d\gamma) + C_A(\theta,d/d\gamma_1)\,C_B(\gamma_1,d/d\gamma)$$
$$+C_B(\pi-\theta,d/d\gamma_1)\,C_A(\gamma_1,d/d\gamma_2)\,C_B(\gamma_2,d/d\gamma)\}j_{A0}(r,\gamma), \tag{9}$$

where

$$\left.\begin{aligned} C_A(\theta,d/d\gamma) &= \int_0^{\infty} j_{A0}(\rho,\theta)\,H_{op}(\rho,d/d\gamma)\,d\rho, \\ C_B(\gamma_i,d/d\gamma) &= \int_0^{\infty} j_{B0}(\rho,\gamma_i)\,H_{op}(\rho,d/d\gamma)\,d\rho, \end{aligned}\right\} \tag{10}$$

and γ_i is to be put equal to π after all the differentiations have been carried out. The integral C_A or C_B can be evaluated in an asymptotic series by means of Erdelyi's

(1956) theorem, if we express the half-plane current j_{A0} or j_{B0} in closed form

$$j_0(r,\gamma) = \frac{c}{\pi^{3/2}}\left(\frac{\exp(-ikr-\frac{1}{4}i\pi)}{(2kr)^{\frac{1}{2}}}\sin\tfrac{1}{2}\gamma + \exp(\tfrac{1}{4}i\pi)\sin\gamma\exp(-ikr\cos\gamma)\right.$$
$$\left.\times\int_0^{(2kr)^{\frac{1}{2}}\cos\frac{1}{2}\gamma}\exp(-it^2)\,dt\right). \tag{11}$$

The part of the integral containing the Fresnel integral can also be developed in series form by repeated integration by parts. The details of these calculations are tedious and are found in Tan (1966).

Because of the restriction on the expression for $G(r,\rho)$ in equation (4), equations (8) and (9) converge only for $r < 2a$. However, this is not a very severe restriction on the method because it is known (Moullin and Phillips 1952) that the currents j_A and j_B approach their respective half-plane values j_{A0} and j_{B0} when r is greater than a small fraction of $2a$. Thus, for any but the smallest apertures, this method would yield good results.

III. The Aperture Field

The diffracted electric and magnetic field everywhere in space can be calculated from the screen currents. For our purpose, to derive the aperture electric field for comparison with other theories and with experiment, it is more convenient to start from equations (1) and (2). We confine ourselves to the case of normal incidence and, multiplying both sides of equation (1) or (2) by

$$-(\pi k/c)\,H_0^{(2)}(k(r+x))$$

and integrating with respect to r, we have

$$E_d(x) = E_0(x) + \int_0^\infty j(\rho)\,E_s(\rho,x)\,d\rho, \tag{12}$$

where E_0 is the half-plane field, $E_s(\rho,x)$ is the field diffracted by one half-plane due to a unit line current at ρ on the other half-plane, and x in the aperture is measured from the aperture edge. Then

$$E_0(x) = -2\pi^{-\frac{1}{2}}\exp(\tfrac{1}{4}i\pi)\int_{(kx)^{\frac{1}{2}}}^\infty \exp(-it^2)\,dt \tag{13}$$

and

$$E_s(\rho,x) = \frac{4ik}{c}\exp\{-ik(\rho+2a-x)\}\int_{(2kx)^{\frac{1}{2}}}^\infty \frac{\exp(-it^2)}{t^2+2k(\rho+2a-x)}\,dt. \tag{14}$$

The integral in E_s can be approximated by a Fresnel integral to a degree of accuracy that is inadequate only if both x and $2a$ are very much smaller than one wavelength (Clemmow 1950). Thus

$$E_s(\rho,x) \simeq \frac{4ik}{c}\frac{\exp\{-ik(\rho+2a-x)\}}{\{2k(\rho+2a)\}^{\frac{1}{2}}}\int_{(2kx)^{\frac{1}{2}}}^\infty \exp(-it^2)\,dt. \tag{15}$$

Using for the first iteration

$$j(\rho) \simeq j_0(\rho, \tfrac{1}{2}\pi)$$

from equation (11), we have that

$$E_1(x) = \int_0^\infty j_0(\rho, \tfrac{1}{2}\pi)\, E_s(\rho, x)\, \mathrm{d}\rho$$

$$= \frac{2}{\pi}\frac{\exp(-2\mathrm{i}ka)}{(2ka)^{\frac{1}{2}}}\int_{(2kx)^{\frac{1}{2}}}^{\infty} \exp(-\mathrm{i}t^2)\,\mathrm{d}t + \frac{5\mathrm{i}}{4\pi}\frac{\exp(-2\mathrm{i}ka+\mathrm{i}kx)}{(2ka)^{3/2}}\int_{(2kx)^{\frac{1}{2}}}^{\infty} \exp(-\mathrm{i}t^2)\,\mathrm{d}t$$

$$+O\{(ka)^{-5/2}\}, \tag{16}$$

TABLE 1

COMPARISON OF SOLUTIONS FOR E-POLARIZED PLANE SLIT APERTURE ELECTRIC FIELD VECTOR

Halfwidth of slit $a = 0\cdot72$ cm ($ka = \sqrt{2}$)

η (degrees)	Distance from Aperture Centre $\bar{x} = a\cos\eta$ (cm)	Solutions			
		Exact* (Mathieu Fns.)	Integral Eqn. (Eqn. (19))	Rayleigh Integral Approximation†	Edge Current Approximation‡
10	0·709	0·209+i0·146	0·199+i0·136	0·726−i0·027	−1·490+i3·680
20	0·677	0·420+i0·292	0·411+i0·287	0·764+i0·051	0·073+i1·838
30	0·624	0·635+i0·433	0·627+i0·433	0·823+i0·145	0·675+i1·282
40	0·552	0·850+i0·566	0·845+i0·569	0·898+i0·243	1·045+i1·048
50	0·463	1·057+i0·688	1·054+i0·693	0·981+i0·338	1·319+i0·940
60	0·360	1·243+i0·790	1·242+i0·798	1·063+i0·444	1·533+i0·890
70	0·246	1·392+i0·871	1·392+i0·878	1·133+i0·487	1·691+i0·870
80	0·125	1·489+i0·921	1·490+i0·929	1·179+i0·545	1·791+i0·863
90	0·000	1·522+i0·938	1·524+i0·947	1·195+i0·564	1·825+i0·861

* Hsu (1959).

† Tan (1966).

‡ Karp and Russek (1956).

where we have used integration by parts and also Erdelyi's theorem. For the first iteration, the current induced on one half-plane by j_0 on the other half-plane is

$$j_1(\rho) = \int_0^\infty j_0(\eta, \tfrac{1}{2}\pi)\left(-\frac{\exp\{-\mathrm{i}k(2a+\rho+\eta)\}}{\pi\rho^{\frac{1}{2}}(2a+\rho+\eta)}(2a+\eta)^{\frac{1}{2}}\right)\mathrm{d}\eta\,. \tag{17}$$

j_1 in turn gives rise to a diffracted field $E_2(x)$ given by

$$E_2(x) = \int_0^\infty j_1(\rho)\, E_s(\rho, x)\, \mathrm{d}\rho$$

$$\simeq -(2/\pi)^{\frac{1}{2}}\exp(-\tfrac{1}{4}\mathrm{i}\pi)\frac{\exp(-4\mathrm{i}ka+\mathrm{i}kx)}{2\pi ka}\int_{(2kx)^{\frac{1}{2}}}^{\infty}\exp(-\mathrm{i}t^2)\,\mathrm{d}t\,. \tag{18}$$

E_2 is thus the result of two iterations of the integral equation. This iterative process can be carried further, but the approximation made in equation (15) would have to be improved. Thus, to an order of $(2ka)^{-3/2}$ the electric field in the aperture is given by

$$E(x) = 1+E_0(x)+E_0(2a-x)+E_1(x)+E_1(2a-x)+E_2(x)+E_2(2a-x). \quad (19)$$

IV. Results and Conclusions

Table 1 gives a numerical comparison of the aperture field as calculated by the various methods for $ka = \sqrt{2}$. It is seen that with only two iterations the results of equation (19) give very good agreement with the exact solution, even for this small slit width. The accuracy is even better for larger slit widths (Tan 1966). The two

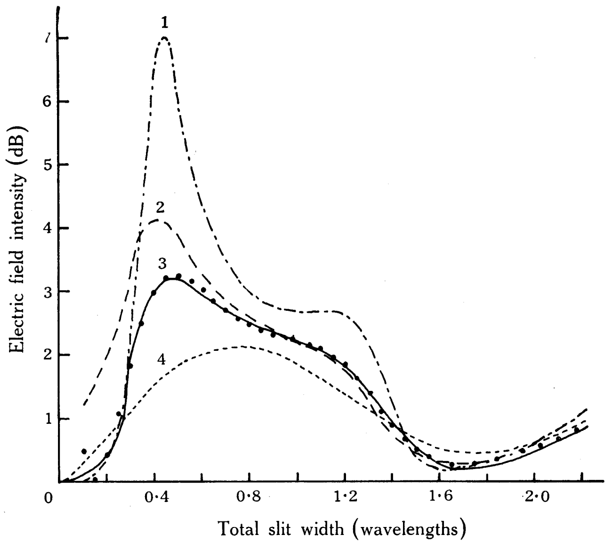

Fig. 2.—Comparison of methods of calculating electric field intensity at the centre of the slit aperture for various slit widths: curve 1, Hadlock's approximation (Hsu 1959); 2, Karp and Russek's (1956) approximation; 3, Mathieu function series (Hsu 1959); 4, second Rayleigh integral approximation. The points (●) lying close to curve 3 are the values obtained from equation (19).

approximation methods are rather poor, especially near the aperture edge, where even the phase is incorrect. To indicate the range of accuracy of equation (19), Figure 2 compares a number of methods of calculating the electric field at the centre of the slit aperture for various slit widths. It is seen that results from equation (19) are accurate for slit widths larger than one-fifth of a wavelength. The relative performances of the approximate theories (curves 1, 2, and 4) are evident from Figure 2.

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