0^+ STATES OF ^{8}Be

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Summary

The light even nuclei with $A \geqslant 10$ have 0^+ excited states near 6 MeV, probably with large α-particle reduced widths. A similar state in ^{8}Be would be very broad. Evidence for 0^+ excited states in ^{8}Be has been obtained here using many-level R-matrix fits to known α–α scattering data, but the excitation energies depend strongly on the assumed channel radius. For a simultaneous fit to the ^{9}Be(p, d)^{8}Be cross section, assuming these higher states are not strongly populated, the channel radius is restricted to (7^{+2}_{-1}) fm, implying a 0^+ excited state at (6 ∓ 3) MeV of width (9 ∓ 4) MeV.

I. Introduction

The nucleus ^{8}Be has been studied in considerable detail (Lauritsen and Ajzenberg-Selove 1966) and its level structure below 16 MeV appears to be very simple, consisting of a 0^+ ground state, a broad 2^+ state at $2\cdot 9$ MeV, and a very broad 4^+ state at $11\cdot 4$ MeV. Other light even nuclei from ^{10}Be to ^{16}O have 0^+ excited states at excitation energies of order 6 MeV, probably with large α-particle reduced widths and not belonging to the lowest shell model configuration. In this paper, we consider the possible existence of such a state in ^{8}Be; because of the large energy available for α-particle decay, one would expect such a state to be very broad and this would make its identification difficult. The properties of these 0^+ excited states in light nuclei are more fully discussed in Section II, where it is considered how a similar state in ^{8}Be could fit in with theories of nuclear structure that have been applied to ^{8}Be.

Evidence for such an excited 0^+ state in ^{8}Be could come from α–α elastic scattering and from reactions that proceed through states of ^{8}Be to give three final particles. The analysis of α–α scattering in this region is particularly simple as the channel spin is zero and there is only one open channel, so the phase shifts δ_l for relative orbital angular momentum l are fairly well known. The present procedure is to use a many-level one-channel R-matrix formalism, given in Section III, to fit the observed s-wave phase shift δ_0 for a range of values of channel radius a_0 and for a particular choice of the boundary condition parameter B_0. The same fit can be obtained for any other value of B_0 by adjusting the level parameters. This is described in Section IV.

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In order to limit the acceptable values of a_0 and the level parameters, the R-matrix formalism of Section III with parameters that fit δ_0 is also required to give a fit to the measured ^{9}Be(p, d)^{8}Be cross section in the region of the ground state main peak and its ghost (Section V). For this purpose certain reasonable "model" restrictions are imposed on the values of B_0 and the feeding factors for the higher 0^+ levels in this reaction.

The final acceptable values of the R-matrix parameters, and the derived properties of the ground and excited 0^+ states, are given in Section VI. Appendixes I, II, and III contain a justification of the use of the one-channel approximation, the relations between the level parameters and B_0 that are needed in order to make the fits independent of the choice of B_0, and a discussion of the best choice of B_0. Some of the formulae in Section III and the appendixes are given for general l values, so that they can also be used in a later paper in which 2^+, 4^+, ... states of ^{8}Be will be discussed.

II. Systematics and Models for ^{8}Be

Table 1 gives the excitation energies of 0^+ excited states observed in the even nuclei from ^{10}Be to ^{16}O, as well as the energies of the corresponding $T = 1$ states in their odd–odd isobars (Ajzenberg-Selove and Lauritsen 1959; Lauritsen and Ajzenberg-Selove 1966). Also given are the Q values Q_α for α-particle decay. There is no obvious correlation of the level position with either the mass number A or Q_α. Over a wider range of A values, the excitation energies of low-lying 0^+ states in closed shell nuclei appear to vary as $A^{-2/3}$ (Sheline and Wildermuth 1960; Meyerhof 1966).

These states probably do not belong to the lowest shell model configuration $(1s)^4(1p)^{A-4}$. This is obviously the case for the ^{16}O state, which appears to contain an appreciable 4p–4h component (e.g. see Brown and Green 1966). Also 2p–2h configurations have been suggested for the ^{14}N state by Unna and Talmi (1958), for the ^{10}Be and ^{10}B states by True and Warburton (1961), and for the ^{12}C state by Cohen and Kurath (1965).

There is some evidence that these excited 0^+ states have large α-particle reduced widths. Only for the ^{12}C state is the α-channel the only open channel, and here the measured width leads to an α-particle reduced width near the "single-particle" value, even when a channel radius considerably larger than the conventional one is used (Barker and Treacy 1962). An appreciable reduced width has been obtained for the ^{16}O state from the ^{6}Li(^{12}C, d)^{16}O reaction (Loebenstein *et al.* 1967; see also Bethge *et al.* 1967).

There is no obvious reason why a similar 0^+ excited state belonging to a higher configuration should not exist in ^{8}Be in the region of 6 MeV excitation energy. Such a state with an α-particle reduced width near the single-particle value would be very broad* with a width of order 10 MeV.

* Early evidence from α–α scattering and from ^{7}Li(p, $\gamma\alpha$)^{4}He for a narrower 0^+ state near 7·6 MeV with width about 1 MeV has not been substantiated (see Ajzenberg-Selove and Lauritsen 1959; Lauritsen and Ajzenberg-Selove 1966).

It is frequently stated that models of ^{8}Be predict only the well-known 0^+, 2^+, and 4^+ states below 16 MeV. This is not true for all models. Thus, in the cluster model of Wildermuth and Kanellopoulos (1959), completely antisymmetric states are constructed from single-particle oscillator wave functions, the lowest eigenstates consisting of two α-particles in relative 3s, 2d, and 1g oscillations, i.e. 0^+, 2^+, and 4^+ states. The next higher α-particle states are the 0^+, 2^+, 4^+, 6^+ sequence coming from the 4s, 3d, 2g, and 1i oscillations, and it was estimated that these lay well above the 1g state (at the order of 20 MeV excitation energy); however, by changing the two-particle interaction it would be possible to reduce the energies of these states.

Table 1

EXCITATION ENERGIES OF 0^+ EXCITED STATES OF LIGHT EVEN NUCLEI AND OF CORRESPONDING $T = 1$ STATES IN THEIR ODD–ODD ISOBARS

Nucleus	E_x (MeV)	Q_α (MeV)	Nucleus	E_x (MeV)	Q_α (MeV)
^{8}Be		0·09	^{14}C	6·59	−12·02
^{10}Be	6·18	−7·42	^{14}N ($T = 1$)	6·31*	−11·04*
^{10}B ($T = 1$)	5·82*	−6·28*	^{14}O	(5·91)†	−10·08
^{12}C	7·66	−7·38	^{16}O	6·06	−7·16

* Measured from the lowest $T = 1$ state.

† Ball and Cerny (1967) suggest that the 5·91 MeV level is probably 0^+.

A shell model calculation including only the lowest $(1s)^4(1p)^4$ configuration does not predict a 0^+ excited state of ^{8}Be below about 16 MeV (see Cohen and Kurath 1965; Barker 1966). A shell model calculation with configuration mixing, including only a limited class of states, predicted in addition to the lowest 0^+, 2^+, 4^+ levels also a 0^+, 2^+, 4^+, 6^+ band starting with a 0^+ level at 9·66 MeV (Gupta, Khadkikar, and Parikh 1966).

It seems therefore that neither the cluster model nor the shell model excludes the existence of a 0^+ excited state of ^{8}Be below 10 MeV.

III. R-matrix Formulae in the One-channel Approximation

To describe processes involving the ^{8}Be nucleus, the formalism of R-matrix theory is employed. This provides a general and convenient representation for two-stage reactions. For practical applications it is necessary to assume that only a finite number of levels and channels are involved.

Lane and Thomas (1958) have given the R-matrix formulae for the cross section of a nuclear reaction in the general many-level many-channel case. For α–α scattering below the ^{7}Li+p threshold, i.e. for channel energies $E < 17 \cdot 35$ MeV, there is only one open channel for a given total angular momentum J, i.e. the $\alpha+\alpha$ channel with relative orbital angular momentum $l = J$, and use of the one-channel approximation seems appropriate. This is justified in Appendix I.

In the one-channel approximation there is a simple connection between the nuclear phase shift δ_l and the R_l function (Lane and Thomas 1958),

$$R_l = \{P_l \cot(\delta_l+\phi_l)+S_l-B_l\}^{-1}. \tag{1}$$

Here P_l, S_l, and $-\phi_l$ are respectively the penetration factor, shift factor, and hard-sphere phase shift, which are energy dependent and can be calculated for a given channel radius a_l; B_l is a real constant boundary condition parameter. Also R_l must be of the form (Lane and Thomas 1958)

$$R_l = \sum_\lambda \gamma_{\lambda l}^2/(E_{\lambda l}-E), \tag{2}$$

in terms of the eigenenergies $E_{\lambda l}$ and the reduced widths $\gamma_{\lambda l}^2$.

The present procedure is to make a q-level approximation and to choose the parameters $E_{\lambda l}$ and $\gamma_{\lambda l}^2$ for given a_l and B_l to give the best fit to the experimental values of δ_l by minimizing

$$X_l = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\delta_l^{\text{exp.}}(E_i)-\delta_l(E_i)}{\epsilon_l(E_i)}\right|^2. \tag{3}$$

Here $\delta_l^{\text{exp.}}(E_i)$ and $\epsilon_l(E_i)$ are the measured phase shift and error at the energy E_i $(i = 1 \dots N)$ respectively, and $\delta_l(E_i)$ is the phase shift calculated from (1) and (2):

$$\delta_l(E) = -\phi_l+\arctan\left[P_l \div \left\{\left(\sum_{\lambda=1}^{q}\gamma_{\lambda l}^2/(E_{\lambda l}-E)\right)^{-1}-S_l+B_l\right\}\right]. \tag{4}$$

The IBM 360/50 computer of the Australian National University was used for the minimization. From equation (4), the dependence of $\delta_l(E)$ on E can be made independent of the choice of B_l by suitably adjusting the values of the level parameters $E_{\lambda l}$ and $\gamma_{\lambda l}^2$. The resulting relations between these parameters and B_l are given in equations (A12) and (A13) of Appendix II.

In this paper we are interested in the case $l = 0$, and for this we may use the fact that the ^{8}Be ground state energy E_g has been measured very accurately in α–α scattering, in order to obtain a relation between the parameter values a_0, B_0, $E_{\lambda 0}$, and $\gamma_{\lambda 0}^2$. Since $\delta_0 = 90°$ at $E = E_g$ and since ϕ_0 is negligible at this energy, one obtains from (4)

$$\left(\sum_{\lambda=1}^{q}\gamma_{\lambda 0}^2/(E_{\lambda 0}-E_g)\right)^{-1} = S_0(E_g)-B_0. \tag{5}$$

Thus it is convenient to do the initial fitting to δ_0 with the choice

$$B_0 = S_0(E_g), \tag{6}$$

so that (5) can be satisfied with

$$E_{10} = E_g. \tag{7}$$

Then the relations (A12) and (A13) can be used to obtain the parameter values that will give the same fit for any other value of B_0. The width Γ_g of the ^{8}Be ground state, defined as the difference of the energies at which δ_0 equals 45° and 135°, is similarly given by a simple formula if one makes the choice (6), (7):

$$\Gamma_g = 2\gamma_{10}^2 P_0(E_g)/\{1+\gamma_{10}^2 S_0'(E_g)\}, \tag{8}$$

where the prime denotes the energy derivative.

In various reactions, ^{8}Be is formed as a product nucleus which then decays into two α-particles. The dependence on the ^{8}Be excitation energy of the cross section for such reactions has been obtained, by a reasonable generalization of the R-matrix formula for the one-level approximation, for the case where many levels of ^{8}Be with the same spin and parity may contribute (Barker 1967). The form appropriate for discussing the low-lying levels of ^{8}Be with spin J, for which only the $\alpha+\alpha$ channel with $l = J$ is open, is

$$\sigma_\alpha \propto P_l \sum_x \left| \frac{\sum_{\lambda=1}^{q} \{G_{\lambda x}^{\frac{1}{2}} \gamma_{\lambda l}/(E_{\lambda l}-E)\}}{1-(S_l-B_l+\mathrm{i}P_l)\sum_{\lambda=1}^{q} \{\gamma_{\lambda l}^2/(E_{\lambda l}-E)\}} \right|^2, \tag{9}$$

where $G_{\lambda x}$ is a real positive feeding factor (usually a slowly varying function of E) and x labels the quantum numbers for the formation process that give incoherent contributions to σ_α. For the case $l = 0$, normalization of (9) to the ^{8}Be ground state main peak leads to a simple relation for the G_{1x} when the choice (6), (7) is made. Simultaneously with the requirement that the $E_{\lambda l}$, $\gamma_{\lambda l}^2$ should be such as to make δ_l given by (4) independent of B_l, the $G_{\lambda x}$ can be required to make σ_α independent of B_l. The resulting relation is equation (A14) of Appendix II.

IV. R-matrix Parameters from α–α Scattering Data

(*a*) *Experimental Data*

Experimental values of the phase shifts $\delta_l^{\mathrm{exp.}}$ and their errors ϵ_l have been obtained by several authors from analyses of the α–α elastic scattering cross sections measured at various α-particle beam energies up to 120 MeV, corresponding to channel energies E up to 60 MeV. The phase shifts must be real for $E < 17\cdot35$ MeV but may be complex at higher energies.

For the lower energies, we use the values of $\delta_0^{\mathrm{exp.}}$ and ϵ_0 given for $E = 0\cdot2$–$1\cdot5$ MeV (Heydenberg and Temmer 1956), $E = 1\cdot92$–$5\cdot94$ MeV (Tombrello and Senhouse 1963), and $E = 6\cdot15$–$11\cdot45$ MeV (Nilson *et al.* 1958). Bredin *et al.* (1959) have extracted real phase shifts from their data for $E = 11\cdot55$–$19\cdot2$ MeV, but their values are given only graphically and no convenient values of ϵ_0 are given. The above data have all been reanalysed by Berztiss (1965) assuming real phase shifts; the $\delta_0^{\mathrm{exp.}}$ values are similar to those previously obtained but the associated errors (obtained using a different criterion) are sometimes quite different. We use Berztiss's values of $\delta_0^{\mathrm{exp.}}$ for $E = 11\cdot55$–$17\cdot1$ MeV, and for the accompanying errors round off his values to $\epsilon_0 = 5°$. Some values of $\delta_0^{\mathrm{exp.}}$ appear to be relatively less precise, e.g. those at $6\cdot15$, $7\cdot6$, and $15\cdot15$ MeV, and we have doubled the corresponding ϵ_0 values.

Complex phase shifts are available at higher energies, e.g. $E = 26 \cdot 7$ to $59 \cdot 93$ MeV (Darriulat *et al.* 1965), but we have not tried to fit δ_0 above the ^{7}Li+p threshold.

From α–α scattering at very low energies Benn *et al.* (1966) have observed the ground state of ^{8}Be at a channel energy $E = E_g = (92 \cdot 12 \pm 0 \cdot 05)$ keV with a width $\Gamma_g = (6 \cdot 8 \pm 1 \cdot 7)$ eV.

(*b*) *Three-level* R-*matrix Fits to* δ_0

R-matrix fits to δ_0 in the low energy region have previously been made in the one-level one-channel approximation, e.g. Barker and Treacy (1962) obtained acceptable fits for $E \lesssim 3$ MeV for channel radii a_0 between $3 \cdot 2$ and $4 \cdot 2$ fm, but deviations increased at higher energies. These deviations may be attributed to the effects of higher 0^+ levels, so we try a many-level approximation in order to fit δ_0 over a larger energy range. As we wish to retain the one-channel approximation, fits are restricted to $E \lesssim 17$ MeV, as discussed in Appendix I.

TABLE 2

PARAMETER VALUES FOR BEST FITS TO $\delta_0^{\text{exp.}}$ IN THE THREE-LEVEL APPROXIMATION FOR VARIOUS CHANNEL RADII

$B_0 = S_0(E_g)$ and $E_{10} = E_g = 92 \cdot 12$ keV

a_0 (fm)	E_{max} (MeV)	N	B_0	γ_{10}^2 (MeV)	E_{20} (MeV)	γ_{20}^2 (MeV)	E_{30} (MeV)	γ_{30}^2 (MeV)	X_0	Γ_g (eV)
5·5	17·1	36	−1·428	0·295	12·58	1·003	30·5	1·74	0·44	5·53
6·0	17·1	36	−1·491	0·193	9·80	0·835	34·7	2·27	0·44	5·46
6·5	17·1	36	−1·550	0·126	7·79	0·712	28·5	1·85	0·43	5·22
7·0	17·1	36	−1·605	0·081	6·28	0·609	22·4	1·38	0·47	4·78
7·5	14·9	33	−1·657	0·052	5·20	0·521	18·8	1·14	0·45	4·27
8·0	13·8 12·75	32 31	−1·707	0·036	4·39	0·454	16·1	1·00	0·47	4·01
8·5	10·9	26	−1·754	0·022	3·74	0·395	13·4	0·79	0·44	3·28
9·0	9·55	23	−1·799	0·016	3·25	0·338	11·9	0·77	0·48	3·13

In Section II it was suggested that the second 0^+ level might be expected near 6 MeV, so that a third or higher 0^+ level could also be contributing appreciably in $E \lesssim 17$ MeV. To allow for the higher 0^+ levels, we use a three-level approximation, where the third "level" is considered to include the effects of all levels above the second. Since from equation (4) the third level is expected to occur for $\delta_0 + \phi_0 \simeq 450°$, we restrict the energy range over which fits are made by requiring $\delta_0^{\text{exp.}} + \phi_0 \lesssim 440°$. Since ϕ_0 increases with a_0, this restriction becomes effective for the larger channel radii $a_0 > 7 \cdot 0$ fm.

The parameter values that give best fits to $\delta_0^{\text{exp.}}$ for various channel radii are given in Table 2. These values are obtained by taking $B_0 = S_0(E_g)$ so that $E_{10} = E_g$, and varying γ_{10}^2, E_{20}, γ_{20}^2, E_{30}, and γ_{30}^2 to minimize X_0 given by equation (3). The range of data fitted is specified by the maximum E_i value E_{max} and the number of data points N used in each fit. It is roughly within the range of channel radii given in Table 2 that acceptable fits to both scattering and reaction data can be found.

The minimum X_0 is about 0·45 for each a_0, the irregular fluctuations being due to the different ranges of data being fitted. In fact, to obtain a suitable minimum X_0 for $a_0 = 8\cdot0$ fm, we had to average the fits obtained for the two cases of $E_{\max} = 13\cdot8$ and 12·75 MeV, as experimental phase shifts are available only at widely spaced energies in this region and they do not vary smoothly with energy. Table 2 also includes calculated values of Γ_g. These all tend to be lower than the experimental value of $(6\cdot8\pm1\cdot7)$ eV, the discrepancy increasing as a_0 increases.

Table 3

PARAMETER VALUES FOR FITS TO $\delta_0^{\text{exp.}}$ IN THE THREE-LEVEL APPROXIMATION FOR $a_0 = 7\cdot0$ fm AND VARIOUS FIXED VALUES OF γ_{10}^2

$E_{\max} = 17\cdot1$ MeV, $B_0 = S_0(E_g) = -1\cdot605$, $E_{10} = E_g = 92\cdot12$ keV

γ_{10}^2 (MeV)	E_{20} (MeV)	γ_{20}^2 (MeV)	E_{30} (MeV)	γ_{30}^2 (MeV)	X_0	Γ_g (eV)
0·05	6·20	0·568	20·8	1·04	0·90	3·10
0·06	6·22	0·582	21·2	1·13	0·66	3·66
0·07	6·25	0·595	21·7	1·24	0·52	4·21
0·08	6·28	0·608	22·3	1·36	0·47	4·73
0·09	6·31	0·620	23·0	1·50	0·51	5·24
0·10	6·34	0·631	23·8	1·65	0·62	5·74
0·11	6·37	0·642	24·6	1·83	0·80	6·22
0·12	6·40	0·652	25·7	2·04	1·04	6·69

The parameter values in Table 2 correspond to the best fits for each a_0. Variations of the parameters about these values can still lead to acceptable fits. We impose the somewhat arbitrary condition $X_0 \lesssim 1$ for an acceptable fit. Then for the particular case $a_0 = 7\cdot0$ fm, Table 3 gives parameter values for fits with $X_0 \lesssim 1$ obtained by taking a set of fixed values of γ_{10}^2 and varying only E_{20}, γ_{20}^2, E_{30}, and γ_{30}^2. It is seen that acceptable fits can be obtained for a wide range of γ_{10}^2 values (and consequently of Γ_g values), but the corresponding values of E_{20} and γ_{20}^2 stay fairly constant, while we are not particularly interested in the parameter values for the third "level". In this way, even for the larger channel radii, acceptable fits can be obtained with the calculated Γ_g in the experimental range.

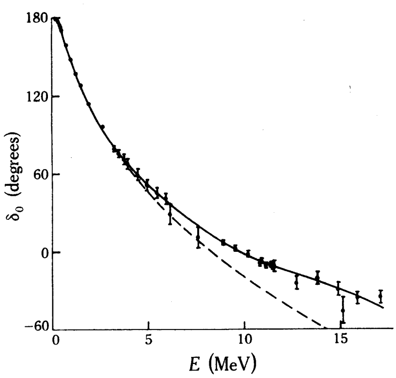

Fig. 1.—The α–α scattering s-wave phase shift δ_0 as a function of ^{8}Be channel energy E. The points are experimental values and the solid curve is the R-matrix three-level fit for the channel radius $a_0 = 7\cdot0$ fm and other parameters as in Table 6. The dashed curve is the best one-level fit for $E \lesssim 3$ MeV obtained with $a_0 = 3\cdot5$ fm (Barker and Treacy 1962).

These fits to $\delta_0^{\text{exp.}}$ are illustrated in Figure 1, where the calculated curve is for $a_0 = 7\cdot0$ fm, $\gamma_{10}^2 = 0\cdot087$ MeV, and other parameter values obtained by interpolation in Table 3, so that $X_0 = 0\cdot49$. These values give the best overall fit to the scattering and reaction data involving the 0^+ levels of ^{8}Be (see Table 6 below).

Thus acceptable three-level fits to the α–α scattering data, including δ_0, E_g, and Γ_g, can be obtained for a wide range of channel radii, including at least $5\cdot5$ to $9\cdot0$ fm, and the corresponding values of E_{20} are seen from Table 2 to vary widely.

(c) *Width of* ^{8}Be *Ground State*

Before making use of data from other reactions involving ^{8}Be to limit the range of acceptable a_0 values, we discuss more fully the value of Γ_g. The only directly measured experimental value is that of Benn *et al.* (1966) giving $\Gamma_g = (6\cdot8\pm1\cdot7)$ eV.

Values calculated from three-level R-matrix fits to δ_0 are given in Tables 2 and 3. Some values previously calculated (Barker and Treacy 1962) using the one-level approximation ($\Gamma_g \simeq 6\cdot8$ eV) or the effective range expansion ($\Gamma_g \simeq 6\cdot7$ eV) were considerably larger than the values of Tables 2 and 3, mainly because they were calculated using an older value of the ground state energy of 94 keV; if calculated at the newer value of $92\cdot12$ keV they should be reduced by 17% to give $\Gamma_g \simeq 5\cdot6$ eV. The values of Γ_g obtained from the effective range expansion can vary considerably, depending on the order of the polynomial expansion assumed for the function K and on the energy range over which the experimental data are fitted. Thus with K assumed to be a quadratic function of E, a good fit ($X = 0\cdot25$, where the definition of X is similar to that of X_l in equation (3)) is obtained to the data for $E \leqslant 1\cdot5$ MeV giving $\Gamma_g = 5\cdot1$ eV, but a much poorer fit is obtained for $E \leqslant 5\cdot0$ MeV ($X = 0\cdot82, \Gamma_g = 4\cdot4$ eV). The lower limit of the energy range is taken as $E = 0\cdot2$ MeV in all cases. A cubic function gives good fits for $E \leqslant 1\cdot5$ MeV ($X = 0\cdot19$, $\Gamma_g = 4\cdot1$ eV) and for $E \leqslant 5\cdot0$ MeV ($X = 0\cdot31$, $\Gamma_g = 5\cdot8$ eV).

Tombrello (1966) also considered the effect of varying the range of data fitted and concluded that the ground state width is extremely ill determined by this method; however, he did not impose the necessary restriction $K = h$ at resonance and this accounts for the wide spread of his values. Rasche (1967; personal communication) used the effective range expansion to fit data for $E \leqslant 1\cdot5$ MeV with K a quadratic function of E and obtained $\Gamma_g = (5\cdot1\pm0\cdot4)$ eV. Ali and Afzal (1967) used a cubic function and also fitted the data for $E \leqslant 1\cdot5$ MeV, but with a phenomenological α–α potential as an intermediary. They did not impose the condition $K = h$ at $E = E_g = 92\cdot12$ keV; instead they found that $K = h$ at $E_R \simeq 95\cdot1$ keV, giving $\Gamma_R \simeq 6\cdot4$ eV. This leads to $\Gamma_g \simeq 4\cdot8$ eV if calculated at $E = E_g$.

A hard-core effective range theory has been developed by Kermode (1965) to treat α–α scattering, and applied by him (Kermode 1967) to obtain $\Gamma_g = (6\cdot14\pm 0\cdot04)$ eV. He assumed his function y_0 to be a quadratic function of E and fitted data for $E \leqslant 12$ MeV. The hard-core theory is capable of fitting δ_0 up to higher energies than the conventional theory, as the expansion of K in the latter certainly breaks down before δ_0 decreases to 0°, which happens at $E \simeq 10$ MeV. To obtain the best value of Γ_g, however, one requires to extrapolate δ_0 to low energies and it is not necessarily advantageous to use an expansion that gives a fit to high energies. Thus for a fit up to only $1\cdot5$ MeV, the quadratic hard-core theory with hard-core radius $\simeq 1\cdot5$ fm gives $X = 0\cdot28$ and $\Gamma_g = 5\cdot4$ eV, while cubic fits for $E \leqslant 1\cdot5$ MeV and $E \leqslant 5\cdot0$ MeV give respectively $X = 0\cdot16$, $\Gamma_g = 4\cdot3$ eV and $X = 0\cdot24$, $\Gamma_g = 5\cdot5$ eV. These values are similar to those obtained from the conventional theory. For Kermode's fit up to 12 MeV, X is considerably larger ($X = 0\cdot48$).

Thus from effective range expansions, the best fits to δ_0 values measured at energies $E \geqslant 0\cdot2$ MeV give Γ_g values in the range 4–6 eV. Acceptable fits to δ_0 would provide a wider range of Γ_g values, probably similar to that indicated in Tables 2 and 3, which are for R-matrix fits to δ_0. In the following section, the range of Γ_g values is considerably reduced by requiring the R-matrix parameters to fit reaction data as well.

V. R-matrix Parameters from Reaction Data

(a) *Choice of Reaction and Experimental Data*

From R-matrix fits to the α–α scattering data we have found a wide range of parameters that are acceptable for describing the low-lying 0^+ states of ^{8}Be. In order to limit the range of these parameters, we wish to use experimental data from reactions that appear to proceed through an intermediate stage involving states of ^{8}Be. To do this it is necessary to separate out the contribution from 0^+ states of ^{8}Be. One difficulty is that, in such reactions giving three final products, the order in which the particles are emitted is usually not certain so that contributions from broad levels cannot be disentangled from the background due to alternative modes of decay. Only for a few special reactions involving β- or γ-decay, such as ^{8}Li(β^-)^{8}Be(α)^{4}He and ^{7}Li(p, γ)^{8}Be(α)^{4}He, can one be reasonably sure which "particle" is emitted first. The β-decay, however, does not populate 0^+ states; also any γ-decay populating 0^+ states would populate 2^+ states as well and these would interfere if γ–α coincidences were measured. It seems unlikely therefore that the contribution of 0^+ states of ^{8}Be to a reaction cross section can be obtained in the region of the broad excited 0^+ state, and one is restricted to using the cross section in the neighbourhood of the ground state. In spite of the extremely small value of Γ_g, the contribution from the ground state does show structure in addition to the ground state main peak. This is in the form of a subsidiary peak or ghost near 1 MeV (Beckner, Jones, and Phillips 1961; Barker and Treacy 1962), and analysis of the size and shape of the ghost can be useful in limiting the R-matrix parameters.

The most accurate information on the ghost peak appears to be that of Hay *et al.* (1967) obtained from the ^{9}Be(p, d)^{8}Be reaction at a beam energy of $5\cdot2$ MeV. From deuteron spectra measured at three angles, the contribution having an angular distribution similar to that of the ground state main peak was separated out for ^{8}Be channel energies $E \leqslant 2\cdot5$ MeV. The strong forward peak in the ground state angular distribution suggested that the reaction proceeds as a direct pickup of a p-wave neutron from ^{9}Be, so this contribution was divided by a neutron penetration factor in order to give a spectral density. This presumably excludes the contributions from the $2 \cdot 9$ MeV 2^+ state of ^{8}Be and from the competing mode of decay* (^{9}Be(p, α)^{6}Li(d)^{4}He), but may contain some or all of the contributions from higher 0^+ levels in addition to that of the ground state.

(b) *Three-level* R-*matrix Fits to Ghost*

We attempt to fit the spectral density obtained by Hay *et al.* (1967), using the form (9) for the cross section. The parameters a_0, B_0, $E_{\lambda 0}$, and $\gamma_{\lambda 0}^2$ are assumed to have values that give acceptable fits to $\delta_0^{\text{exp.}}$, leaving only the $G_{\lambda x}$ to be varied.

Since only one x value is expected to contribute (corresponding to pickup of a $p_{3/2}$ neutron), the calculated spectral density in the three-level approximation can be written

$$\rho_0(E) = cP_0 \left| \frac{\sum_{\lambda=1}^{3} \{g_\lambda \gamma_{\lambda 0}/(E_{\lambda 0}-E)\}}{1-(S_0-B_0+\mathrm{i}P_0) \sum_{\lambda=1}^{3} \{\gamma_{\lambda 0}^2/(E_{\lambda 0}-E)\}} \right|^2, \tag{10}$$

where we have put $G_\lambda = g_\lambda^2 P_n$, with g_λ constant and P_n the neutron penetration factor, and c is a normalization constant. A fit obtained initially for one particular value of B_0 can be obtained for any other value of B_0 by using the relations of Appendix II.

At this point, however, it is useful to make an approximation based on a model for the levels of ^{8}Be and ^{9}Be involved in the ^{9}Be(p, d)^{8}Be reaction, and this has the effect of limiting the range of B_0 values over which a given fit can be obtained. In the shell model, the ground state of ^{9}Be belongs to the lowest configuration $(1s)^4(1p)^5$, so that direct pickup of a p-wave neutron is expected to populate the higher 0^+ states of ^{8}Be only through the admixtures of lowest configuration $(1s)^4(1p)^4$ that they contain. More admixture is expected in the second state than in the third, so we make the reasonable restrictions

$$|g_2/g_1| \lesssim 0 \cdot 3, \qquad |g_3/g_1| \ll 0 \cdot 3, \tag{11}$$

corresponding to 10% or less intensity of the lowest configuration in the second state and much less than 10% in the third. If the condition (11) is imposed for all B_0 then a given fit to the spectral density can be obtained for only a limited range of B_0 values. Since changing B_0 changes the composition of the states $\lambda = 1, 2, 3$, there will be some value of B_0 that makes the shell model argument above most accurate. The question of the best choice of B_0 is discussed in Appendix III, where it is concluded that it probably lies in the region $S_0(E_{10})$ to $S_0(E_{20})$, where $S_0(E_{20}) \simeq 0$. The initial fitting is done here with $B_0 = S_0(E_{10}) = S_0(E_g)$ and with $g_3 = 0$, then the relations of Appendix II are used to obtain parameters that give the same fit for $B_0 = 0$. It is found that these parameters satisfy (11), provided the initial

* A reaction for which the ghost peak is obscured by a competing mode of decay is ^{6}Li(^{3}He, p)^{8}Be, as shown by the results of Lorenz (1966).

parameters satisfy (11), so that within this range of B_0 values the dependence of the fit on B_0 is not significant.

If (10) is normalized to make $\int \rho_0(E)\,\mathrm{d}E$ over the ground state main peak equal to unity then for $B_0 = S_0(E_g)$

$$cg_1^2 = \pi^{-1}\{1+\gamma_{10}^2 S_0'(E_g)\}. \tag{12}$$

For other B_0, c is given by a complicated expression depending on all the $E_{\lambda 0}$, $\gamma_{\lambda 0}$, and g_λ.

As the criterion of best fit we minimize

$$Y_0 = \frac{1}{24}\sum_{i=1}^{24}\left|\frac{\rho_0^{\text{exp.}}(E_i)-\rho_0(E_i)}{\eta_0(E_i)}\right|^2, \tag{13}$$

where the E_i are 24 equally spaced values of E from $0\cdot2$ to $2\cdot5$ MeV, and $\rho_0^{\text{exp.}}(E)\pm\eta_0(E)$ are the boundaries of the band given in Figure 5(ii) of Hay *et al.* (1967). Since the η_0 are therefore not probable errors there is no expectation for Y_0 to be near unity for good fits; indeed the best fits we get give $Y_0 \simeq 0\cdot25$ and fits with $Y_0 \lesssim 0\cdot5$ appear to be acceptable when judged by eye.

TABLE 4

PARAMETER VALUES FOR FITS TO $\rho_0^{\text{exp.}}$ IN THE THREE-LEVEL APPROXIMATION FOR $a_0 = 7\cdot0$ fm AND VARIOUS SETS OF PARAMETER VALUES GIVING ACCEPTABLE FITS TO $\delta_0^{\text{exp.}}$

γ_{10}^2 (MeV)	X_0	Y_0	$B_0 = S_0(E_g)$ g_2/g_1	$B_0 = S_0(E_g)$ g_3/g_1	$B_0 = 0$ g_2/g_1	$B_0 = 0$ g_3/g_1
0·0775	0·48	0·55	−0·16	0·0	−0·24	−0·01
0·08	0·47	0·42	−0·13	0·0	−0·21	−0·01
0·0825	0·47	0·33	−0·10	0·0	−0·18	−0·02
0·085	0·48	0·28	−0·07	0·0	−0·15	−0·02
0·0875	0·49	0·27	−0·04	0·0	−0·13	−0·02
0·09	0·51	0·29	−0·02	0·0	−0·10	−0·02
0·0925	0·53	0·34	0·01	0·0	−0·07	−0·03
0·095	0·55	0·43	0·04	0·0	−0·04	−0·03
0·0975	0·59	0·55	0·07	0·0	−0·02	−0·03

The fitting procedure is to use, for each value of a_0 and for $B_0 = S_0(E_g)$, sets of level parameters that give $X_0 \lesssim 1$ (such as those in Table 3 for $a_0 = 7\cdot0$ fm), and to vary g_2/g_1 (with $g_3 = 0$) so as to minimize Y_0 for each set. As an example, values so obtained for $a_0 = 7\cdot0$ fm are shown in Table 4, for those sets of level parameters that give $Y_0 \lesssim 0\cdot5$. The level parameters are specified by the values of γ_{10}^2 and the corresponding X_0. Table 4 also includes values of g_2/g_1 and g_3/g_1 for both cases $B_0 = S_0(E_g)$ and $B_0 = 0$; this change of B_0 does not produce large changes in either g_2/g_1 or g_3/g_1, and all the entries in the table satisfy the restrictions (11).

For $a_0 = 7\cdot0$ fm, the smallest value of Y_0 is obtained for approximately the same level parameters as those that give the smallest value of X_0. This is not the case for other channel radii, as may be seen from Table 5, where the smallest value of Y_0 for each of the different channel radii is given together with the corresponding values of γ_{10}^2 and of X_0. The values of g_2/g_1 and g_3/g_1, for both $B_0 = S_0(E_g)$ and $B_0 = 0$, satisfy (11) for channel radii between 6 and 9 fm, but this applies to the smallest Y_0 values and a wider range of channel radii could yield fits with acceptable values of Y_0. The Y_0 values in Table 5 show a shallow minimum at $a_0 \simeq 7\cdot5$ fm,

TABLE 5

PARAMETER VALUES FOR BEST FITS TO $\rho_0^{\text{exp.}}$ IN THE THREE-LEVEL APPROXIMATION FOR VARIOUS CHANNEL RADII AND FOR PARAMETER VALUES GIVING ACCEPTABLE FITS TO $\delta_0^{\text{exp.}}$

a_0 (fm)	γ_{10}^2 (MeV)	X_0	Y_0	$B_0 = S_0(E_g)$ g_2/g_1	$B_0 = S_0(E_g)$ g_3/g_1	$B_0 = 0$ g_2/g_1	$B_0 = 0$ g_3/g_1
5·5	0·250	0·63	0·36	0·58	0·0	0·48	−0·09
6·0	0·170	0·51	0·33	0·27	0·0	0·19	−0·05
6·5	0·120	0·44	0·29	0·07	0·0	−0·01	−0·03
7·0	0·087	0·50	0·27	−0·05	0·0	−0·13	−0·02
7·5	0·064	0·53	0·25	−0·13	0·0	−0·22	−0·01
8·0	0·048	0·59	0·26	−0·18	0·0	−0·27	0·00
8·5	0·036	0·71	0·29	−0·22	0·0	−0·31	0·00
9·0	0·027	0·69	0·30	−0·25	0·0	−0·34	0·01

but there is a more pronounced minimum in the corresponding X_0 values at $a_0 \simeq 6\cdot5$ fm. In order to obtain a best simultaneous fit to $\delta_0^{\text{exp.}}$ and $\rho_0^{\text{exp.}}$, and to allow a more pictorial if somewhat less accurate presentation of the results, we introduce the quantity $Z_0 = 0\cdot5X_0 + Y_0$, and take the smallest Z_0 as giving the best fit, provided (11) is satisfied. The conditions $X_0 \lesssim 1$, $Y_0 \lesssim 0\cdot5$ for acceptable fits are replaced by $Z_0 \lesssim 0\cdot7$.

TABLE 6

PARAMETER VALUES FOR BEST FITS TO $\delta_0^{\text{exp.}}$ AND $\rho_0^{\text{exp.}}$ IN THE THREE-LEVEL APPROXIMATION FOR VARIOUS CHANNEL RADII

$B_0 = S_0(E_g)$, $E_{10} = E_g = 92\cdot12$ keV, $g_3/g_1 = 0$

a_0 (fm)	γ_{10}^2 (MeV)	E_{20} (MeV)	γ_{20}^2 (MeV)	E_{30} (MeV)	γ_{30}^2 (MeV)	g_2/g_1	Γ_g (eV)	X_0	Y_0	Z_0
6·0	0·173	9·78	0·833	30·5	1·71	0·30	5·00	0·49	0·33	0·58
6·5	0·121	7·78	0·709	27·8	1·74	0·08	5·03	0·44	0·29	0·51
7·0	0·087	6·30	0·616	22·8	1·45	−0·05	5·07	0·49	0·27	0·51
7·5	0·064	5·23	0·536	19·5	1·28	−0·14	5·12	0·52	0·26	0·52
8·0	0·047	4·42	0·470	16·5	1·10	−0·19	5·15	0·58	0·26	0·55
8·5	0·036	3·77	0·411	13·8	0·90	−0·22	5·19	0·70	0·29	0·64
9·0	0·027	3·28	0·346	12·3	0·91	−0·26	5·18	0·68	0·30	0·64

In Table 6 is given the complete set of parameter values showing the best fits, in this sense, for each channel radius for which the condition (11) is satisfied. Complementary to Table 6 is Figure 2, where contours of constant Z_0 are shown as functions of g_2/g_1 and of Γ_g for various channel radii. Γ_g is used as abscissa rather than γ_{10}^2 in order to make the contours comparable for different a_0. Acceptable fits correspond to regions within the contours $Z_0 = 0\cdot7$ and the lines $g_2/g_1 = \pm0\cdot3$.

From Table 6 and Figure 2 it is seen that the best overall fit, with smallest Z_0 and smallest $|g_2/g_1|$, is obtained for a_0 near $7\cdot0$ fm, and that acceptable fits can be obtained for a_0 between about $6\cdot0$ and $9\cdot0$ fm. Exclusion of smaller channel

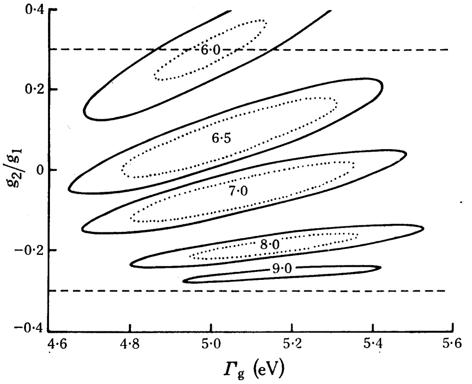

Fig. 2.—Acceptable regions for R-matrix three-level fits to $\delta_0^{\text{exp.}}$ and $\rho_0^{\text{exp.}}$ for various channel radii a_0. The values of a_0 (in fm) are indicated within the sets of contours, which are for $Z_0 = 0\cdot7$ (solid curves) and $Z_0 = 0\cdot6$ (dotted curves). The acceptable regions are within the contours $Z_0 = 0\cdot7$ and between the lines $g_2/g_1 = \pm0\cdot3$ (dashed lines).

radii depends on the restrictions (11), e.g. for $a_0 = 5\cdot5$ fm, $Z_0 \lesssim 0\cdot7$ only for $g_2/g_1 \gtrsim 0\cdot5$. Larger channel radii are excluded because Z_0 does not become sufficiently small, e.g. for $a_0 = 11$ fm, the minimum Z_0 is $0\cdot95$. The best fits to $\delta_0^{\text{exp.}}$ and $\rho_0^{\text{exp.}}$, for the parameter values of Table 6 for $a_0 = 7\cdot0$ fm, are shown in Figures 1 and 3 respectively.

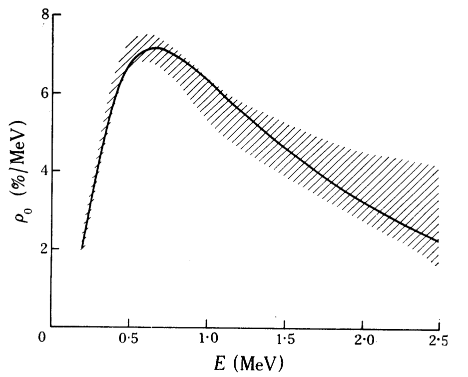

Fig. 3.—Spectral density ρ_0 associated with ^{8}Be ground state as a function of ^{8}Be channel energy E. The hatched region is as obtained from the reaction ^{9}Be(p, d)^{8}Be, and the curve is the fit for $a_0 = 7\cdot0$ fm and other parameters as in Table 6.

VI. Conclusions

Parameter values that give the best three-level R-matrix fits to the s-wave α–α scattering data and to the ^{9}Be(p, d)^{8}Be ghost data, subject to the restriction (11), are given in Table 6, and an indication of the variations of these values for which acceptable fits are possible may be obtained from Figure 2 in conjunction with Tables 2, 3, and 6. These values are all for $B_0 = S_0(E_g)$, but almost identical fits exist for any B_0 in the reasonable range between this value and zero. The best fit is obtained for channel radius $a_0 \simeq 7\cdot0$ fm, and acceptable fits are possible for a_0 between about 6 and 9 fm.

For the ^{8}Be ground state, the width Γ_g required to fit these data is found to be $(5\cdot1\pm0\cdot4)$ eV, with a considerably narrower range than is obtained from fitting $\delta_0^{\text{exp.}}$ alone.

The initial problem of this paper concerned the properties of the second 0^+ level of ^{8}Be, in particular to see if its position and width agreed with expectations based on properties of other light nuclei. From Table 6, the dimensionless reduced width of the second 0^+ state, defined by $\theta_{20}^2 = \gamma_{20}^2(\hbar^2/M_0\, a_0^2)^{-1}$ (where M_0 is the reduced mass of the $\alpha+\alpha$ channel), is close to $1\cdot4$ for all the values of a_0. This is the same as the value of the dimensionless reduced width obtained for the second 0^+ state of ^{12}C for a channel radius of $6\cdot5$ fm (Barker and Treacy 1962). Also the ratio $\gamma_{10}^2/\gamma_{20}^2$ of the reduced widths of the first and second 0^+ states of ^{8}Be is seen from Table 6 to be about $0\cdot1$–$0\cdot2$. This is similar to the ratio of the reduced widths of the ground and first excited states of ^{16}O, found to be in the range 0–$0\cdot36$ (Loebenstein *et al.* 1967).

We define the energy E_e and width Γ_e of the second 0^+ level of ^{8}Be as being the peak energy and width at half maximum of the expression (10) when $g_1 = g_3 = 0$, $g_2 \neq 0$, i.e. when only the second level is fed. These values depend to some extent on the choice of B_0. For the three channel radii $a_0 = 6\cdot0$, $7\cdot0$, and $9\cdot0$ fm, the values of E_e are $9\cdot7$, $6\cdot0$, and $3\cdot1$ MeV for $B_0 = S_0(E_g)$ ($9\cdot1$, $5\cdot6$, and $2\cdot8$ MeV for $B = 0$), and the values of Γ_e are $13\cdot6$, $9\cdot5$, and $5\cdot0$ MeV for $B_0 = S_0(E_g)$ ($13\cdot1$, $9\cdot0$, and $4\cdot7$ MeV for $B = 0$). In particular the value $E_e \simeq 6$ MeV corresponding to the best fit to the scattering and reaction data is consistent with expectations based on the systematics of the light even nuclei.

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Appendix I

Validity of One-channel Approximation

By one-channel approximation, we mean that the elastic scattering cross section can be expressed in terms of an R-function that has the usual form as a function of energy, although the constant level parameters may have modified meanings.

To justify the use of the R-matrix one-channel approximation for α–α scattering, we use the formulae given by Lane and Thomas (1958), especially in Section X of their paper. With only one retained channel (the $\alpha+\alpha$ channel with $l=J$, which is labelled by l), the reduced R-matrix $\boldsymbol{R}_{rr}$ contains only the one element $(\boldsymbol{R}_{rr})_{ll} \equiv R_{ll}$ and the collision matrix element for α–α scattering becomes

$$U_{ll} = \exp\{2\mathrm{i}(\omega_l-\phi_l)\}\{1+2\mathrm{i}P_l/(R_{ll}^{-1}-L_l^0)\}, \tag{A1}$$

where ω_l is the Coulomb phase shift and $L_l^0 = S_l^0+\mathrm{i}P_l$, with $S_l^0 = S_l-B_l$. The nuclear phase shift δ_l is defined by

$$U_{ll} = \exp\{2\mathrm{i}(\omega_l+\delta_l)\}. \tag{A2}$$

One can therefore express δ_l in terms of R_{ll} or vice versa.

Lane and Thomas (1958) show that R_{ll} can be expressed in the form

$$R_{ll} = \sum_{\lambda\mu} \gamma_{\lambda l}\gamma_{\mu l}A_{\lambda\mu}, \tag{A3}$$

where the level matrix $\boldsymbol{A}$ is defined by

$$(A^{-1})_{\lambda\mu} = (E_\lambda-E)\delta_{\lambda\mu}-\xi_{\lambda\mu} \tag{A4}$$

with

$$\xi_{\lambda\mu} = \sum_c \gamma_{\lambda c}\gamma_{\mu c}L_c^0, \tag{A5}$$

the sum being over all channels c with total angular momentum $J=l$ except the retained channel l.

If all the channels c with nonzero $\gamma_{\lambda c}$ are closed then the $\xi_{\lambda\mu}$ are real, making R_{ll} and hence δ_l real. In this case (A1) and (A2) give

$$R_{ll} = \{P_l \cot(\delta_l+\phi_l)+S^0\}^{-1}, \tag{A6}$$

which is of the form assumed in Section III provided R_{ll} can be expressed in the form (2). This is possible if the $L_c^0 \equiv S_c^0$ are linear functions of E throughout the energy range considered. Then

$$R_{ll} = \sum_\lambda \bar{\gamma}_{\lambda l}^2/(\bar{E}_\lambda - E), \tag{A7}$$

where the $\bar{E}_\lambda$ are the q roots (for the q-level approximation) of the qth-order equation

$$\det|(E_\lambda-E)\delta_{\lambda\mu}-\xi_{\lambda\mu}| = 0, \tag{A8}$$

and

$$\bar{\gamma}_{\lambda l}^2 = -\left[\left\{\frac{\mathrm{d}}{\mathrm{d}E}\left(\sum_{\mu\nu}\gamma_{\mu l}\gamma_{\nu l}A_{\mu\nu}\right)^{-1}\right\}_{E=\bar{E}_\lambda}\right]^{-1}. \tag{A9}$$

Thus the one-channel approximation for α–α scattering may be justified over a certain energy range provided that only the $\alpha+\alpha$ channel is open and that for all other channels the S_c are linear functions of E. The parameters that enter the formulae are then, however, the $\bar{E}_\lambda$ and $\bar{\gamma}_{\lambda l}^2$ rather than the original E_λ and $\gamma_{\lambda l}^2$, the relation between them being given by (A8), (A9).

Thus for $l = 0$ one should not try to fit δ_0 above the energy $E = 17\cdot35$ MeV at which the ^{7}Li(0)+p channel opens, as the ^{8}Be ground state at least probably has a large reduced width for this channel—the spectroscopic factor from a shell model calculation (Barker 1966) is $\mathscr{S} \simeq 1\cdot5$. Also for this channel S is sufficiently linear for E from 0 to 17 MeV (confirmed by a two-channel fit to δ_0 over this range), although its curvature increases rapidly near threshold. Other channels are not expected to restrict further the range of validity of the one-channel approximation.

We should also comment on the connection between $\bar{E}_\lambda$, $\bar{\gamma}_{\lambda l}^2$ and E_λ, $\gamma_{\lambda l}^2$. Lane and Thomas (1958) considered one case in which these are simply related; in this they made ξ diagonal by choosing B_c to make S_c^0 small throughout the energy region of interest. This is not applicable here as S for the ^{7}Li(0)+p channel varies from about $-4\cdot0$ to $-1\cdot5$ as E goes from 0 to 17 MeV. A different approximation can make ξ diagonal and this may have more relevance for the present case. This is to assume that, apart from the $\alpha+\alpha$ channel, there is no channel c that has nonvanishing $\gamma_{\lambda c}$ for more than one level λ. This would be the case for nucleon channels for instance if the lowest 0^+ state belongs to the lowest configuration, the second 0^+ state to a 2p–2h configuration, the third to a 4p–4h configuration, etc. In this approximation

$$\bar{E}_\lambda = E_\lambda - \frac{\sum_c \gamma_{\lambda c}^2 S_c^0(E_\lambda)}{1+\sum_c \gamma_{\lambda c}^2 S_c'(E_\lambda)}, \qquad \bar{\gamma}_{\lambda l}^2 = \frac{\gamma_{\lambda l}^2}{1+\sum_c \gamma_{\lambda c}^2 S_c'(E_\lambda)}. \tag{A10}$$

One may choose B_c to make $\gamma_{\lambda c}^2 S_c^0(E_\lambda) = 0$ for each c, giving $\bar{E}_\lambda = E_\lambda$. Then (A7), (A10) are just the formulae (X, 2.3), (X, 2.4) of Lane and Thomas (1958), and their following discussion regarding the modified normalization of the $\gamma^2_{\lambda c}$ applies. The present fits give values of $\bar{\gamma}^2_{\lambda l}$, and, to the extent that closed channels are not normally included explicitly in R-matrix fits to levels, this corresponds to the quantity usually quoted as the reduced width of a level.

Similar arguments justify the use of a similar one-channel approximation for interpreting reactions, such as that discussed in Section V, as long as the ^{8}Be energies remain below about 17 MeV.

Appendix II

Variation of Level Parameters with Change of B

In order that the dependence on E of $\delta(E)$ given by equation (4) should be independent of B, we require the level parameters E'_λ and γ'^2_λ corresponding to some other value B' of B to satisfy

$$\left(\sum_{\lambda=1}^{q} \gamma'^2_\lambda/(E'_\lambda - E)\right)^{-1} + B' = \left(\sum_{\lambda=1}^{q} \gamma^2_\lambda/(E_\lambda - E)\right)^{-1} + B \tag{A11}$$

(for convenience we drop the suffix l). This requires the E'_λ to be the roots of the qth-order equation

$$E'^q_\lambda + \sum_{n=1}^{q} E'^{q-n}_\lambda \frac{(-1)^n}{n!} \sum_{\lambda_1 \ldots \lambda_n} \{1 - n(B'-B)\gamma^2_{\lambda_1}/E_{\lambda_1}\} E_{\lambda_1} E_{\lambda_2} \ldots E_{\lambda_n} = 0\,, \tag{A12}$$

and the γ'^2_λ to be given by

$$\gamma'^2_\lambda = \left(\sum_{n=1}^{q} E'^{q-n}_\lambda \frac{(-1)^{n-1}}{(n-1)!} \sum_{\lambda_1 \ldots \lambda_n} (\gamma^2_{\lambda_1}/E_{\lambda_1}) E_{\lambda_1} E_{\lambda_2} \ldots E_{\lambda_n}\right) \div \left(\prod_{\mu \neq \lambda} (E'_\lambda - E'_\mu)\right), \tag{A13}$$

where the Greek indices run from 1 to q, except that in $\sum_{\lambda_1 \ldots \lambda_n}$ terms for which any two of $\lambda_1 \ldots \lambda_n$ are the same are omitted.

In order that the dependence on E of σ_α given by equation (9) should be simultaneously independent of B, we similarly require

$$G'^{\frac{1}{2}}_{\lambda x} = \left(\prod_{\mu} (E'_\lambda - E_\mu) \sum_{\nu} G^{\frac{1}{2}}_{\nu x} \gamma_\nu/(E'_\lambda - E_\nu)\right) \div \left(\gamma'_\lambda \prod_{\mu \neq \lambda} (E'_\lambda - E'_\mu)\right). \tag{A14}$$

In the one-level case (A12), (A13) reduce to the well-known relations

$$E'_1 = E_1 - (B'-B)\gamma^2_1\,, \qquad \gamma'^2_1 = \gamma^2_1\,, \tag{A15}$$

while (A14) becomes

$$G'_{1x} = G_{1x}. \tag{A16}$$

Appendix III

Choice of B_0 *Value*

There are two criteria to consider in selecting the best value of B_0 to use in the present calculation. One lies within R-matrix theory and comes from the use of a three-level approximation. The other is dependent on the use of a nuclear model and is related to the physical assumption that the higher 0^+ levels are not fed appreciably in the ^{9}Be(p, d)^{8}Be reaction.

Within R-matrix theory, although the same fit to the data can be obtained with a q-level approximation for any value of B (we omit the suffix l), there may be some value of B that makes the parameters E_λ, γ_λ^2 giving this fit closest to their "correct" values, i.e. closest to those obtained with an ∞-level fit with the same B. Lane and Thomas (1958) point out that for the one-level approximation this is the case for $B \simeq S(E_1)$. As an illustration of how this should be generalized for more

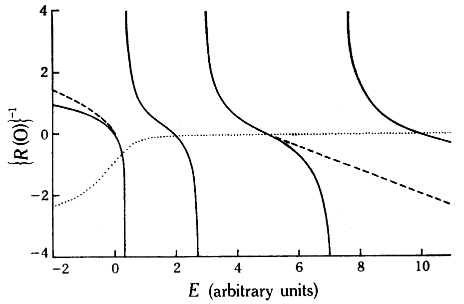

Fig. 4.—Illustrative example of three-level approximation (dashed curves) to "correct" R-function (solid curves). The ordinate is R^{-1} (for $B = 0$) and the abscissa is energy E. The shift factor S is shown as a dotted curve. Fitting is over the energy range $E = 0\cdot2$ to $4\cdot5$.

than one level, we consider an idealized case where the correct R-function is assumed to be a four-level R-function and one tries to fit this with a three-level approximation over a restricted energy range. For the correct R-function for arbitrary B we take

$$R(B) = \sum_{\lambda=1}^{4} \gamma_\lambda^2(B)/(E_\lambda(B)-E) \tag{A17}$$

with

$$\{R(B)\}^{-1}+B = \{R(0)\}^{-1}, \tag{A18}$$

in order that δ should be independent of B. The three-level approximation is taken as

$$\bar{R}(B) = \sum_{\lambda=1}^{3} \bar{\gamma}_\lambda^2(B)/(\bar{E}_\lambda(B)-E) \tag{A19}$$

with

$$\{\bar{R}(B)\}^{-1}+B = \{\bar{R}(0)\}^{-1}. \tag{A20}$$

In order to correspond to the type of fit to δ used in this paper we assume that the parameters of $\bar{R}(B)$ are chosen to make $\bar{R}(B)$ approximate $R(B)$ as closely as possible over an energy range from $E_g = E_1(S(E_g))$ to just below $E_3(0)$. This is illustrated schematically in Figure 4, where the solid curves give $\{R(0)\}^{-1}$ as a function of E. From (A17), (A18), these curves cut the line $\{R(0)\}^{-1} = B$ at the points

$E = E_\lambda(B)$ (Lane and Thomas 1958, equation (IV, 2.7)). Also $\gamma_\lambda^2(B)$ is given by the slope of the curves at $E = E_\lambda(B)$, since

$$\gamma_\lambda^2(B) = -([\mathrm{d}\{R(0)\}^{-1}/\mathrm{d}E]_{E=E_\lambda(B)})^{-1}$$

(Lane and Thomas 1958, equation (IV, 2.9)). The dashed curves in Figure 4 give $\{\bar{R}(0)\}^{-1}$, which closely approximates $\{R(0)\}^{-1}$ within the energy range used in the fit. The dotted curve gives S as a function of E.

Thus, provided $\bar{E}_\lambda(B)$ lies within the energy range used in the fit, both $\bar{E}_\lambda(B)$ and $\bar{\gamma}_\lambda^2(B)$ should be close to the correct values.* In the present example, $\bar{E}_2(B)$ and $\bar{\gamma}_2^2(B)$ should be accurate for all B, while $\bar{E}_1(B)$ and $\bar{\gamma}_1^2(B)$ should be accurate for any $B \leqslant S(E_g)$. Actually they will be accurate for more positive values of B than this, as the two curves do not diverge rapidly. From realistic numerical fits (in which the two curves are made to agree exactly at $E = E_g$) we have found that $\bar{E}_1(B)$ lies within 50 keV of $E_1(B)$ and $\bar{\gamma}_1^2(B)$ within 10% of $\gamma_1^2(B)$ for $B \leqslant 1 \cdot 5$. Thus provided B is not too positive, we can assume that the parameters obtained for the first and second levels are close to their correct values.

A possible restriction on the choice of B also comes from the use of condition (11), which is based on the assumption that the ^{8}Be and ^{9}Be levels are well described by shell model wave functions with little configuration mixing. Then the eigenfunction X_λ of the R-matrix theory should best describe the internal part of the actual wave function if $B = S(E_\lambda)$, as then X_λ joins smoothly onto an outgoing external wave function. Thus to describe the first level we should want to take $B = S(E_1)$ and, for the second level, $B = S(E_2)$, so the best compromise value of B should be somewhere in the region $S(E_1)$ to $S(E_2)$. Any B value in this range is acceptable according to the first criterion discussed above.

* In the one-level approximation $\{\bar{R}(0)\}^{-1}$ is a linear function of E and it is obvious that only a limited range of B values can give accurate parameter values.