EQUILIBRIUM POLOIDAL MAGNETIC FIELDS IN ROTATING STARS

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Summary

Solutions are presented for equilibrium poloidal dipole magnetic fields in a uniformly rotating upper main-sequence star with no meridional circulation currents. An upper limit is found for the ratio of kinetic energy of rotation to the magnetic field energy. The perturbations to the structure of the star are also given.

I. Introduction

Ferraro (1937) has established that, if a star has no internal mass motions, the presence of a poloidal magnetic field tends strongly to keep the star in a state of "isorotation" in which the angular velocity is constant around any given field line. Such a rotation law implies that meridional circulation currents will be generated, but these will tend to destroy the state of isorotation by convecting angular momentum. Since the high conductivity of the stellar material means that the magnetic field is "frozen" into the material, the circulation will also convect magnetic flux and so distort the field, which will eventually react on the circulation and tend to suppress it.

In the absence of rotation, an arbitrary field will, in general, itself cause meridional circulation. An equilibrium poloidal field has been calculated approximately by Monaghan (1966*a*) for which the meridional circulation ceases in a nonrotating star, assuming infinite conductivity. The existence of this solution suggests that an equilibrium state may exist in a rotating star in which the poloidal field stops the circulation that would be induced by the rotation alone and maintains a state of isorotation.

In the present paper we consider an upper main-sequence star in uniform (solid body) rotation. Assuming that the conductivity is infinite, equilibrium solutions for a poloidal dipole magnetic field with no meridional circulation have been found. It has been found that, as the ratio of kinetic energy of rotation to magnetic energy is increased from zero, the equilibrium field is progressively distorted and "submerged", the surface field strength being reduced to zero when the rotational energy is comparable with the magnetic energy.

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II. Basic Equations

A star having infinite electrical conductivity and rotating with uniform angular velocity $\mathbf{\Omega}$ with a poloidal magnetic field $\mathbf{H}_\mathrm{p}$ and in which no circulation of matter occurs is described by the following set of equilibrium equations:
hydrostatic equilibrium

$$\frac{\nabla P}{\rho}+\nabla\phi = \frac{(\nabla\times\mathbf{H}_\mathrm{p})\times\mathbf{H}_\mathrm{p}}{4\pi\rho}+\Omega^2\tilde{\boldsymbol{\varpi}}, \tag{1}$$

Poisson's equation

$$\nabla^2\phi = 4\pi G\rho, \tag{2}$$

divergence of field

$$\nabla.\mathbf{H}_\mathrm{p} = 0, \tag{3}$$

conservation of energy

$$\nabla.\mathbf{F} = \epsilon\rho, \tag{4}$$

radiative thermal equilibrium

$$\mathbf{F} = -(4ac/3\kappa\rho)T^3\,\nabla T, \tag{5}$$

convective thermal equilibrium

$$\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}T} = 4-\frac{1\cdot5\,\beta^2}{4-3\beta} = (n+1)_\mathrm{ad}. \tag{6}$$

Here P, ρ, T, and κ represent respectively the gas pressure, density, temperature, and opacity; ϕ is the gravitational potential, $\mathbf{F}$ the radiative energy flux, β the ratio of radiation pressure to total pressure, ϵ the rate of energy generation per unit mass, $\tilde{\boldsymbol{\varpi}}$ the vector displacement from the axis of rotation, and the other symbols have their usual meaning.

It will be assumed that the star is perturbed only slightly from a spherically symmetric model that has been obtained previously. The model chosen is of an initial main-sequence star (with uniform composition) of 15 solar masses as described by Stothers (1963). In Stothers's notation, the composition is

$$X = 0\cdot70, \qquad Y = 0\cdot27, \qquad Z = 0\cdot03, \qquad X_\mathrm{CNO} = \tfrac{1}{2}Z.$$

The opacity is due to electron scattering and is given by $\kappa = 0\cdot19(1+X)$. The energy generation law is taken as

$$\epsilon = \epsilon_0\, XX_\mathrm{CNO}\,\rho T^\nu, \tag{7}$$

where $\nu = 15$ and $\log\epsilon_0 = -106\cdot6$.

Radiation pressure is important in stars of this mass. Consequently, the equation of state is

$$P = (\mathscr{R}/w)\rho T+\tfrac{1}{3}aT^4 = (\mathscr{R}/w)\rho T/\beta, \tag{8}$$

where $\mathscr{R}$ is the gas constant and w is the mean molecular weight of the gas.

We will assume that the field is symmetric about the axis of rotation and use spherical polar coordinates (r, θ, φ), where θ is measured from the positive axis of rotation. Equation (3) can be satisfied by writing

$$H_r = r^{-2}\,\partial\psi/\partial\mu\,, \qquad H_\theta = r^{-1}(1-\mu^2)^{-\frac{1}{2}}\,\partial\psi/\partial r\,, \tag{9}$$

where $\mu = \cos\theta$, ψ is a scalar function of r and θ, and H_r and H_θ are the r and θ components of $\mathbf{H}_\mathrm{p}$.

According to the assumption that the star is only slightly perturbed from the spherical, we will use a first-order perturbation method with $\lambda = \mathscr{B}^2/GM^2$ as expansion parameter, where M is the mass of the star and $\mathscr{B}$ is a constant of the dimensions of ψ defined by

$$\psi = \mathscr{B}b\,. \tag{10}$$

The set of dimensionless variables (similar to those of Schwarzschild 1947) defined as follows will be used:

$$\left.\begin{aligned} r &= Rx\,, & \phi &= (GM/R)(\pi_0+\lambda\pi_1)\,, \\ P &= \frac{GM^2}{4\pi R^4}\left(p_0+\lambda\frac{\beta_0 p_0}{t_0}p_1\right), & T &= \frac{w}{\mathscr{R}}\frac{GM}{R}\left(t_0+\lambda\beta_0 t_1\right), \\ & & 4\pi r^2 F_r = L\{f_0+\lambda(xt_0^4/Cp_0)f_1\}\,, & \end{aligned}\right\} \tag{11}$$

where the subscript 0 denotes unperturbed variables, 1 denotes first-order perturbations, F_r is the radial component of $\mathbf{F}$, R and L are the radius and luminosity of the star respectively, and

$$C = \frac{3\kappa}{64\pi^2 ac}\left(\frac{\mathscr{R}}{wG}\right)^4 \frac{L}{M^3}\,.$$

It is also convenient to define the parameter ω^2 by

$$\omega^2\lambda = R^3\Omega^2/GM\,.$$

$\omega^2\lambda$ is thus a measure of the ratio of the kinetic energy of rotation to the gravitational energy, while λ is a measure of the ratio of total magnetic energy to gravitational energy. Both must be assumed small.

With the variables thus defined, the following set of first-order perturbation equations can be derived.

$$\frac{\partial\pi_1}{\partial x}+\frac{\partial p_1}{\partial x} = \left(\frac{4}{\beta_0}-3\right)\frac{V}{x}\left(t_1-\frac{p_1}{n+1}\right)+D_r\,, \tag{12a}$$

$$\frac{\partial\pi_1}{\partial\mu}+\frac{\partial p_1}{\partial\mu} = -(1-\mu^2)^{-\frac{1}{2}}xD_\mu\,, \tag{12b}$$

$$\frac{\partial^2\pi_1}{\partial x^2}+\frac{2}{x}\frac{\partial\pi_1}{\partial x}+\frac{1}{x^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\pi_1}{\partial\mu}\right) = \frac{\beta_0 p_0}{t_0^2}\left(p_1-(4-3\beta_0)t_1\right), \tag{12c}$$

$$\frac{\partial f_1}{\partial x} = \left(\frac{4V}{n+1} - V - 1\right)\frac{f_1}{x} + \frac{1}{x}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial t_1}{\partial \mu}\right), \tag{12d}$$

the radiative equation

$$\frac{\partial t_1}{\partial x} = -\frac{f_1}{x} - \frac{V}{\beta_0(n+1)x}\left(p_1 - \{4\beta_0 + (n+1)(1-\beta_0)\}t_1\right), \tag{12e}$$

and the convective equation

$$\{(n+1)-4\}\,\mathrm{d}[(\beta_0/t_0)\{p_1-(n+1)t_1\}] = (\beta_0/t_0)\{p_1-(n+1)t_1\}\,\mathrm{d}[(n+1)-4], \tag{12f}$$

where

$$D_r = -\frac{1}{x^2(1-\mu^2)}\frac{\partial b}{\partial x}\left(\frac{\partial^2 b}{\partial x^2} + \frac{1-\mu^2}{x^2}\frac{\partial^2 b}{\partial \mu^2}\right)\frac{t_0}{\beta_0 p_0} + (1-\mu^2)\omega^2 x,$$

$$D_\mu = -\frac{1}{x^3(1-\mu^2)^{\frac{1}{2}}}\frac{\partial b}{\partial \mu}\left[\frac{\partial^2 b}{\partial x^2} + \frac{1-\mu^2}{x^2}\frac{\partial^2 b}{\partial \mu^2}\right]\frac{t_0}{\beta_0 p_0} - \mu(1-\mu^2)^{\frac{1}{2}}\omega^2 x,$$

and

$$V = -(x/p_0)\,\mathrm{d}p_0/\mathrm{d}x, \qquad (n+1) = (t_0/p_0)\,\mathrm{d}p_0/\mathrm{d}t_0.$$

The perturbation to the adiabatic relation, equation (12f), can be integrated to give

$$p_1-(n+1)t_1 = \alpha(t_0/\beta_0)\{(n+1)-4\}, \tag{13}$$

where α is a constant of integration.

Separation of the independent variables is achieved by writing

$$b(x,\mu) = (1-\mu^2)\,B(x), \tag{14}$$

and expanding the other variables in the form

$$Q_1(x,\mu) = Q_{10}(x) + Q_{12}(x)\,\mathrm{P}_2(\mu), \tag{15}$$

where $\mathrm{P}_2(\mu) = (\frac{3}{2}\mu^2 - \frac{1}{2})$ is the second Legendre polynomial.

The equations (12) can be reduced to first order by putting

$$\eta = \mathrm{d}B/\mathrm{d}x, \qquad q_1 = x^2\,\partial\pi_1/\partial x. \tag{16}$$

Then q_1 corresponds to the perturbation in mass fraction. It is also convenient to define

$$\Gamma_1 = \pi_1 + p_1 - \tfrac{1}{2}\omega^2 x^2(1-\mu^2). \tag{17}$$

Substituting in (12) according to equations (14)–(17) and separating the coefficients of $\mathrm{P}_2(\mu)$, we obtain the following set of equations.

$$\frac{\mathrm{d}\Gamma_{12}}{\mathrm{d}x} = \left(\frac{4}{\beta_0} - 3\right)\frac{V}{x}\left(t_{12} - \frac{p_{12}}{n+1}\right) + \frac{\eta\Gamma_{12}}{B}, \tag{18a}$$

$$\frac{d\eta}{dx} = \frac{2B}{x^2}+\frac{3}{2}\frac{\beta_0 p_0}{t_0}\frac{x^2\Gamma_{12}}{B}, \tag{18b}$$

$$\frac{dB}{dx} = \eta, \tag{18c}$$

$$\frac{d\pi_{12}}{dx} = \frac{q_{12}}{x^2}, \tag{18d}$$

$$\frac{dq_{12}}{dx} = 6\pi_{12}+\frac{\beta_0 p_0 x^2}{t_0}\Big(p_{12}-(4-3\beta_0)t_{12}\Big), \tag{18e}$$

$$\frac{df_{12}}{dx} = \left(\frac{4V}{n+1}-V-1\right)\frac{f_{12}}{x}-\frac{6t_{12}}{x}, \tag{18f}$$

the radiative equation

$$\frac{dt_{12}}{dx} = \frac{V}{\beta_0(n+1)x}\Big(t_{12}\{4\beta_0+(n+1)(1-\beta_0)\}-p_{12}\Big)-\frac{f_{12}}{x}, \tag{18g}$$

and the convective equation

$$p_{12} = (n+1)t_{12}, \tag{18h}$$

where

$$p_{12} = \Gamma_{12}-\pi_{12}-\tfrac{1}{3}\omega^2x^2.$$

The terms independent of μ remaining in (12) give the set

$$\frac{d\Gamma_{10}}{dx} = \left(\frac{4}{\beta_0}-3\right)\frac{V}{x}\left(t_{10}-\frac{p_{10}}{n+1}\right)-\frac{\eta\Gamma_{12}}{B}, \tag{19a}$$

$$\frac{d\pi_{10}}{dx} = \frac{q_{10}}{x^2}, \tag{19b}$$

$$\frac{dq_{10}}{dx} = \frac{\beta_0 p_0 x^2}{t_0}\Big(p_{10}-(4-3\beta_0)t_{10}\Big), \tag{19c}$$

$$\frac{df_{10}}{dx} = \left(\frac{4V}{n+1}-V-1\right)\frac{f_{10}}{x}, \tag{19d}$$

the radiative equation

$$\frac{dt_{10}}{dx} = \frac{V}{\beta_0(n+1)x}\Big(t_{10}\{4\beta_0+(n+1)(1-\beta_0)\}-p_{10}\Big)-\frac{f_{10}}{x}, \tag{19e}$$

and the convective equation

$$p_{10} = (n+1)t_{10}+\alpha\{(n+1)-4\}t_0/\beta_0, \tag{19f}$$

where

$$p_{10} = \Gamma_{10}-\pi_{10}+\tfrac{1}{3}\omega^2x^2.$$

III. Boundary Conditions

The boundary conditions are exactly as stated by Monaghan (1966*a*) and are restated here in the final form appropriate to equations (18) and (19).

(1) *At $x = 0$*

(i) $\mathrm{d}p_{12}/\mathrm{d}x = p_{12}/x = 0$ and similarly for t_{12} and π_{12}.

(ii) $\mathrm{d}p_{10}/\mathrm{d}x = \mathrm{d}t_{10}/\mathrm{d}x = \mathrm{d}\pi_{10}/\mathrm{d}x = 0\,.$

(2) *At $x = 0$*

$B'' - 2B/x^2 = 0$ and B/x^2, B'/x are finite.

(3) *At $x = 1$*

(i) $3\pi_{12} + \mathrm{d}\pi_{12}/\mathrm{d}x = 0\,,$

(ii) $\pi_{10} = \mathrm{d}\pi_{10}/\mathrm{d}x = 0\,.$

(4) *At $x = 1$*

(i) $p_{12} - 4t_{12} = 0\,,$

(ii) $p_{10} - 4t_{10} = 0\,.$

(5) *At $x = 1$*

$B + \mathrm{d}B/\mathrm{d}x = 0\,.$

(6) *At $x = x_c$, the boundary of the convective core*

(i) B, $\mathrm{d}B/\mathrm{d}x$, p_{12}, t_{12}, π_{12}, $\mathrm{d}\pi_{12}/\mathrm{d}x$, $\mathrm{d}p_{12}/\mathrm{d}x$, $\mathrm{d}^2B/\mathrm{d}x^2$, and $\mathrm{d}^2\pi_{12}/\mathrm{d}x^2$ are continuous;

(ii) p_{10}, t_{10}, π_{10}, $\mathrm{d}p_{10}/\mathrm{d}x$, $\mathrm{d}\pi_{10}/\mathrm{d}x$, and $\mathrm{d}^2\pi_{10}/\mathrm{d}x^2$ are continuous.

IV. Solution of Equations

The nonlinear equations (18) initially had to be solved with zero rotation by an approximation method described by Monaghan (1966*a*). Thereafter solutions could be found directly.

In the convective core, where the adiabatic relation holds, the equations in Γ_{12}, η, and B decouple from those in π_{12} and q_{12}, and equation (18*a*) can be integrated to give

$$\Gamma_{12} = kB\,,$$

where k is a constant. A transformation of the form

$$\left.\begin{aligned}(B,\eta,\omega) &= a(B^*,\eta^*,\omega^*)\,,\\ (\Gamma_1,\pi_1,q_1,t_1,p_1,f_1) &= a^2(\Gamma_1^*,\pi_1^*,q_1^*,t_1^*,p_1^*,f_1^*)\,,\end{aligned}\right\} \tag{20}$$

where a is a constant, can be used to normalize the solution such that

$$\Gamma_{12} = B\,.$$

Singularities at $x = 0$ are avoided, and boundary conditions (1) (i) and (2) in Section III are satisfied by starting the solutions with the expansions

$$B = a_1 x^2, \qquad \pi_{12} = a_2 x^2.$$

The constants a_1 and a_2 determine the solution in the core. f_{12} is not determined in the core and its value at the core boundary is treated as a free parameter, since no condition is placed on this value by the energy equation (see below). This value and the boundary conditions (6) (i) then determine the envelope solution in terms of the core solution. The three outer boundary conditions (3) (i), (4) (i), and (5) then determine the three parameters a_1, a_2, and $(f_{12})_{\text{core}}$. The singularities at $x = 1$ mean that the solutions are divergent near $x = 1$, but it was found to be sufficiently accurate to apply the boundary conditions at $x = 0 \cdot 98$, rather than use series expansions from $x = 1$.

With η, B, and Γ_{12} determined from (18), equations (19) are linear and there is no difficulty in determining the correct values of the parameters involved. Solutions at $x = 0$ are started, according to the boundary conditions (1) (ii), with

$$\pi_{10} = a_3, \qquad \Gamma_{10} = a_4.$$

There is also the constant α in the adiabatic relation to be determined. However, the value of f_{10} at the core boundary is determined in terms of the core solution, as discussed below, and so the boundary conditions (6) (ii) completely determine the envelope solution in terms of the core solution. Once again the three outer boundary conditions (3) (ii) and (4) (ii) are applied at $x = 0 \cdot 98$ and determine the correct values of a_3, a_4, and α.

The value of f_{10} at the surface of the core is determined by the condition that the total change of flux through the core surface is due to the first-order change in energy generation. The energy generation in the unperturbed model is confined to the core. Integration of equation (4) gives

$$\int F_r \, \mathrm{d}S = \int \rho\epsilon \, \mathrm{d}V, \tag{21}$$

where the first integral is over the core surface and the second is over the core volume. Using formula (7) for ϵ we get, in the variables of (11)

$$\int \rho\epsilon \, \mathrm{d}V = \epsilon_0 X X_{\text{CNO}} \left(\frac{w}{\mathscr{R}}\right)^{\nu} \frac{G^{\nu} M^{\nu+2}}{4\pi R^{\nu+3}} (I_0 + \lambda I_1), \tag{22}$$

where

$$I_0 = \int_0^{x_c} \beta_0^2 p_0^2 t_0^{\nu-2} x^2 \, \mathrm{d}x,$$

$$I_1 = \int_0^{x_c} \beta_0^2 p_0^2 t_0^{\nu-3} [2p_{10} + \{(\nu+6)\beta_0 - 8\} t_{10}] x^2 \, \mathrm{d}x,$$

and $x = x_c$ is the core boundary. The terms dependent on angle make no contribution when the integration over θ is carried out. Similarly the left-hand side of (21) becomes

$$\int F_r \,\mathrm{d}S = L_0\left\{1+\lambda\left(\frac{xt_0^4}{Cp_0}f_{10}\right)_{x=x_c}\right\}. \tag{23}$$

Again there is no contribution by the angular-dependent terms. There will thus be no condition on f_{12}. Substituting (22) and (23) in (21), we get

$$(f_{10})_{x=x_c} = \left(\frac{Cp_0}{xt_0^4}\right)_{x=x_c}\frac{I_1}{I_0}. \tag{24}$$

The correct values of the parameters specifying the solution of the nonlinear equations were obtained as follows. The equations were integrated for four linearly independent trial sets of (a_1, a_2, f_{12}) and the outer values of these solutions compared with the boundary conditions. If the trial values are "sufficiently" accurate then a simultaneous linear extrapolation will give values of (a_1, a_2, f_{12}) for which the boundary conditions are more accurately satisfied. The boundary conditions can be satisfied to the desired accuracy by successive extrapolation.

The same procedure was used for the linear equations, except that only one extrapolation was necessary to satisfy the boundary conditions exactly. Since the whole process was done automatically on a computer, there was no advantage in superposing solutions of the homogeneous equations with particular solutions.

V. Results

The solution with zero rotation was first found using the "pseudopolytropic" approximation, followed by successive corrections (Monaghan 1966*a*). The accuracy of this solution was sufficient to provide values of the parameters (a_1, a_2, f_{12}) for which the method described in the previous section would converge to the solution of the nonlinear equations (18). The results provide a check on the accuracy of the approximation. The solutions for B are compared in Table 1: in column 2, the pseudopolytropic approximation; in column 3, the approximate solution after one cycle of corrections has been added; in column 4, the direct solution of the nonlinear equations. The comparison shows that the field structure can be obtained quite accurately by the approximation method. Of the other variables, π_{12} and q_{12} were obtained with similar accuracy, but two cycles of correction were needed to achieve 10% accuracy of p_{12}, t_{12}, and f_{12} in the outer layers of the star. Successive corrections did not improve this accuracy significantly.

For small amounts of rotation the approximation method can be used, but for rotation fast enough to distort the field significantly the method breaks down. This is because, in the first approximation, the field is unaffected by rotation, since the equations in B are decoupled from the other equations. Direct solutions of the nonlinear equations can still be obtained by extrapolating from the values of (a_1, a_2, f_{12}) obtained for solutions with slower rotation.

TABLE 1

COMPARISON OF APPROXIMATE SOLUTIONS WITH DIRECT SOLUTION

For the case $\omega^2 = 0$

(1) x	(2) B 1st Approx.	(3) B After 1st Correction	(4) B Direct Solution	(1) x	(2) B 1st Approx.	(3) B After 1st Correction	(4) B Direct Solution
0·0	0·0	0·0	0·0	0·6	−0·124	−0·108	−0·106
0·1	−0·016	−0·016	−0·015	0·7	−0·112	−0·095	−0·094
0·2	−0·055	−0·053	−0·053	0·8	−0·100	−0·084	−0·083
0·3	−0·097	−0·092	−0·091	0·9	−0·089	−0·075	−0·074
0·4	−0·124	−0·115	−0·113	0·98	−0·082	−0·069	−0·068
0·5	−0·131	−0·117	−0·115				

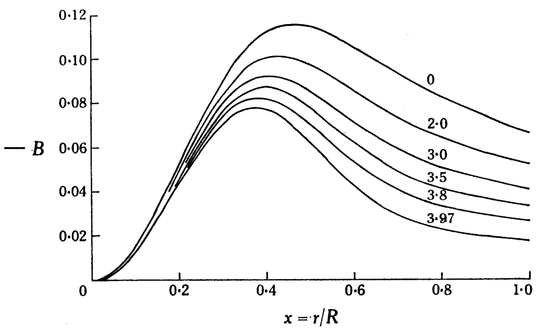

Fig. 1.—Solutions for B for different amounts of rotation, specified by ω^2 (see text). The curves are labelled with the respective values of ω^2.

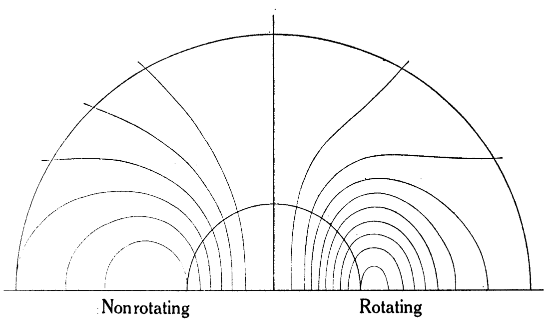

Fig. 2.—Equilibrium magnetic field lines in the uniformly rotating star. The left-hand quadrant shows the field in the absence of rotation. The right-hand quadrant shows the field for the case $\omega^2 = 3\cdot97$.

Solutions have been obtained for a series of values of ω^2 ranging from 0 up to about 4. The variation of B with x for different values of ω^2 is given in Figure 1. The corresponding field lines are given by

$$\psi = \mathscr{B}B\sin^2\theta = \text{constant},$$

and are illustrated in Figure 2 for the cases $\omega^2 = 0$ and $3\cdot97$. As ω^2 approaches its largest value, it can be seen that the surface field strength becomes progressively smaller relative to the field inside the star.

From equation (9), the surface polar field strength is

$$H_s = 2\mathscr{B}B(1)/R^2, \tag{25}$$

and the field strength H_c at the centre of the star, where $| B | = -a_1 x^2$, is

$$H_c = 2\mathscr{B}a_1/R^2. \tag{26}$$

We can use either of these expressions to eliminate $\mathscr{B}$ from the definition of λ. H_c will be the more useful parameter here, so we obtain

$$\lambda = R^4 H_c^2/4GM^2 a_1^2. \tag{27}$$

Now the angular velocity is obtained from the definition of ω^2:

$$\Omega^2 = R\omega^2 H_c^2/4Ma_1^2. \tag{28}$$

In Table 2 are listed values of a_1 and $B(x = 1)$ for different values of ω^2. We see that the angular velocity increases steadily with ω^2 for a given value of H_c.

TABLE 2

APPROACH TO LIMITING ROTATION

(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
ω^2	a_1	$B(1)$	I_B	ω^2	a_1	$B(1)$	I_B
0	1·62	$6\cdot66\times10^{-2}$	$12\cdot42\times10^{-2}$	3·8	1·43	$2\cdot64\times10^{-2}$	$7\cdot17\times10^{-2}$
2·0	1·55	$5\cdot13\times10^{-2}$	$9\cdot90\times10^{-2}$	3·9	1·42	$2\cdot27\times10^{-2}$	$6\cdot88\times10^{-2}$
3·0	1·50	$4\cdot09\times10^{-2}$	$8\cdot60\times10^{-2}$	3·97	1·40	$1\cdot75\times10^{-2}$	$6\cdot55\times10^{-2}$
3·5	1·46	$3\cdot32\times10^{-2}$	$7\cdot74\times10^{-2}$				

These results are best presented in terms of the total energies of the magnetic field and the rotation. The energy density of the field is $H^2/8\pi$. Integrating this through all space, the total field energy is

$$E_M = R^3 H_s^2 I_B/12B^2(1), \tag{29}$$

where

$$I_B = \int_0^1 \left\{\frac{2B^2}{x^2} + \left(\frac{dB}{dx}\right)^2\right\} dx + B(1)$$

and the integral outside the star was evaluated using the exterior solution $B = B(1)/x$. I_B is tabulated in column 4 of Table 2. The kinetic energy of rotation is

$$E_\Omega = \tfrac{1}{2}\int \rho\Omega^2\varpi^2\, dV = \tfrac{1}{3}MR^2\Omega^2 I_\Omega, \tag{30}$$

where

$$I_\Omega = \int_0^1 t^{-1}\beta p x^4\, dx = 0\cdot159.$$

From (29) and (30) the ratio of rotational energy to magnetic energy is

$$E_\Omega/E_M = I_\Omega\,\omega^2/I_B. \tag{31}$$

In Figure 3, the dependence of the surface field strength on the energy of rotation for a given magnetic energy has been displayed by plotting $B^2(1)/I_B$ against ω^2/I_B. Also plotted is ω^2. From this we see that the surface field is reduced to zero at $\omega^2/I_B \simeq 65$, corresponding to $\omega^2 \simeq 4\cdot 0$ and $E_\Omega/E_M \simeq 10$. From (28) we get the limiting relation

$$\Omega \simeq 4\cdot 2\times 10^{-12} H_c(\bar{R}/\bar{M})^{\frac{1}{2}}. \tag{32}$$

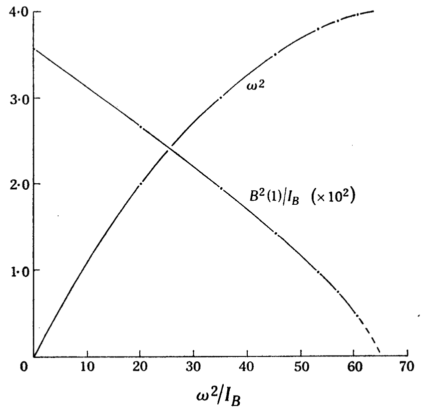

Fig. 3.—Showing the dependence of the surface field strength, $H_s^2 = (12E_M/R^3)\,B^2(1)/I_B$, and ω^2 on the ratio of rotational to magnetic energy, $E_\Omega/E_M = I_\Omega\,\omega^2/I_B$.

The perturbations to the structure of the star can be obtained from the solutions for the remaining variables of equations (18) and (19). Observable changes in the star will depend on the values of the perturbations at the surface, and in Table 3 are listed the values at $x = 1$ of p_{10}, p_{12}, f_{10}, and f_{12} for five values of ω^2.

TABLE 3

PERTURBATIONS AT THE SURFACE

For the case $x = 1$

ω^2	0	2·0	3·0	3·5	3·97
p_{10}	−0·1034	0·319	0·549	0·666	0·772
p_{12}	−0·0304	−0·684	−1·016	−1·179	−1·332
f_{10}	−0·0178	−0·180	−0·263	−0·301	−0·340
f_{12}	0·0437	0·387	0·564	0·647	0·729

The radial energy flux at the surface is (using the surface series solutions for the zero-order model)

$$F_R = F_{R0}\{1+\lambda(4f_{10}+4f_{12}P_2)\}, \tag{33}$$

F_{R0} being the surface energy flux of the unperturbed model. The total luminosity is

$$L = L_0(1+4f_{10}\lambda). \tag{34}$$

The position of the distorted surface is defined by the boundary condition $P = T = 0$.

To the first order in λ this condition gives, at $x = 1$,

$$p_1 + x_1(\mathrm{d}p_0/\mathrm{d}x)_{x=1} = 0\,, \tag{35}$$

where the new surface is given by

$$x = 1 + \lambda x_1 = 1 + \lambda(x_{10} + x_{12}\,\mathrm{P}_2)\,.$$

From the zero-order model and (35) we get

$$x_1 = p_1/\beta_0 = 1\cdot 04\,p_1\,. \tag{36}$$

In all cases the total luminosity of the star is decreased, the surface is brighter near the poles than near the equator, and the surface is distorted into an oblate spheroid. The nonrotating model shows an overall contraction, as was found by Monaghan (1966*b*) for the polytrope $n = 3$. Rotation increases the oblateness and causes an overall expansion, but the polar radius is further decreased. The distortion of the surface corresponds to an eccentricity

$$e^2 = -3x_{12}\,\lambda\,. \tag{37}$$

For zero rotation

$$e^2 = 5\cdot 3(R^4H_{\mathrm{s}}^2/GM^2)\,,$$

the coefficient in this expression falling between the value $15\cdot 1$, found by Monaghan (1966*b*) for the nonrotating polytrope $n = 3$, and the value ~ 2 for the same case corrected to keep the central temperature of the polytrope constant.

It is interesting to note the mechanical effect of the combined field and rotation as revealed by the density perturbation. In the nonrotating model the star is compressed simultaneously towards the centre and the equatorial plane, so that the core density increases, but the maximum density perturbation occurs at $x \simeq 0\cdot 25$ on the equatorial plane. The effect of rotation is to increase the compression of the core, so that the density perturbation maximum shifts rapidly to $x = 0$. The increase in density is accompanied by a decrease in the central temperature and an increase in the central pressure.

VI. Discussion

The distortion produced in the equilibrium magnetic field by rotation reflects the pattern of meridional circulation that was found by Sweet (1950) in a uniformly rotating star of uniform composition in the absence of a magnetic field. However, the results must have been affected by the angular dependence of the field assumed in equation (14), which selects the dipole component of the field. As the effects produced by rotation become more important the field structure should become more complicated. These complications cannot easily be treated, because the field cannot be expanded as a multipole series of the form

$$b = \Sigma\, B_{2n}(x)\,\mathrm{P}_{2n}(\mu)\,,$$

say, since b enters the hydrostatic equation nonlinearly.

The assumption of infinite electrical conductivity overestimates the coupling between the circulation currents and the field, but this overestimate will be least for a large-scale field of the form considered here, since this would have the slowest diffusion rate if the conductivity were finite. The limiting case with zero surface field that was found should therefore be an indication of the stage at which a more realistic field would be dragged below the surface. From the limiting relation (32), a slowly rotating star with $\Omega \simeq 10^{-6}$ would require an internal field strength of about 5×10^{5} gauss to halt circulation, while for a star with $\Omega \simeq 5\times10^{-4}$, near the maximum allowable angular velocity, a field of about 2×10^{8} gauss would be required.

The validity of the equilibrium solutions obtained here depends on the time scale of approach to equilibrium. Large-scale changes of the field structure due to the circulation currents require the circulation to transport matter a distance of the order of the stellar radius or more within the lifetime of the star. This requires (Roxburgh 1964)

$$\Omega^2 \gtrsim 10^{-13}\bar{R}\bar{L}/\bar{M},$$

where the bars denote solar units. For very massive stars this implies an angular velocity close to the dynamical limit. For, say, an A5 star with $\log\bar{M} = 0\cdot34$, $\log\bar{R} = 0\cdot25$, and $\log\bar{L} = 1\cdot3$ (Allen 1963) it implies $\Omega \gtrsim 10^{-6}$. It is unlikely that the star would be so near its limiting state that the field would be highly concentrated towards the centre, so, with the minimum interior field strength found for $\Omega = 10^{-6}$, the surface field strength might be of the order of 10^{4} gauss. A-stars characteristically have angular velocities much greater than this, of the order of 10^{-4}. The corresponding surface field is 10^{6} gauss.

In conclusion, if a star has sufficient angular velocity for a magnetic field to approach the equilibrium state within the star's lifetime, these results indicate that an observable field much stronger than those that have been observed would be required to halt the induced meridional circulation. Otherwise, the results give an indication of the mechanical effects on the star of a large-scale poloidal field.

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VIII. References

Allen, C. W. (1963).—"Astrophysical Quantities." (Athlone Press, Univ. London.)
Ferraro, V. A. C. (1937).—*Mon. Not. R. astr. Soc.* **97**, 458.
Monaghan, J. J. (1966*a*).—*Mon. Not. R. astr. Soc.* **132**, 1.
Monaghan, J. J. (1966*b*).—*Mon. Not. R. astr. Soc.* **134**, 275.
Roxburgh, I. W. (1964).—*Mon. Not. R. astr. Soc.* **128**, 157.
Schwarzschild, M. (1947).—*Astrophys. J.* **106**, 427.
Stothers, R. (1963).—*Astrophys. J.* **138**, 1074.
Sweet, P. A. (1950).—*Mon. Not. R. astr. Soc.* **110**, 548.