NONLINEAR ADIABATIC PULSATIONS OF MASSIVE STARS

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Summary

The pulsational properties of a sequence of massive stars have been investigated in this paper. The second-order approximation to the equation of motion governing the adiabatic radial oscillations of these stars has been determined allowing for the variation of the radiation pressure throughout the star. In each case the form of the radial velocity curve at the surface of the star has been established taking into account the influence of higher modes of oscillation.

I. Introduction

This investigation into the pulsational properties of massive stars, with any uniform composition, has been based on a sequence of four stellar models $\mathscr{M} = 10$, 15, 20, and 30, where

$$\mathscr{M} = \mu^2 M/M_\odot,$$

μ being the mean molecular weight of the stellar material. These models were initially constructed by Van der Borght (1964*a*) to study the evolution of massive stars. Radiation pressure has been taken into account fully with electron scattering as the main source of opacity:

$$\kappa = 0\cdot 2004(1+X),$$

where X is the abundance of hydrogen. Furthermore, if β defines the ratio of the gas pressure to the total pressure we have

$$\Gamma_1 = \beta + \tfrac{2}{3}(4-3\beta)^2/\{\beta+8(1-\beta)\}$$

for the adiabatic exponent in the presence of radiation. Table 1 gives the range of β throughout the star in each case (Van der Borght 1964*a*). These stellar models have provided the equilibrium values of the pressure p, the density ρ, the mass $M(r)$ contained within a sphere of radius r, as well as the values of g and β throughout the star.

In this paper we consider the anharmonic radial pulsations of these models and determine the radial velocity curve at the surface of the star when the variation of the radiation pressure throughout the star is taken into account. In addition, having taken a sequence of models with different mass, yet of similar chemical composition, we have been able to form comparative conclusions about the effect of increasing mass, as well as observing the effect of including higher modes of oscillation, on the form of the radial velocity curve. The pulsational stability of these models

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has also been investigated and will be the subject of a later study concerned with the effect of including the nonadiabatic terms in the equation of motion.

An unexpected facet of stellar pulsations was encountered with the massive star of $\mathscr{M} = 20$ when resonance was established between the first and second modes of oscillation.

Table 1

VARIATION OF β

$$\beta = \frac{\text{gas pressure}}{\text{total pressure}}$$

$\mathscr{M}$	β Surface	β Centre	$\mathscr{M}$	β Surface	β Centre
10	0·904	0·794	20	0·790	0·654
15	0·843	0·714	30	0·706	0·571

II. Linear Equation

To establish the exact form of the radial velocity curve of a pulsating star it is first necessary to determine the solution of the equation of motion governing the pulsations. However, for the stellar models under consideration in this paper, no general solution of this partial differential equation can be established, and accordingly we have introduced an expansion involving the eigenfunctions of the linear problem as a method of solution for this equation. Hence as a first step we consider the linear problem and derive the eigenfunctions for the respective stellar models.

The linear differential equations governing small radial adiabatic pulsations may be written in the form (Schwarzschild and Härm 1959)

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta r}{r}\right) = -\frac{1}{\Gamma_1}\frac{\delta p}{p} - 3\frac{\delta r}{r}, \tag{1}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta p}{p}\right) = \left\{\frac{\delta p}{p} + \left(4+\omega^2\frac{x^3}{q}\right)\frac{\delta r}{r}\right\}\frac{\rho}{p}\frac{GM(r)}{Rx}, \tag{2}$$

where δr and δp are the amplitudes of the pulsations in position and pressure with $\delta r/r$ and $\delta p/p$ defining the relative amplitudes respectively. In these equations p, ρ, and r refer to the equilibrium values of pressure, density, and radius respectively, R is the radius of the star, and the nondimensional variables x and q, together with ω^2, have been introduced such that

$$x = \frac{r}{R}, \qquad q = \frac{M(r)}{M}, \qquad \text{and} \qquad \omega^2 = \frac{R^3}{GM}\left(\frac{2\pi}{\text{period}}\right).$$

The boundary conditions associated with equations (1) and (2) require $\delta r = 0$ at the centre for $r = 0$, and $\delta p = 0$ at the surface for $r = R$. These conditions arise from the physical requirements that the Lagrangian displacement and the Lagrangian variation in pressure should be zero at the centre and surface respectively.

The solution of the eigenvalue problem, represented by equations (1) and (2) subject to the above boundary conditions, was undertaken using the method given by Van der Borght (1964*b*). Table 2 gives for each stellar model the first six eigenvalues ω_k, with ω_1 corresponding to the fundamental mode of oscillation. Furthermore, the associated eigenfunctions $\eta_k = \delta r/r$ together with their first derivatives $\eta_k' = \mathrm{d}\eta_k/\mathrm{d}x$ were derived and tabulated in normalized form for the later calculation of the coefficients $C_{ij,k}$ as detailed in the next section.

Table 2

EIGENVALUES FOR FIRST SIX MODES OF OSCILLATION

Model	Mode					
$\mathscr{M}$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
10	2·0507	3·6487	4·9764	6·2587	7·5160	8·7563
15	1·8657	3·5699	4·8924	6·1628	7·4069	8·6355
20	1·7457	3·5187	4·8358	6·0976	7·3343	8·5572
30	1·5971	3·4525	4·7602	6·0125	7·2430	8·4604

III. Nonlinear Radial Oscillations

From the basic Lagrangian equations, as given by Ledoux and Walraven (1958), the equation of motion for radial adiabatic oscillations can be written in the form

$$\frac{\partial^2 r}{\partial t^2} = -\frac{r^2}{\rho_0 r_0^2}\frac{\partial}{\partial r_0}\left\{p_0\left(\frac{3r_0^2}{\partial(r^3)/\partial r_0}\right)^{\Gamma_1}\right\} - G\frac{M(r_0)}{r^2}, \tag{3}$$

where the zero subscript indicates the equilibrium value.

Substituting

$$r = r_0(1+\xi), \qquad \xi(r_0, t) \tag{4}$$

into equation (3) and using primes to denote differentiation with respect to r_0, it follows that

$$\rho_0 r_0 \frac{\partial^2 \xi}{\partial t^2} = -(1+\xi)^2 \frac{\partial}{\partial r_0}\left(p_0(1+\xi)^{-2\Gamma_1}(1+\xi+r_0\xi')^{-\Gamma_1}\right) - g_0\rho_0(1+\xi)^{-2}.$$

Neglecting third and higher order terms of ξ on expansion, and recalling that Γ_1 varies with r_0, we obtain

$$\begin{aligned}\rho_0 r_0 \frac{\partial^2 \xi}{\partial t^2} = {} & r_0 p_0 \Gamma_1 \xi'' + (p_0\Gamma_1' r_0 + 4p_0\Gamma_1 - g_0\rho_0\Gamma_1 r_0)\xi' + (4g_0\rho_0 + 3p_0\Gamma_1' - 3g_0\rho_0\Gamma_1)\xi \\ & -(9p_0\Gamma_1\Gamma_1' + \tfrac{15}{2}p_0\Gamma_1' - \tfrac{9}{2}g_0\rho_0\Gamma_1^2 + \tfrac{9}{2}g_0\rho_0\Gamma_1 + 2g_0\rho_0)\xi^2 \\ & -(6p_0\Gamma_1'\Gamma_1 r_0 - p_0\Gamma_1' r_0 + 12p_0\Gamma_1^2 - 4p_0\Gamma_1 - 3g_0\rho_0\Gamma_1^2 r_0 + g_0\rho_0 r_0\Gamma_1)\xi\xi' \\ & -(3p_0 r_0\Gamma_1^2 - p_0 r_0\Gamma_1)\xi\xi'' \\ & -(p_0\Gamma_1'\Gamma_1 r_0^2 + \tfrac{1}{2}p_0\Gamma_1' r_0^2 + 4p_0\Gamma_1^2 r_0 + 2p_0\Gamma_1 r_0 - \tfrac{1}{2}g_0\rho_0\Gamma_1^2 r_0^2 - \tfrac{1}{2}g_0\rho_0\Gamma_1 r_0^2)\xi'^2 \\ & -(p_0 r_0^2\Gamma_1 + p_0 r_0^2\Gamma_1^2)\xi'\xi''. \end{aligned} \tag{5}$$

The eigenfunctions $\eta_i(r_0)$ of the linear problem have been introduced by adopting a solution of the form

$$\xi = \sum_{i=1}^{n} \eta_i(r_0)\, q_i(t)\,. \tag{6}$$

Noting that the $\eta_i(r_0)$ determined in Section II satisfy the orthogonality conditions

$$\int_0^R \rho_0 r_0^4 \eta_i \eta_j \,\mathrm{d}r_0 = 0 \qquad \text{if} \qquad i \neq j\,, \tag{7}$$

and taking $n = 6$, that is, considering the first six modes of oscillation, we now proceed to establish the differential equations that will give the q's as functions of time.

On substitution of (6) into (5) the linear terms in ξ on the right-hand side of (5) reduce to

$$-\sum_{i=1}^{6} \sigma_i^2 \rho_0 r_0 \eta_i q_i$$

when an alternative form of the linear problem

$$\eta_i'' + \left(\frac{4}{r_0} + \frac{\Gamma_1'}{\Gamma_1} - \frac{g_0\rho_0}{p_0}\right)\eta_i' + \left(\frac{\sigma_i^2\rho_0}{\Gamma_1 p_0} + \frac{3\Gamma_1'}{r_0\Gamma_1} - \frac{3g_0\rho_0}{r_0 p_0} + \frac{4g_0\rho_0}{r_0 p_0 \Gamma_1}\right)\eta_i = 0\,, \tag{8}$$

where $\sigma_i^2 = \omega_i^2 GM/R^3$, is taken into account.

Following this substitution we multiply the resulting equation successively by

$$r_0^3 \eta_k(r_0)\,\mathrm{d}r_0\,, \qquad k = 1, 2, \ldots, 6$$

and then integrate over the volume of the star, observing that this procedure enables us to apply the orthogonality conditions (7). Finally, by writing $\tau = \sigma_1 t$ for the time variable, and with $x = r/R$, the resulting differential equations take the form

$$\frac{\mathrm{d}^2 q_k}{\mathrm{d}\tau^2} = -(\sigma_k^2/\sigma_1^2)q_k - \sum_{i,j} C_{ij,k}\, q_i q_j\,, \qquad k = 1, 2, \ldots, 6\,, \tag{9}$$

where

$$\sigma_k^2 = (GM/R^3)\omega_k^2\,, \qquad I_k = \frac{\omega_1^2 G}{\mathscr{R}} \int_0^1 \frac{\beta\bar{p}}{\bar{t}}\, x^4 \eta_k^2 \,\mathrm{d}x\,,$$

and

$$\begin{aligned} C_{ij,k} = \mathscr{M}^{-1}\Bigg[&\int_0^1 \left\{2\Gamma_1'(9\Gamma_1 + \tfrac{15}{2})x^2 - (9\Gamma_1^2 - 9\Gamma_1 - 4)\frac{\beta G\bar{m}}{\mathscr{R}\bar{t}}\right\}\bar{p}x\eta_i\eta_j\eta_k\,\mathrm{d}x \\ &+ \int_0^1 \left\{\Gamma_1' x^2(6\Gamma_1 - 1) - \Gamma_1(1 - 3\Gamma_1)\left(4x - \frac{\beta G\bar{m}}{\mathscr{R}\bar{t}}\right)\right\}\bar{p}x^2(\eta_i\eta_j' + \eta_i'\eta_j)\eta_k\,\mathrm{d}x \\ &+ \int_0^1 \Gamma_1(3\Gamma_1 - 1)\bar{p}x^4(\eta_i\eta_j'' + \eta_i''\eta_j)\eta_k\,\mathrm{d}x \\ &+ \int_0^1 \left\{2\Gamma_1' x^2(\Gamma_1 + \tfrac{1}{2}) - \Gamma_1\left(\frac{G\bar{m}(\Gamma_1 + 1)}{\mathscr{R}\bar{t}} - 4x(2\Gamma_1 + 1)\right)\right\}\bar{p}x^3\eta_i'\eta_j'\eta_k\,\mathrm{d}x \\ &+ \int_0^1 \Gamma_1(\Gamma_1 + 1)\bar{p}x^5(\eta_i\eta_j'' + \eta_i''\eta_j)\eta_k\,\mathrm{d}x\Bigg]\Big((1 + \delta_j^i)I_k\Big)^{-1}\,, \end{aligned}$$

δ_j^i being the Kronecker delta.

Moreover, we have introduced the new variables $\bar{p}$, $\bar{t}$, and $\bar{m}$ as tabulated in Van der Borght (1964*a*), which enables us to evaluate the coefficients of the six differential equations (9) directly from the data provided by the stellar models.

TABLE 3

VALUES OF COEFFICIENTS σ_k^2/σ_1^2 AND $C_{ij,k}$ OCCURRING IN EQUATIONS (9) FOR SIX MODES OF OSCILLATION

For the model $\mathscr{M} = 20$

i, j	$k = 1$	2	3	4	5	6
			σ_k^2/σ_1^2			
	1·0000	4·0629	7·6736	12·2005	17·6513	24·0284
			$C_{ij,k}$			
1, 1	3·3194	−14·6495	31·7457	−49·9432	57·1571	−47·4572
1, 2	−0·2103	−0·5776	−5·6690	4·4219	−6·1191	2·4007
1, 3	0·0215	−1·2276	−2·9899	−6·4225	0·7110	−5·9993
1, 4	−0·0065	−0·0940	−2·8324	−6·1965	−8·8701	−5·1163
1, 5	−0·0012	−0·0880	−0·6617	−5·5409	−11·3863	−14·4451
1, 6	0·0010	−0·0670	−0·5849	−2·7893	−11·2610	−19·9250
2, 2	−0·0087	−2·2999	−8·4981	−7·8841	−3·3469	−8·2862
2, 3	−0·0108	−1·0348	−7·9207	−16·3942	−14·5574	−14·9762
2, 4	−0·0014	−0·3322	−4·0605	−17·1232	−28·9952	−30·5427
2, 5	−0·0015	−0·1522	−1·9184	−11·6683	−34·4221	−51·4463
2, 6	−0·0015	−0·1616	−1·6132	−8·8369	−31·4068	−64·8958
3, 3	−0·0035	−0·8571	−7·0089	−21·5485	−32·2827	−34·6511
3, 4	−0·0034	−0·4662	−4·9857	−20·6151	−45·5674	−62·5901
3, 5	−0·0021	−0·2798	−3·3428	−17·7354	−50·9637	−88·7908
3, 6	−0·0026	−0·2836	−2·9855	−16·2186	−53·3670	−108·0211
4, 4	−0·0028	−0·4907	−4·9584	−22·1682	−57·6872	−96·4446
4, 5	−0·0034	−0·4340	−4·6840	−23·0341	−67·6441	−127·7014
4, 6	−0·0039	−0·4527	−4·7801	−24·6854	−77·5751	−157·4310
5, 5	−0·0043	−0·5588	−5·7257	−28·1125	−85·0959	−168·9295
5, 6	−0·0057	−0·6591	−6·7403	−33·5582	−104·1346	−213·0241
6, 6	−0·0076	−0·8699	−8·6996	−42·8874	−133·1334	−274·7825

Indeed one can readily verify the following relationships between the two sets of variables

$$\rho = \frac{M_{\odot}\beta\bar{p}}{R^3\mathscr{R}\bar{t}\mu^2}, \qquad p = \frac{M_{\odot}^2\bar{p}}{R^4\mu^4}, \qquad g = \frac{G}{R^2x^2}\bar{m}\frac{M_{\odot}}{\mu^2},$$

$$M(r) = \bar{m}(M_{\odot}/\mu^2), \qquad \text{and} \qquad \mathscr{M} = \mu^2(M/M_{\odot}),$$

where the mean molecular weight $\mu = (2X+0\cdot75\,Y+0\cdot5\,Z)^{-1}$. After making

allowance for the separate formulation of $\mathrm{d}\beta/\mathrm{d}x$ in the core and envelope (Van der Borght 1964*a*), that is,
in the convective core

$$\frac{\mathrm{d}\beta}{\mathrm{d}x}=\frac{G\beta(1-\beta)}{\mathscr{R}x^2}\frac{\bar{m}}{\bar{t}}\frac{3\beta^2}{32-24\beta-3\beta^2}$$

and in the radiative envelope

$$\frac{\mathrm{d}\beta}{\mathrm{d}x}=\frac{1}{\mathscr{R}}\frac{\beta(1-\beta)}{\bar{t}}\frac{1}{x^2}\left(\frac{\kappa\mu^2L}{4\pi c(1-\beta)M_\odot}-\bar{m}G\right),$$

we computed values of Γ_1' throughout the star using

$$\frac{\mathrm{d}\Gamma_1}{\mathrm{d}x}=\frac{(21\beta^2-48\beta+32)}{3(8-7\beta)^2}\frac{\mathrm{d}\beta}{\mathrm{d}x}.$$

To calculate η_i'' we expressed (8) in terms of the new variables and then used the values of η_i and η_i' as determined previously from (1) and (2).

The coefficients $C_{ij,k}$ of the differential equations (9) have been tabulated in Table 3 for the star $\mathscr{M}=20$ only, as the coefficients determined in the case of the other three stars under consideration follow a similar pattern. We should remark at this stage that the radius R of the star has been eliminated from these calculations and that whereas the quantities X, Y, and Z, determining the uniform composition of the star, are not directly involved in the calculations they do enter implicitly through the quantity $\mathscr{M}$.

IV. Numerical Solutions and Radial Velocity Curves

The system of simultaneous second-order differential equations (9) has been solved by numerical integration for each stellar model. The following initial values were adopted

$$q_1(0)=0\cdot03\,,\qquad \frac{\mathrm{d}q_k(0)}{\mathrm{d}\tau}=0\,,\qquad \text{for}\quad k=1,2,\ldots,6\,,$$

and an iterative procedure was used to determine the values of

$$q_k(0)\,,\qquad k=2,3,\ldots,6\,,$$

under the requirement that the solutions for the second and higher modes of oscillation should be periodic and have the same period as the first mode $q_1(\tau)$.

As evidenced in Table 4, which lists the values of $q_k(0)$ as determined above, a factor that no doubt assisted in the simultaneous solution of the differential equations is the relative independence of the $q_k(0)$ values when additional higher modes are taken into account.

As mentioned in Section I, a resonance interaction was encountered between the first and second modes of oscillation for the star $\mathscr{M}=20$. From Table 3 it is seen that

$$\sigma_2^2/\sigma_1^2=\omega_2^2/\omega_1^2=4\cdot0629$$

and some significance can apparently be placed on the value of this ratio.

The radial velocity curve at the surface of the star is determined, for one period of τ, by

$$\frac{dq(\tau)}{d\tau} = \sum_{k=1}^{6} \frac{dq_k(\tau)}{d\tau}\eta_k(1) = \sum_{k=1}^{6} \frac{dq_k(\tau)}{d\tau}$$

as the $\eta_k(x)$ have been normalized at the surface $x = 1$.

The skewness of the radial velocity curve is established by the factor K, the ratio of the rise of the radial velocity from minimum to maximum to the total period, and can be considered to some extent as a measure of the effect of including the

TABLE 4

RESULTS OF NUMERICAL COMPUTATIONS

Values are given of $q_k(0)$ adopted to ensure periodicity, skewness of the radial velocity curve K, and period of oscillation τ

No. of Modes	$q_1(0)$	$q_2(0)$	$q_3(0)$	$q_4(0)$	$q_5(0)$	$q_6(0)$	K	τ
				$\mathscr{M} = 10$				
First								
4	0·0300	−0·0036	−0·0048	0·0044			0·302	6·290
5	0·0300	−0·0036	−0·0048	0·0040	−0·0030		0·302	6·290
6	0·0300	−0·0036	−0·0048	0·0040	−0·0030	0·0025	0·310	6·290
				$\mathscr{M} = 15$				
4	0·0300	−0·0158	−0·0048	0·0046			0·254	6·300
5	0·0300	−0·0159	−0·0046	0·0043	−0·0121		0·159	6·300
6	0·0300	−0·0158	−0·0047	0·0044	−0·0083	0·0025	0·159	6·300
				$\mathscr{M} = 20$				
4	0·0300	0·0632	−0·0078	0·0009			0·671	6·335
5	0·0300	0·0632	−0·0079	0·0010	0·0011		0·679	6·335
6	0·0300	0·0641	−0·0077	0·0012	0·0074	0·0077	0·758	6·335
				$\mathscr{M} = 30$				
4	0·0300	0·0146	−0·0036	0·0046			0·717	6·350
5	0·0300	0·0146	−0·0043	0·0046	−0·0036		0·701	6·350
6	0·0300	0·0146	−0·0051	0·0047	−0·0034	0·0027	0·693	6·350

nonlinear terms in the equation of motion. We would expect the value of K from the linear theory to be one-half. Values of K, taking into account the first four, five, and six modes of oscillation, are tabulated in Table 4.

The main significance of these results is that, overall, the inclusion of higher modes of oscillation has relatively little effect on the value of K and the general shape of the radial velocity curve, whereas an increase in mass (and radiation pressure) produces an appreciable change in both the skewness and the form of the radial velocity curve. This trend is illustrated in Figure 1. There is also an increase in the period of oscillation corresponding to an increase in mass; for the star $\mathscr{M} = 10$ we have the value of $\tau = 6{\cdot}29$ for one period, and for the star $\mathscr{M} = 30$,

$\tau = 6\cdot35$, as compared with the linear case where $\tau = 6\cdot28$ for one period. Values of the period τ are also given in Table 4.

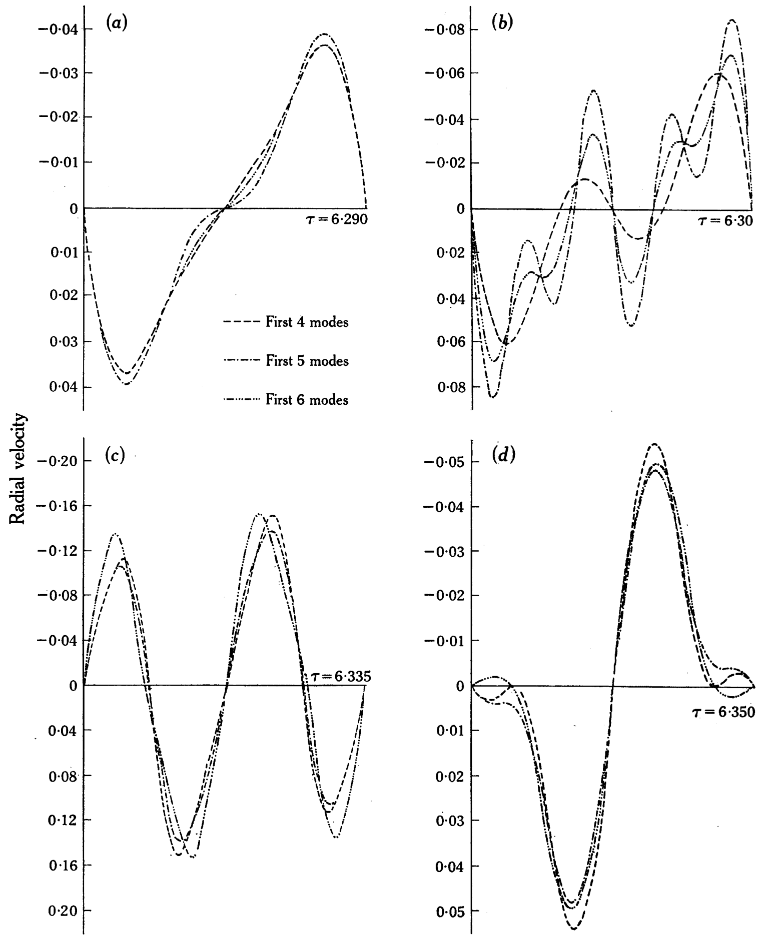

Fig. 1.—Radial velocity curves determined at the surface of the star taking into account the influence of the first four, five, and six modes of oscillation for (*a*) $\mathscr{M} = 10$, (*b*) $\mathscr{M} = 15$, (*c*) $\mathscr{M} = 20$, and (*d*) $\mathscr{M} = 30$.

Finally, it should be stressed that these results apply to stars having any uniform composition. However, if we were to consider a "normal" composition, say $X = 0\cdot70$, $Y = 0\cdot27$, $Z = 0\cdot03$, then these results would equally apply to stars in the mass range $26\cdot2\,M_\odot$ to $78\cdot5\,M_\odot$.

V. Acknowledgment

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