EFFECT OF HORIZONTAL AND VERTICAL MAGNETIC FIELDS ON KELVIN–HELMHOLTZ INSTABILITY

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[*Manuscript received April* 9, 1968]

Summary

This paper discusses the effect of a general oblique magnetic field on the stability of two superposed fluids in relative horizontal motion. The stable and unstable cases at the interface ($z = 0$) between two uniform fluids with constant densities and velocities of streaming are separately discussed. The combined effect of horizontal and vertical magnetic fields is to increase the wavelength at which the Kelvin–Helmholtz instability sets in.

I. Introduction

Michael (1955) has discussed the stability of a combined current and vortex sheet in a perfectly conducting fluid, while the effect on the Kelvin–Helmholtz instability of a magnetic field transverse to the direction of streaming has been considered by Northrop (1956). In the present note we study the effect of both horizontal and vertical magnetic fields on the Kelvin–Helmholtz instability. A general equation formulating this effect is first obtained. We then suppose that the two uniform fluids, of densities ρ_1 and ρ_2 and velocities of streaming U_1 and U_2, are separated by a horizontal boundary at $z = 0$. The stable and unstable cases are then separately discussed.

II. Basic Equations

The fluid is considered to be heterogeneous, inviscid, and of zero resistivity. We also suppose that the streaming takes place in the x direction with a velocity U.

The equations of motion and continuity are

$$\rho \,.\, \mathrm{d}\boldsymbol{q}/\mathrm{d}t = -\nabla p - \rho \boldsymbol{g} + \mu \boldsymbol{j} \times \boldsymbol{H}; \tag{1}$$

where $\boldsymbol{q}$ is the velocity vector, p the pressure, μ the magnetic permeability, and $\boldsymbol{g}$ the acceleration due to gravity; and

$$\nabla \,.\, \boldsymbol{q} = 0, \tag{2}$$

as the fluid is considered to be incompressible.

Since the density of a particle moving with the fluid remains constant,

$$\partial\rho/\partial t + (\boldsymbol{q} \,.\, \nabla)\rho = 0. \tag{3}$$

Using Maxwell's equations for a perfect conductor ($\eta = 0$)

$$\partial H/\partial t = \nabla \times (\boldsymbol{q} \times \boldsymbol{H}). \tag{4}$$

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Let the actual density at any point due to a disturbance be $\rho+\delta\rho$ and let δp denote the corresponding increment in pressure. Also let the components of velocity in the perturbed state be $U+u$, v, and w. Further, if $H_\perp$ and $H_\parallel$ denote the vertical and horizontal magnetic fields respectively and $\boldsymbol{h} = (h_x, h_y, h_z)$ is the perturbation in H, we have

$$\rho\left(\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+w\frac{\mathrm{d}U}{\mathrm{d}z}\right)+\frac{\mu H_\perp}{4\pi}\left(\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}\right) = -\frac{\partial}{\partial x}.\delta p\,, \tag{5}$$

$$\rho\left(\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}\right)-\frac{\mu H_\parallel}{4\pi}\left(\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}\right) = -\frac{\partial}{\partial y}.\delta p\,, \tag{6}$$

$$\rho\left(\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right)-\frac{\mu H_\parallel}{4\pi}\left(\frac{\partial h_z}{\partial x}-\frac{\partial h_x}{\partial z}\right)-\frac{\mu H_\perp}{4\pi}\left(\frac{\partial h_z}{\partial y}-\frac{\partial h_y}{\partial z}\right) = -\frac{\partial}{\partial z}.\delta p-g.\delta\rho\,, \tag{7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0\,, \tag{8}$$

$$\frac{\partial}{\partial t}.\delta\rho+U\frac{\partial}{\partial x}.\delta\rho = -w.\frac{\mathrm{d}\rho}{\mathrm{d}z}, \tag{9}$$

and

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\boldsymbol{h} = \left(\frac{\partial}{\partial x}.H_\parallel+\frac{\partial}{\partial y}.H_\perp\right)\boldsymbol{q}+h_z.\mathrm{D}U\,. \tag{10}$$

Analysing the disturbances into normal modes, we seek solutions whose dependence on x, y, and t is given by

$$\exp\{\mathrm{i}(k_x.x+k_y.y+n.t)\}\,, \tag{11}$$

where k_x, k_y, and n are constants; k_x is the wave number along the x direction, k_y the wave number along the y direction, and k the resultant wave number. Using the above perturbation, equations (5)–(10) become

$$\mathrm{i}\rho(n+k_x U)u+\rho(\mathrm{D}U)w+(\mu H_\perp/4\pi)(\mathrm{i}k_x.h_y-\mathrm{i}k_y.h_x) = -\mathrm{i}k_x.\delta p\,, \tag{12}$$

$$\mathrm{i}\rho(n+k_x U)v-(\mu H_\parallel/4\pi)(\mathrm{i}k_x.h_y-\mathrm{i}k_y.h_x) = -ik_y.\delta p\,, \tag{13}$$

$$\mathrm{i}\rho(n+k_x U)w+(\mu H_\parallel/4\pi)(\mathrm{D}h_x-\mathrm{i}k_x.h_z)+(\mu H_\perp/4\pi)(\mathrm{D}h_y-\mathrm{i}k_y.h_z) = -\mathrm{D}.\delta p-g.\delta\rho\,, \tag{14}$$

$$\mathrm{i}k_x.u+\mathrm{i}k_y.v+\mathrm{D}w = 0\,, \tag{15}$$

$$\mathrm{i}(n+k_x U).\delta\rho = -w.\mathrm{D}\rho\,, \tag{16}$$

$$h_x = \frac{k_x H_\parallel+k_y H_\perp}{n+k_x U}\left(u-\frac{\mathrm{i}\,\mathrm{D}U}{n+k_x U}w\right), \tag{17a}$$

and

$$h_y = \frac{k_x H_\parallel+k_y H_\perp}{n+k_x U}v\,, \qquad h_z = \frac{k_x H_\parallel+k_y H_\perp}{n+k_x U}w\,. \tag{17b}$$

Multiplying equations (12) and (13) by $-\mathrm{i}k_x$ and $-\mathrm{i}k_y$ respectively and adding, and substituting the values of h_x and h_y, we get

$$\rho(n+k_x U)\mathrm{D}w-k_x\rho(\mathrm{D}U)w = \mathrm{i}k^2.\delta p\,. \tag{18}$$

Multiplying equations (12) and (13) by $-ik_y$ and $+ik_x$ respectively and adding, we obtain the z component of vorticity ζ as

$$\zeta = \mathrm{i}k_x v - \mathrm{i}k_y u = \{k_y(\mathrm{D}U)/(n+k_x U)\}w. \tag{19}$$

Also

$$\mathrm{i}k^2 u = -(k_x \mathrm{D}w + k_y \zeta) = -[k_x \mathrm{D}w + \{k_y^2 \mathrm{D}U/(n+k_x U)\}w] \tag{20a}$$

and

$$\mathrm{i}k^2 v = (k_x \zeta - k_y \mathrm{D}w) = [\{k_x k_y(\mathrm{D}U)/(n+k_x U)\}w - k_y \mathrm{D}w]. \tag{20b}$$

Eliminating δp between equations (14) and (18) and using (9), (19), and (20), we get finally

$$\begin{aligned}\mathrm{D}(\rho(n+k_x U)\mathrm{D}w - \rho k_x(\mathrm{D}U)w) &= k^2\rho(n+k_x U)w + gk^2\frac{\mathrm{D}\rho}{n+k_x U}w \\ &+ k_x^2\frac{\mu H_\parallel^2}{4\pi}\left\{\mathrm{D}\left(\frac{\mathrm{D}w}{n+k_x U}\right) - \frac{k^2 w}{n+k_x U}\right\} - k_x^3\frac{\mu H_\parallel^2}{4\pi}\mathrm{D}\left(\frac{\mathrm{D}U}{(n+k_x U)^2}w\right) \\ &+ k_y^2\frac{\mu H_\perp^2}{4\pi}\left\{\mathrm{D}\left(\frac{\mathrm{D}w}{n+k_x U}\right) - \frac{k^2 w}{n+k_x U}\right\} - k_x k_y^2\frac{\mu H_\perp^2}{4\pi}\mathrm{D}\left(\frac{\mathrm{D}U}{(n+k_x U)^2}w\right) \\ &+ k_x k_y\frac{\mu H_\parallel H_\perp}{2\pi}\left\{\mathrm{D}\left(\frac{\mathrm{D}w}{n+k_x U}\right) - \frac{k^2 w}{n+k_x U}\right\} - k_x^2 k_y\frac{\mu H_\parallel H_\perp}{2\pi}\mathrm{D}\left(\frac{\mathrm{D}U}{(n+k_x U)^2}w\right).\end{aligned} \tag{21}$$

Equation (21) is thus a general equation formulating the effects of both horizontal and vertical magnetic fields on the Kelvin–Helmholtz instability. The particular cases of horizontal and vertical fields (cf. Chandrasekhar 1961, pp. 510 and 512) can be derived in the limit of vanishing $H_\perp$ and $H_\parallel$ respectively.

III. Two Uniform Fluids in Relative Motion Separated by a Horizontal Boundary

The two uniform fluids of densities ρ_1 and ρ_2 and velocities of streaming U_1 and U_2 are supposed to be separated by a horizontal boundary at $z = 0$. Then, in the two regions of constant ρ and U, equations (19) and (21) give

$$\zeta = 0, \tag{22}$$

and

$$\left(\rho(n+k_x U) - \frac{k_x^2}{n+k_x U}\frac{\mu H_\parallel^2}{4\pi} - \frac{k_y^2}{n+k_x U}\frac{\mu H_\perp^2}{4\pi} - \frac{k_x k_y}{n+k_x U}\frac{\mu H_\parallel H_\perp}{2\pi}\right)\left(\mathrm{D}^2 - k^2\right)w = 0. \tag{23}$$

Since w must be bounded both when $z \to +\infty$ (in the upper fluid) and $z \to -\infty$ (in the lower fluid), the solutions of equation (23) can be written as

$$w_1 = A(n+k_x U_1)\mathrm{e}^{+kz}, \qquad z < 0, \tag{24a}$$

$$w_2 = A(n+k_x U_2)\mathrm{e}^{-kz}, \qquad z > 0, \tag{24b}$$

$w/(n+k_x U)$ being continuous at the interface.

Integrating equation (23) across the interface at $z = 0$, we get

$$\Delta_0\{\rho(n+k_x U)\mathrm{D}w\} = k_x^2 \frac{\mu H_\parallel^2}{4\pi} \Delta_0\left(\frac{\mathrm{D}w}{n+k_x U}\right) + k_y^2 \frac{\mu H_\perp^2}{4\pi} \Delta_0\left(\frac{\mathrm{D}w}{n+k_x U}\right)$$

$$+ k_x k_y \frac{\mu H_\parallel H_\perp}{2\pi} \Delta_0\left(\frac{\mathrm{D}w}{n+k_x U}\right) + gk^2 \Delta_0(\rho)\left(\frac{w}{n+k_x U}\right)_0, \tag{25}$$

where $\Delta_0(f)$ is the change that a quantity f experiences at $z = 0$, and $(w/(n+k_x U))_0$ is the unique value that this quantity has at $z = 0$.

Substituting the values of w_1 and w_2 from equations (24) in (25), we obtain the characteristic equation

$$\rho_2(n+k_x U_2)^2 + \rho_1(n+k_x U_1)^2 = gk(\rho_1-\rho_2) + k_x^2(\mu H_\parallel^2/2\pi) + k_y^2(\mu H_\perp^2/2\pi) + k_x k_y(\mu H_\parallel H_\perp/\pi). \tag{26}$$

Letting $\alpha_1 = \rho_1/(\rho_1+\rho_2)$ and $\alpha_2 = \rho_2/(\rho_1+\rho_2)$, the roots of equation (26) are

$$n = -k_x(\alpha_1 U_1 + \alpha_2 U_2) \pm \left(gk(\alpha_1-\alpha_2) + k_x^2 \frac{\mu H_\parallel^2}{2\pi(\rho_1+\rho_2)} + k_y^2 \frac{\mu H_\perp^2}{2\pi(\rho_1+\rho_2)}\right.$$

$$\left. + k_x k_y \frac{\mu H_\parallel H_\perp}{\pi(\rho_1+\rho_2)} - k_x^2 \alpha_1 \alpha_2 (U_1-U_2)^2\right)^{\frac{1}{2}}. \tag{27}$$

The Kelvin–Helmholtz instability occurs only if n contains a negative imaginary part.

(a) Stable Case

In this case $\alpha_1 > \alpha_2$. It is observed from equation (27) that the combined effect of horizontal and vertical magnetic fields is to suppress the Kelvin–Helmholtz instability if

$$k_x^2 \alpha_1 \alpha_2 (U_1-U_2)^2 \leqslant F, \tag{28a}$$

where

$$F = gk(\alpha_1-\alpha_2) + k_x^2 \frac{\mu H_\parallel^2}{2\pi(\rho_1+\rho_2)} + k_y^2 \frac{\mu H_\perp^2}{2\pi(\rho_1+\rho_2)} + k_x k_y \frac{\mu H_\parallel H_\perp}{\pi(\rho_1+\rho_2)}. \tag{28b}$$

In any direction, instability occurs when

$$k_x^2 \alpha_1 \alpha_2 (U_1-U_2)^2 > F. \tag{29}$$

Hence for a given difference in velocity U_1-U_2 and a given direction of the wave vector k, instability occurs for all wave numbers

$$k > g(\alpha_1-\alpha_2)\left(\alpha_1 \alpha_2 (U_1-U_2)^2 \cos^2\theta - \frac{\mu H_\parallel^2}{2\pi(\rho_1+\rho_2)} \cos^2\theta - \frac{\mu H_\perp^2}{2\pi(\rho_1+\rho_2)} \sin^2\theta\right.$$

$$\left. - \frac{\mu H_\parallel H_\perp}{\pi(\rho_1+\rho_2)} \sin\theta \cos\theta\right)^{-1}, \tag{30}$$

where θ is the angle between the directions of k and U. The minimum wave number

$k_{\min}$ is given by

$$k_{\min} = 2g(\alpha_1-\alpha_2)[\{v_A^2-\alpha_1\alpha_2(U_1-U_2)^2-v_B^2\}^2+4v_A^2v_B^2]^{\frac{1}{2}}$$
$$\times(\{\alpha_1\alpha_2(U_1-U_2)^2-v_A^2+v_B^2\}^2+4v_A^2v_B^2+\{\alpha_1\alpha_2(U_1-U_2)^2-v_A^2-v_B^2\}$$
$$[\{v_A^2-\alpha_1\alpha_2(U_1-U_2)^2-v_B^2\}^2+4v_A^2v_B^2]^{\frac{1}{2}})^{-1}, \tag{31}$$

and occurs when θ is given by

$$2\theta = n\pi+\tan^{-1}[2v_A v_B/\{v_A^2-v_B^2-\alpha_1\alpha_2(U_1-U_2)^2\}], \tag{32}$$

where

$$v_A^2 = \mu H_{\parallel}^2/2\pi(\rho_1+\rho_2) \qquad \text{and} \qquad v_B^2 = \mu H_{\perp}^2/2\pi(\rho_1+\rho_2).$$

The minimum wave number occurs for odd n. Thus it is clear that for a given U_1-U_2 instability occurs for the least wave number given by equation (31), when k is inclined at an angle θ given by equation (32) with the direction of U for odd n. We will have instability for $k > k_{\min}$.

(*b*) *Unstable Case*

In this case $\alpha_2 > \alpha_1$. It is clear from equation (27) that the combined effect of horizontal and vertical magnetic fields will suppress the Kelvin–Helmholtz instability if

$$k_x^2\alpha_1\alpha_2(U_1-U_2)^2 < F. \tag{33}$$

Also, if

$$k_x^2\alpha_1\alpha_2(U_1-U_2)^2 = F,$$

then

$$n = -k_x(\alpha_1 U_1+\alpha_2 U_2), \tag{34}$$

which leads to stability.

In any direction, instability occurs under the condition (29). Here $k_{\min}$ occurs for even n and is given by

$$k_{\min} = 2g(\alpha_2-\alpha_1)[\{v_A^2-\alpha_1\alpha_2(U_1-U_2)^2-v_B^2\}^2+4v_A^2v_B^2]^{\frac{1}{2}}$$
$$\times(\{\alpha_1\alpha_2(U_1-U_2)^2-v_A^2-v_B^2\}^2+4v_A^2v_B^2+\{v_A^2+v_B^2-\alpha_1\alpha_2(U_1-U_2)^2\}$$
$$\times[\{v_A^2-\alpha_1\alpha_2(U_1-U_2)^2-v_B^2\}^2+4v_A^2v_B^2]^{\frac{1}{2}})^{-1}, \tag{35}$$

for the same value of θ given by equation (32). We will have instability for $k > k_{\min}$.

IV. References

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