VACUUM POLARIZATION INTERACTION BETWEEN PROTONS, EFFECTIVE RANGE PARAMETERS, AND TRIPLET PHASES*

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Summary

The scattering cross section is formulated in terms of the Coulomb, vacuum polarization, and nuclear amplitudes. Comparison is made with available data; the search variables being the effective range parameters and a strength factor Λ for the vacuum polarization, as well as two triplet parameters. The strength factor Λ attained the value $0 \cdot 92$ at optimum fit and new values of the effective range parameters are obtained. Scattering experiments are suggested that should have appreciable sensitivity to the vacuum polarization amplitude. Estimates of the triplet phases and spin polarization are given.

I. Introduction

In the mid 1930's the quantum electrodynamical concept of vacuum polarization emerged from the work of Dirac (1934), Furry and Oppenheimer (1934), Heisenberg (1934), Serber (1935), Uehling (1935), and Pauli and Rose (1936). From the work of Furry and Oppenheimer, as well as that of Uehling, it appeared that the modification of the Coulomb field between charged particles should manifest itself in the displacement of atomic energy levels as well as in a deviation from pure Mott scattering between low energy protons. In recent years precise calculations and measurements have indicated, for atomic energy levels and similar systems, that the vacuum polarization phenomenon operates very much as expected although some small discrepancies may remain.

In contradistinction to the previous examples there exist only relatively crude indications of the vacuum polarization interaction between protons. They have been adduced by Foldy and Erikson (1955), Erikson, Foldy, and Rarita (1956), and Durand (1957). Related discussions have also been given by De Wit and Durand (1958), Heller (1960), Slobodrian (1966), and Kermode (1968). These communications suggest that the vacuum polarization interaction is manifestly operative but quantitative conclusions need reinforcement.

I have, therefore, considered the experimental aspects of this question dichotomously. To what extent can the body of precise low energy proton–proton scattering data (Brolley, Seagrave, and Beery 1964; Knecht, Dahl, and Messelt 1966) be reanalysed to illuminate this question? What additional experiments, specifically designed to evaluate the strength of the vacuum polarization, are practicable?

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II. Procedure

Since only differential cross sections exist in the family of data under consideration it suffices to construct a quasi-phenomenological cross section relation. The cross section has therefore been related to the sum of three amplitudes representing the Coulomb field, the vacuum polarization, and the strong interaction. In the customary notation the Coulomb amplitude is

$$A_{\mathrm{C}} = -(\gamma/2k\sin^2\tfrac{1}{2}\theta)\exp\{-2\mathrm{i}\gamma\ln(\sin\tfrac{1}{2}\theta)\}. \tag{1}$$

The Coulomb parameter γ and the wave number k are computed relativistically. The vacuum polarization amplitude can be formulated to first order in the fine structure constant by utilizing the Uehling potential $I(r)$, which supplements the Coulomb field to give the total electric potential

$$V_E(r) = (e^2/r)\{1+1\cdot549\times10^{-3}I(r)\}. \tag{2}$$

$I(r)$ can be computed easily from a representation given by Schwinger (1949), namely

$$I(r) = \int_1^\infty \exp(-2\zeta rx)(x^{-2}+\tfrac{1}{2}x^{-4})(x^2-1)^{\frac{1}{2}}\,\mathrm{d}x, \tag{3}$$

with

$$(2\zeta)^{-1} = \hbar/2m_{\mathrm{e}}c.$$

Durand (1957) has given the vacuum polarization amplitude, based on the Uehling potential and summed over all angular momentum states, as

$$A_{\mathrm{vp}} = -\frac{\alpha\gamma}{3\pi k}|\Gamma(1+\mathrm{i}\gamma)|^2\int_0^1(1+\tfrac{1}{2}y)(1-y)^{\frac{1}{2}}\frac{1}{(1-\cos\theta)y+\nu}\left(\frac{2y+\nu}{(1-\cos\theta)y+\nu}\right)^{\mathrm{i}\gamma}$$
$$\times\exp\left\{-2\gamma\tan^{-1}\left(\frac{\nu}{2y}\right)^{\frac{1}{2}}\right\}{}_2F_1\left(-\mathrm{i}\gamma,1+\mathrm{i}\gamma;1;\frac{y}{y+\bar{X}}\right)\mathrm{d}y, \tag{4}$$

where

$$\nu = 4m_{\mathrm{e}}^2c^2/ME, \qquad \bar{X} = \nu/(1-\cos\theta),$$

E is the laboratory energy of the incident proton, and M is its mass. In most previous applications of A_{vp} only limited sums over angular momenta were used to approximate A_{vp}. Here A_{vp} is computed exactly.

Whilst all angular momenta are included in A_{C} and A_{vp}, the strong interaction amplitude A_{n} is computed for $l = 0$ only. An approximation to the p-wave contributions will be introduced later. A convenient parameterization of A_{n} can be obtained from the effective range formulation of Heller (1960) wherein the influence of the vacuum polarization is convoluted. Thus the nuclear phase δ_0^E referred to the total electric field (2) is given by

$$\frac{C^2k}{1-\Phi_0}\left((1+\chi_0)\cot\delta_0^E-\tan\tau_0\right)+\frac{h(\gamma)}{R}+\frac{l_0(\gamma)}{R} = -\frac{1}{a}+\tfrac{1}{2}k^2r_0-Pr_0^3k^4. \tag{5}$$

The notation and form are those of the standard effective range theory (Jackson and Blatt 1950) except for the appearance of the quantities Φ_0, χ_0, τ_0, and l_0. At this point the Foldy–Erikson procedure for handling the vacuum polarization could have been introduced, but Heller's method seems more logical. To some extent this choice is a matter of taste. The basis of the latter method may be briefly recapitulated in order to define Φ_0, χ_0, τ_0, and l_0.

The nuclear phase δ_0^E is now defined in terms of the total electric field (2). If, for the moment, the strong interaction is switched off, the $l = 0$ radial equation can be written solely in terms of the potential given in equation (2). There will exist regular and irregular solutions that have the asymptotic forms

$$S_0(r) = F_0(r) + \tan\tau_0\, G_0(r) \tag{6}$$

and

$$T_0(r) = G_0(r) - \tan\tau_0 F_0(r), \tag{7}$$

where $F_0(r)$ and $G_0(r)$ are the regular and irregular Coulomb functions. From the integral equation for S_0 it may be inferred that

$$\tan\tau_0 = -2\gamma\lambda \int_0^\infty \mathrm{d}r\, \frac{F_0(r)\, I(r)\, S_0(r)}{r}. \tag{8}$$

When the solutions S_0 and T_0 are found the remaining quantities may be computed from the relations:

$$\Phi_0 = -2\gamma\lambda \int_0^\infty \mathrm{d}r\, \frac{F_0(r)\, I(r)\, T_0(r)}{r}, \tag{9}$$

$$\chi_0 = -2\gamma\lambda \int_0^\infty \mathrm{d}r\, \frac{G_0(r)\, I(r)\, S_0(r)}{r}, \tag{10}$$

$$l_0 = -\lambda \int_0^\infty \mathrm{d}r \left\{ \left(\frac{C^2 G_0(r)\, T_0(r)}{1-\Phi_0} \right)_E - \left(\frac{C^2 G_0(r)\, T_0(r)}{1-\Phi_0} \right)_{E=0} \right\} \frac{I(r)}{r}. \tag{11}$$

τ_0 is the $l = 0$ phase shift of the vacuum polarization amplitude defined with respect to the Coulomb field. Thus it may be noted that τ_0, Φ_0, χ_0, and l_0 depend linearly on λ. Application of the Wronskian relation

$$(\cos^2\tau_0)(1+\chi_0)(1-\Phi_0) = 1 \tag{12}$$

obviates the calculation of Φ_0 when τ_0 and χ_0 have been obtained. With these relations in hand, the nuclear amplitude may be written as

$$A_{\mathrm{n}} = [\exp(2\mathrm{i}\tau_0)\{\exp(2\mathrm{i}\delta_0^E) - 1\}]/2\mathrm{i}k. \tag{13}$$

Other contributions to the total scattering amplitude (Breit 1962) have been discussed. The present analysis assumes that in the energy interval under discussion A_{vp} dominates the remaining poorly known terms. It may turn out that in the future a level of precision in experimental data will be reached which demands consideration of some of these ignored terms.

Evidently the square of the sum of the three amplitudes, with suitable antisymmetrization, should provide a cross section to confront the experiments. However, it is desirable to attempt to include the effects of the p-waves in the cross section. Unfortunately their relative weakness imposes great difficulty in assessing their influence on the experimental data so far. It would appear, for the present, that a quasi-theoretical estimate (Knecht, Dahl, and Messelt 1966; henceforth referred to as KDM) based on the work of Noyes (1964) should be used. The approximate increment to the cross section arising from the weak p-waves, which takes Coulomb effects into account, is

$$\Delta\sigma_{\mathrm{p}} = \frac{\gamma^2}{4k^2}\left\{-\frac{2X_1P_1Z_2}{\gamma}+\left(\frac{2Y_1P_1}{\gamma}+\frac{4}{\gamma^2}\right)Z_1+\frac{4P_2Z_3}{\gamma^2}\right\}, \tag{14}$$

the notation being detailed in KDM. Pertinent to the present analysis is the fact that the Z's depend on the triplet phases δ_0, δ_1, and δ_2, and these can be expressed (according to KDM) as

$$\delta_0 = \gamma+4\alpha-2\beta, \qquad \delta_1 = \gamma-2\alpha-\beta, \qquad \delta_2 = \gamma+\tfrac{2}{5}\alpha+\beta, \tag{15}$$

where γ, α, and β are the central, tensor, and spin orbit contributions respectively. Values of α computed by Noyes (cited in KDM) are used in the present work. Convenient parameterizations for the other terms in the restricted energy region under consideration have been given by Sher (personal communication):

$$\gamma = p_{\mathrm{c}}E, \qquad \beta = p_{ls}E^2. \tag{16}$$

If now the Uehling potential were to be assigned an arbitrary strength factor Λ, the amplitude could be written as

$$A = A_{\mathrm{C}}(E,\theta)+\Lambda A_{\mathrm{vp}}(E,\theta)+A_{\mathrm{n}}(E,\tau_0',\Phi_0'\chi_0',l_0',a,r_0,P), \tag{17}$$

where

$$\tau_0' = \Lambda\tau_0, \qquad \Phi_0' = \Lambda\Phi_0, \qquad \chi_0' = \Lambda\chi_0, \qquad \text{and} \qquad l_0' = \Lambda l_0.$$

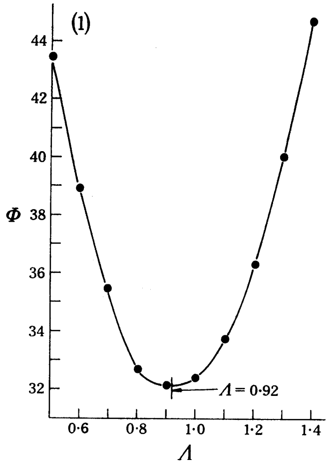
Cross sections based on this construction would provide a best fit to experimental data for $\Lambda = 1$ if only these components applied at their anticipated strengths. Thus a search routine was written to fit the calculated cross section to the measured relative cross sections at the interference minimum (Brolley, Seagrave, and Beery 1964) and the measured absolute cross sections at 1·397, 1·855, 2·425, and 3·037 MeV (KDM). Since the interference minimum data were relative cross sections, a normalization factor N was introduced. Searches were then conducted with the seven parameters Λ, a, r_0, P, N, p_{c}, and p_{ls} released. The minuscule p-wave contribution was not included in the interference region.

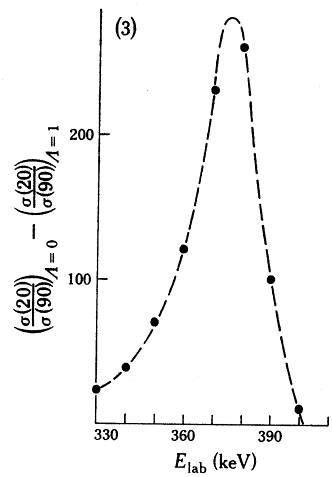
The function to be minimized was taken to be

$$\Phi = \sum_{\mathrm{data}}\{(\sigma_{\mathrm{obs}}-\sigma_{\mathrm{calc}})/\epsilon_{\mathrm{exp}}\}^2. \tag{18}$$

III. Results

The wavefunctions $S_0(r)$ and $T_0(r)$ were easily computed from ~ 1000 f inward to $\sim 0 \cdot 001$ f. A fourth-order dual pass Runge Kutta technique was employed. In the case of $S_0(r)$, at some radius near the origin, procedure was shifted to the two-point technique. In general, it was easy to maintain at least six significant digits of accuracy over the entire range. The results are displayed in Table 1. Useful approximations to these results can be obtained by replacing $S_0(r)$ and $T_0(r)$ in equations (8)–(11) with $F_0(r)$ and $G_0(r)$ respectively.





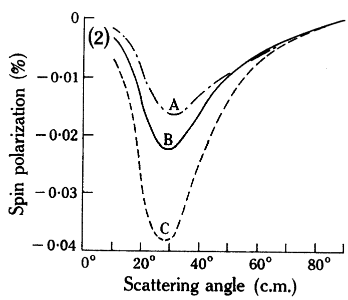


Fig. 1.—Goodness of fit as a function of the strength factor Λ.

Fig. 2.—Predicted angular dependence of spin polarization on the basis of three different sets of triplet phase shifts at $E = 3 \cdot 037$ MeV:
A, Knecht, Dahl, and Messelt;
B, present work;
C, McGregor, Arndt, and Wright.

Fig. 3.—Difference between computed cross section ratios with and without vacuum polarization.

With 55 data points and 7 free parameters the search procedure yielded $\Lambda = 0 \cdot 92$ and $\Phi = 32 \cdot 11$. In order to gain some insight as to the validity of this result, the dependence of Φ on Λ was computed. The results are depicted graphically in Figure 1. In this case there were six parameters to solve for.

For $\Lambda = 1$ effective range parameters were determined. The results are presented in Table 2. From this determination there also resulted a set of triplet phases as well as singlet phases derived from the effective range formula. The singlet phases corresponding to the $\Lambda = 1$ effective range parameters were then obtained. Both singlet and triplet phases are given in Table 3. For convenience Heller's approximations to τ, Φ_0, χ_0, and l_0 were employed to compute the energy at which

TABLE 1

PARAMETERS FOR EFFECTIVE RANGE FORMULA

E (MeV)	$-\tau_0 \times 10^3$	$-\Phi_0 \times 10^3$	$-\chi_0 \times 10^3$	$-l_0 \times 10^3$
0·33766	1·854	2·304	2·295	2·367
0·36248	1·843	2·245	2·237	2·408
0·38348	1·834	2·199	2·191	2·434
0·39425	1·829	2·177	2·169	2·448
0·40517	1·824	2·155	2·147	2·461
1·397	1·491	1·336	1·332	2·601
1·855	1·401	1·193	1·190	2·511
2·425	1·317	1·072	1·069	2·386
3·037	1·246	0·9794	0·9769	2·253

TABLE 2

COMPARISON OF EFFECTIVE RANGE PARAMETERS

Reference	a (fermi)	r (fermi)	P	Q* (keV)
Present work	$-7{\cdot}815 \pm 0{\cdot}004$	$2{\cdot}799 \pm 0{\cdot}016$	$0{\cdot}029 \pm 0{\cdot}012$	
Heller (1967)	$-7{\cdot}817 \pm 0{\cdot}007$	$2{\cdot}810 \pm 0{\cdot}018$	$0{\cdot}035 \pm 0{\cdot}009$	
Noyes (1964)	$-7{\cdot}828 \pm 0{\cdot}008$	$2{\cdot}794 \pm 0{\cdot}026$	$0{\cdot}026 \pm 0{\cdot}014$	
Slobodrian (1968)	$-7{\cdot}7856 \pm 0{\cdot}0078$	$2{\cdot}840 \pm 0{\cdot}009$	$0{\cdot}072 \pm 0{\cdot}005$	$0{\cdot}034 \pm 0{\cdot}004$

* Q is the coefficient of the k^6 term in the effective range formula. It is not used in the present work.

TABLE 3

NUCLEAR PHASE SHIFTS IN RADIANS

E (MeV)	Singlet δ_0^E Present Work	Singlet δ_0^E KDM	Triplet Present Work	Triplet KDM	Triplet MAW*
0·33766	0·22323				
0·36248	0·24096				
0·38252†	0·25500				
0·38348	0·25567				
0·39425	0·26309				
0·40517	0·27055				
			0·00433 (δ_0)	0·00438 (δ_0)	0·00566 (δ_0)
1·397	0·68612	0·68628	−0·00237 (δ_1)	−0·00237 (δ_1)	−0·00350 (δ_1)
			0·000442(δ_2)	0·000332(δ_2)	0·000646(δ_2)
			0·00682	0·00696	0·00838
1·855	0·77406	0·77369	−0·00373	−0·00368	−0·00516
			0·000724	0·000576	0·000995
			0·0103	0·0106	0·0120
2·425	0·84367	0·84395	−0·00565	−0·00553	−0·00740
			0·00113	0·000978	0·00153
			0·0144	0·0151	0·0162
3·037	0·89041	0·89052	−0·00793	−0·00749	−0·00992
			0·00163	0·00155	0·00212

* McGregor, Arndt, and Wright (personal communication).

† Interference minimum.

the cross section has a minimum value, namely 0·87178 mb. The corresponding phase is included.

Spin polarizations implied by the triplet phases were computed according to the method of Hull and Shapiro (1958). The results are plotted in Figure 2.

If the triplet phases be regarded as approximately correct, they may be employed in a search for a set a, r, P, N, $\Lambda = 0$ that minimizes Φ. Thus, for the same set of triplet phases, it is then possible to compute cross sections for $\Lambda = 0$ and 1. The results of such computations are depicted in Figure 3.

For $\Lambda = 1$, A_{n} as well as A_{C} and A_{vp} are plotted in Figure 4 to span the valley of interference.

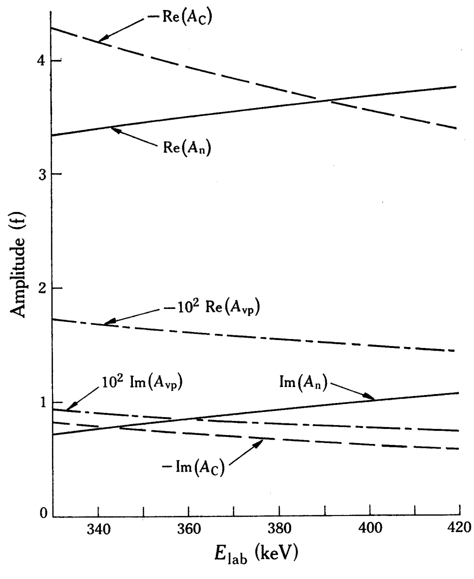


Fig. 4.—Behaviour of the Coulomb (A_{C}), vacuum polarization (A_{vp}), and nuclear (A_{n}) amplitudes at 90° c.m. Cancellations in the real and imaginary parts of $A_{\mathrm{n}}+A_{\mathrm{C}}$ are conspicuous.

IV. Discussion

The present analysis was adumbrated by studies of Noyes (personal communication). From analyses at four separate energies his results were:

E (MeV)	1·397	1·855	2·425	3·037
Λ	1·76±0·31	1·19±0·38	0·86±0·26	0·65±0·34

From this work it is reasonable to infer the significant manifestation of the vacuum polarization interaction within range of the expected strength. However, it is clearly desirable to quantify, in greater degree, our knowledge in this respect. The results depicted in Figure 1 do imply that the interaction is operating at near or equal to its expected strength as calculated on the basis of point charges. Moreover the curve of Figure 1 is conservative in the sense that it is based on some freedom in the fitting of the p-waves. If the p-waves are held fixed at the values determined for $\Lambda = 1$ the valley becomes much steeper.

One, quite naturally, might inquire into the possibility of performing experiments specifically designed to evaluate Λ. A_{vp} is, of course, manifest at all energies at which proton–proton scattering experiments might be performed. However, it attains maximum strength below 500 keV. Moreover the dominant amplitudes, A_{C} and A_{n}, tend to cancel each other strongly in the region 300–400 keV, as is clear from an inspection of Figure 4. The effect of these calculations is to enhance the visibility of the vacuum polarization amplitude (see e.g. Fig. 3). The experimental ramification is that the 90° c.m. valley of interference should be measured with a great multiplicity of precise points. Moreover it is not necessary to measure absolute cross sections. It is desirable to measure at least one forward cross section also at the same time. This not only facilitates data analysis but removes, at least to first order, the requirement for accurate knowledge of beam currents and target densities. Thus, an analysis of the type discussed here is freed of several quantities which would have weakened the conclusions, namely the p-waves and the shape parameter. Such an analysis should yield increased precision for the scattering length since it becomes dominant in the effective range formalism as the energy decreases.

An accurate knowledge of the scattering length is useful in discussions of charge independence. In this connection it may be noted that the scattering length calculated by Slobodrian (1968; see Table 2) is somewhat different from the other values. If use is to be made of this value it is well to bear in mind that it comes from a Foldy–Erikson type of analysis that does not use the lowest energy high precision proton–proton scattering data. Moreover, Noyes (personal communication) and also McGregor, Arndt, and Wright (personal communication) have shown that the 9·918 proton–proton scattering data (Slobodrian *et al.* 1968), used by Slobodrian in his analysis, is not in agreement with the body of generally accepted data.

It is difficult to ascribe rigorous significance to the errors appearing in Table 2. Errors in the present analysis are dominated by the cross section errors assigned by KDM. If the latter errors were true random standard deviations, one would expect $\Phi/(N_{\mathrm{d}}-N_{\mathrm{p}})$, where N_{d} is the number of data points and N_{p} the number of parameters, to approach unity if the fit were good and N_{d} large. In the present case, for $\Lambda = 1$, this ratio is less than unity. The departure from unity is quite acceptable in view of the relatively small number of data points. This can also occur if the experimental errors are large. The KDM errors include estimates for many types of systematic errors and it becomes difficult to give a precise and concise definition of them. Nonetheless, errors in the parameters may be formally computed (Hildebrand 1956). Precise definition of these computed errors must again be difficult. For the present it is convenient to say that the computed errors have a character somewhat similar to the KDM errors. In this connection it is noted that the optimization calculation was driven to the point where $\partial\Phi/\partial a \sim 10^{-6}$. Since the p-waves are so weak, it did not seem worth while to drive the gradients of the related parameters this far. It would be desirable to have more precision and more data in order to sharply delimit the p-waves. Nonetheless it is instructive to compare the p-wave phases derived from the present analysis with those of KDM and McGregor, Arndt, and Wright (personal communication). The triplet phases obtained by the latter group were from a search over a large amount of data at many energies and, *a priori*, are probably the best values in Table 3. Measurements of spin polarization are helpful in this area. Spin polarizations implied by the three sets of triplet phases are noticeably different, as may be seen in Figure 2. Unfortunately the magnitudes are such as to almost preclude observation with contemporary techniques.

The singlet phases of the present analysis are in excellent accord with those of KDM and McGregor, Arndt, and Wright. Also, the energy of the interference minimum, 382·52 keV, agrees well with that of Heller (1967), 382·43 keV.

In summary then, it appears that the vacuum polarization interaction is operating at or close to the expected strength. Present accelerator designs offer the hope of a more quantitative investigation in this area as well as improved values of the scattering length. More high precision proton–proton scattering data below 10 MeV are needed.

V. Acknowledgments

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