COLLISIONAL DAMPING OF PLASMA OSCILLATIONS

By W. P. Wood* and B. W. Ninham*†

[*Manuscript received December* 23, 1968]

Summary

The attenuation of longitudinal electron oscillations in a dilute plasma is evaluated from the Krook approximation to the Boltzmann equation. The predictions of this model are compared with those of various other collision models. In the limit of "few collisions" there is complete qualitative agreement. However, the quantitative agreement among contributions beyond those due to dynamical friction is poor.

I. Introduction

In this paper we consider the effect of collisions upon longitudinal electron oscillations in an unbounded plasma, on the assumption that the collision integral in the Boltzmann equation can be approximated by the simple collision model of Gross and Krook (1956). Our purpose is to compare predictions of this model with results obtained from other calculations using different terms for the collision processes. We have restricted our analysis to the case of "few collisions", defined through the inequality $\lambda/\sqrt{2}a\kappa \ll 1$, where κ is the wave number of the oscillation, $a = (kT/m)^{\frac{1}{2}}$ is the electron thermal speed, and λ is an effective collision frequency. We have considered this limit since those investigations with which we compare our results have also been restricted, either implicitly or explicitly, to this limit. Furthermore the analysis is carried out on the assumption that κ is small, that is, $1/\kappa \gg h$, where h is the Debye shielding distance.

II. Formulation of the Problem

The general Boltzmann equation can be written as

$$\frac{\partial f}{\partial t}+\boldsymbol{v}.\frac{\partial f}{\partial \boldsymbol{r}}+\frac{\boldsymbol{F}}{m}.\frac{\partial f}{\partial \boldsymbol{v}} = C\,, \tag{1}$$

where $f = f(\boldsymbol{r},\boldsymbol{v},t)$ is the one-particle distribution function for electrons of mass m and velocity $\boldsymbol{v}$, and $\boldsymbol{F}$ is a force acting on the electron. The collision term C takes account of the effect of both electron–electron collisions and electron–ion collisions on the electron distribution function. In the limit which concerns us, the ions can be replaced by a smeared-out background of positive charge.

* Department of Applied Mathematics, University of New South Wales, P.O. Box 1, Kensington, N.S.W. 2033.

† Present address: DCRT, National Institute of Health, Bethesda, Maryland 20014, U.S.A.

Aust. J. Phys., 1969, **22**, 453–8

To account for the effect of collisions we shall use the statistical model introduced by Gross and Krook (1956). This collision term has the form

$$C = \sum_{j=\mathrm{e,i}} \delta f_{\mathrm{e}j}/\delta t = \sum_{j=\mathrm{e,i}} n_j K_{\mathrm{e}j}(n\Phi_{j\mathrm{e}}-f), \tag{2}$$

where

$$\Phi_{j\mathrm{e}} = (m/2\pi kT_{j\mathrm{e}})^{3/2}\exp\{-m(\boldsymbol{v}-\boldsymbol{u}_{j\mathrm{e}})^2/2kT_{j\mathrm{e}}\}. \tag{3}$$

The subscripts e and i refer to electrons and ions respectively and the function $n_{\mathrm{e}} = n = n(\boldsymbol{r},t)$ is the local density of electrons at position $\boldsymbol{r}$ and time t. The terms u_{ee} and u_{ie} are ensemble average drift velocities. The $K_{\mathrm{e}j}$ are velocity-independent frequency parameters.

To simplify the analysis we shall asume that momentum is not conserved in electron–ion collisions and hence $u_{\mathrm{ie}} = 0$. This assumption is justified provided $m_{\mathrm{e}}/m_{\mathrm{i}} \ll 1$. Also we shall assume that the electrons emerge from collisions with a temperature equal to that of the undisturbed gas. This is an isothermal approximation in that local temperature fluctuations will be put equal to zero. This is a reasonable approximation for a dilute system for which the collision frequency is small.

With these approximations taken into account we may write

$$\Phi_{\mathrm{ee}} = (m/2\pi kT)^{3/2}\exp\{-m(\boldsymbol{v}-\boldsymbol{u})^2/2kT\} \tag{4}$$

and

$$\Phi_{\mathrm{ie}} = f_0 = (m/2\pi kT)^{3/2}\exp(-mv^2/2kT), \tag{5}$$

with $u_{\mathrm{ee}} = u = u(\boldsymbol{r},t)$ and $n_{\mathrm{i}} = n$ for charge neutrality. The parameters K_{ee} and K_{ei} are related to the collision frequencies λ_{ee} and λ_{ei} respectively through $n_0 K_{\mathrm{ee}} = \lambda_{\mathrm{ee}}$ and $n_0 K_{\mathrm{ei}} = \lambda_{\mathrm{ei}}$. The frequencies λ_{ee} and λ_{ei} are collision frequencies for electron–electron and electron–ion collisions respectively whose explicit values must be determined from a phenomenological argument.

The collision terms specified by equations (2), (4), and (5) imply that for electron–ion collisions number density but not momentum is conserved while for electron–electron collisions both number density and momentum are conserved. In both cases energy is not conserved in collisions on account of the isothermal assumption.

In general the change in momentum of a particle due to the average force acting on it at $(\boldsymbol{r},t)$ is accounted for in equation (1) by the term $(\boldsymbol{F}/m).(\partial f/\partial\boldsymbol{v})$. The effect of the system of particles as a whole acting on an electron may be handled through the introduction of a Hartree potential $\psi(\boldsymbol{r},t)$ given by

$$\psi(\boldsymbol{r},t) = \epsilon\int V(|\boldsymbol{r}-\boldsymbol{r}'|)\{n(\boldsymbol{r}',t)-n_0\}\,\mathrm{d}^3r'. \tag{6}$$

Here $V(|\boldsymbol{r}-\boldsymbol{r}'|)$ is the two-body potential which depends on $|\boldsymbol{r}-\boldsymbol{r}'|$ only, and the parameter ϵ characterizes the strength of the interaction. The quantity n_0 is the equilibrium density. In terms of Fourier components for the space coordinates, equation (6) becomes

$$\psi(\boldsymbol{\kappa},t) = \epsilon V(\boldsymbol{\kappa})\{n(\boldsymbol{\kappa},t)-n_0\}, \tag{7}$$

with

$$V(\boldsymbol{\kappa}) = \int V(\boldsymbol{r})\exp(\mathrm{i}\boldsymbol{\kappa}.\boldsymbol{r})\,\mathrm{d}^3r = V(-\boldsymbol{\kappa}), \tag{8}$$

and

$$n(\kappa, t) = \int n(r, t) \exp(\mathrm{i}\kappa . r)\, \mathrm{d}^3 r .$$

In terms of ψ, the average force F is,

$$F = -\partial\psi/\partial r . \tag{9}$$

III. Approximate Solutions of the Dispersion Relation

Using well known techniques, which involve linearizing equation (1), taking Fourier components with respect to space and Laplace transforms with respect to time, and solving for $n(\kappa, p)$, one obtains the dispersion relation for electron oscillations:

$$D(\kappa, p) = 1 - \int_{-\infty}^{\infty} \left(\lambda - \epsilon V(\kappa)\, n_0 \frac{\mathrm{i}\kappa v}{kT} - p \frac{\lambda_{\mathrm{ee}}}{\mathrm{i}\kappa} \frac{mv}{kT}\right)\left(p + \mathrm{i}\kappa v + \lambda\right)^{-1} f_0\, \mathrm{d}v$$
$$= 0 \tag{10}$$

where $\lambda = \lambda_{\mathrm{ei}} + \lambda_{\mathrm{ee}}$ and p is the Laplace transform with respect to time.

We require solutions of equation (10) valid for small real κ and λ sufficiently small so that the inequality

$$\lambda/\sqrt{2}a\kappa \ll 1$$

is satisfied. The inequality defines the limit of "few collisions". A solution to the dispersion relation in this region can be found as follows. Equation (10) may be written in the form

$$1 + \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{2}} \frac{\mathrm{i}}{\kappa} \int_{-\infty}^{\infty} \left(\lambda - \epsilon \frac{V(\kappa)\, n_0 \sqrt{2} s}{m \quad a} - p \frac{\lambda_{\mathrm{ee}} \sqrt{2} s}{\mathrm{i}\kappa a}\right) \frac{\exp(-s^2)}{s - Z}\, \mathrm{d}s = 0, \tag{11}$$

where

$$v = \sqrt{2}as, \qquad Z = -\left(\frac{p+\lambda}{\sqrt{2}a\mathrm{i}\kappa}\right), \qquad \text{and} \qquad a = \left(\frac{kT}{m}\right)^{\frac{1}{2}}.$$

We make the substitution $p = -\mathrm{i}\omega - \gamma$ and confine our analysis to a wave that propagates in the positive x direction. In terms of the real variables ω and γ we write

$$Z = \frac{\omega}{\sqrt{2}a\kappa} + \mathrm{i}\left(\frac{\lambda - \gamma}{\sqrt{2}a\kappa}\right) \equiv \eta + \mathrm{i}\zeta . \tag{12}$$

We next obtain an expansion for the integral of equation (11) in powers of ζ ($\zeta > 0$). In making such an expansion we assume that the damping coefficient γ, is small, that is, $\gamma \ll \omega$, an assumption which can be justified *a posteriori* when κ is small. The integral of equation (11) has the form

$$I(Z) = \int_{-\infty}^{\infty} g(s)/(s - Z)\, \mathrm{d}s,$$

where $g(s)$ is analytic. Using an identity due to Jackson (1960) we may then write

$$I(Z) = I(\eta + \mathrm{i}\zeta) = \sum_{j=0}^{\infty} \frac{(\mathrm{i}\zeta)^j}{j!}\left(P \int_{-\infty}^{\infty} \frac{g^{(j)}(s)\, \mathrm{d}s}{s - \eta} + \pi \mathrm{i} g^{(j)}(\eta)\right), \tag{13}$$

where P denotes the principal value and the superscript (j) denotes the number of times the function is to be differentiated. Using equation (13) in (11) we have, after some algebra, for the real part of the dispersion relation

$$1+\frac{1}{\sqrt{2a\kappa}}\left(-\sqrt{\pi}\lambda\exp(-\eta^2)-\zeta\lambda\, O(\eta^{-2})\right)$$
$$+\left(\frac{n_0\epsilon V(\kappa)}{ma^2}+\frac{\gamma\lambda_{\mathrm{ee}}}{(\kappa a)^2}\right)\left(-\tfrac{1}{2}\eta^{-2}-\tfrac{3}{4}\eta^{-4}-O(\eta^{-6})-\sqrt{\pi}\zeta(1-2\eta^2)\exp(-\eta^2)\right)$$
$$+\frac{\omega\lambda_{\mathrm{ee}}}{(\kappa a)^2}\left(-\sqrt{\pi}\eta\exp(-\eta^2)-\zeta\, O(\eta^{-3})\right)=0. \tag{14}$$

To first order in λ, neglecting terms in $\lambda\gamma$ and exponential terms (for $\omega\approx\omega_{\mathrm{p}}$, the exponential terms are trivially small), we have for equation (14)

$$1+\left(n_0\epsilon V(\kappa)/ma^2\right)\left(-\tfrac{1}{2}\eta^{-2}-\tfrac{3}{4}\eta^{-4}\right)=0.$$

For an electron gas $\epsilon V(\kappa)=4\pi e^2/\kappa^2$, where e is the charge on the electron. Using this result we obtain

$$\omega^2=\omega_{\mathrm{p}}^2+3a^2\kappa^2, \tag{15}$$

where

$$\omega_{\mathrm{p}}^2=4\pi e^2 n_0/m.$$

Equation (15) is the well known plasma dispersion relation for electron oscillations in a dilute plasma.

Correspondingly the imaginary part of the dispersion relation is

$$\frac{1}{\sqrt{2a\kappa}}\left(-\lambda\eta^{-1}-\tfrac{1}{2}\lambda\eta^{-3}-\tfrac{3}{4}\lambda\eta^{-5}+2\sqrt{\pi}\lambda\zeta\eta\exp(-\eta^2)\right)$$
$$+\left(\frac{n_0\epsilon V(\kappa)}{ma^2}-\frac{\gamma\lambda_{\mathrm{ee}}}{(\kappa a)^2}\right)\left(\sqrt{\pi}\eta\exp(-\eta^2)+\zeta\eta^{-3}+3\zeta\eta^{-5}\right)$$
$$+\frac{\omega\lambda_{\mathrm{ee}}}{(\kappa a)^2}\left(-\tfrac{1}{2}\eta^{-2}-\tfrac{3}{4}\eta^{-4}-\tfrac{15}{8}\eta^{-6}-\sqrt{\pi}\zeta(1-2\eta^2)\exp(-\eta^2)\right)=0. \tag{16}$$

After simplifying the expression in equation (16) and solving for γ to first order in λ we have

$$\gamma=(\tfrac{1}{8}\pi)^{\frac{1}{2}}\omega_{\mathrm{p}}^4(a\kappa)^{-3}\exp(-\omega^2/2a^2\kappa^2)+\tfrac{1}{2}\lambda_{\mathrm{ei}}+\kappa^2h^2(\lambda_{\mathrm{ei}}+\lambda_{\mathrm{ee}}), \tag{17}$$

where h is the Debye shielding distance given by

$$h^2=kT/4\pi e^2 n_0.$$

The first term in equation (17) is the well known Landau damping which exists for collisionless electron oscillations. The term independent of κ is the damping due to electron–ion collisions. The term in κ^2 contains effects of both electron–electron and electron–ion collisions.

IV. Results

This section compares the results obtained above and those obtained from various other calculations using different terms for the collision processes.

For convenience we recast equation (17) to a different form using

$$\gamma = \gamma_h + \gamma_{ei} + \gamma_{ee}, \tag{18}$$

where

$$\gamma_{ei} = \tfrac{1}{2}\lambda_{ei} + \kappa^2 h^2 \lambda_{ei} \tag{19}$$

and

$$\gamma_{ee} = \kappa^2 h^2 \lambda_{ee}, \tag{20}$$

while γ_h is the Landau damping.

Equations (18)–(20) are exact solutions for the damping to order λ in the approximations we have employed. Previous calculations using the Krook collision model have yielded only certain terms of equation (18). Bhatnagar, Gross, and Krook (1954) found γ_{ee} but neglected γ_h and γ_{ei}. In a calculation by Burgers (1962) the terms γ_h and γ_{ei} (to zero order in κ) were found while γ_{ee} was neglected due to the approximations employed. For the sake of comparison with other results derived from different collision terms we have included the effects of electron–ion and electron–electron collisions to first order in the collision frequency as well as the Landau damping term.

Lenard and Bernstein (1958) studied plasma oscillations using a linearized collision term of the form

$$\frac{\delta f}{\delta t} = \beta \frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\boldsymbol{v} f + a^2 \frac{\partial f}{\partial \boldsymbol{v}} \right),$$

where β is an effective collision frequency and $a = (kT/m)^{\frac{1}{2}}$. This equation is a Fokker–Planck type collision term which represents a diffusion in velocity space for the small angle scattering that is characteristic of a Coulomb force. They obtained for the damping $\gamma = \beta$, which necessarily restricted their calculation to electron–ion collisions.

Collision damping of long wavelength plasma oscillations using the Guernsey (1962) kinetic equation has been studied by Gorman and Montgomery (1963). They restricted their calculations to electron–electron collisions and though they were not able to find explicit analytic expressions for the damping they did conclude that γ_{ee} was proportional to κ^2.

The results of Comisar (1963), who based his investigation on a much more realistic Fokker–Planck equation than that used by Lenard and Bernstein, were

$$\gamma_{ei} = \tfrac{1}{2}\lambda_{ei} + 2\kappa^2 h^2 \lambda_{ei}, \tag{21}$$

$$\gamma_{ee} = 0 \cdot 950\, \lambda_{ee}\, \kappa^2 h^2. \tag{22}$$

These results are for the case of "few collisions" which is defined through the inequality $\lambda/\sqrt{2}a\kappa \ll 1$.

Ogasawara (1963) has also computed similar results from the linearized Boltzmann equation. He finds

$$\gamma_{ei} = \tfrac{1}{2}\lambda_{ei} + 0 \cdot 8\, \lambda_{ei}\, \kappa^2 h^2, \tag{23}$$

$$\gamma_{ee} = 0 \cdot 80\, \lambda_{ee}\, \kappa^2 h^2. \tag{24}$$

Finally we mention that the results obtained by Dubois, Gilinsky, and Kivelson (1962) using diagram techniques of many-body theory are exactly the same as in equations (23) and (24), except for a factor in the collision frequency that arises from quantum effects.

V. Discussion

A comparison of the above results is rather interesting if we keep in mind the approximations made in deriving our answers for the Krook model. The term $\frac{1}{2}\lambda_{\mathrm{ei}}$ (zeroth order in κ) for the damping due to electron–ion collisions is the same for all calculations. The approximation we made for electron–ion collisions was that only number density is conserved in collisions. In the determination of equations (21)–(24) the approximation $m_{\mathrm{e}} \ll m_{\mathrm{i}}$ has been made and notwithstanding the sophistication of the initial approach to the problem this condition implies that again only number density is conserved in collisions between electrons and ions, to a first approximation. For terms involving dynamical friction and velocity diffusion (order κ^2 terms) the Krook model is closer to the linearized Boltzmann equation and many-body approaches, which can be seen on comparison of equations (19), (21), and (23). Which of the terms is correct is rather a moot point though the results obtained from the Krook model could hardly be claimed to be exact here since we have not determined the effect of neglecting energy conservation in collisions. However, whether or not one can talk of temperature variations for the condition $\lambda/\sqrt{2}a\kappa \ll 1$ is an open question.

For the contribution to the damping from electron–electron collisions the difference between the Krook model and Fokker–Planck results is only 5% while the difference between equations (20) and (24) is 20%.

Despite the discrepancies between the results it is clear that one cannot correctly treat the damping of collective plasma modes by the Vlasov equation or the random phase approximation since these approaches fail to treat the effects of collisions. For it is possible that the damping of high frequency plasma waves may be dominated by the collision damping rather than the Landau damping. Which process is dominant depends on the magnitude of κh. For example, if $\kappa h = 0\cdot 5$ then $\gamma_h \approx 0\cdot 15\omega_{\mathrm{p}}$, while $\gamma_{\mathrm{ei}}+\gamma_{\mathrm{ee}} \approx 8\cdot 6\times 10^{-8}\omega_{\mathrm{p}}$ (for $n_0 = 10^{16}\,\mathrm{cm}^{-3}$, $T = 10^8\,{}^{\circ}\mathrm{K}$, and $h \approx 7\times 10^{-4}$ cm); for $\kappa h = 0\cdot 1$ we have $\gamma_h \approx 1\cdot 2\times 10^{-19}\omega_{\mathrm{p}}$ and $\gamma_{\mathrm{ei}}+\gamma_{\mathrm{ee}} \approx 5\times 10^{-8}\omega_{\mathrm{p}}$ for the same density and temperature values.

VI. References

Bhatnagar, P. L., Gross, E. P., and Krook, M. (1954).—*Phys. Rev.* **94**, 511.

Burgers, J. M. (1962).—*In* "Fluid Dynamics and Applied Mathematics". (Eds J. B. Diaz and S. I. Pai.) pp. 79–103. (Science Publishers: New York.)

Comisar, G. G. (1963).—*Physics Fluids* **6**, 76, 1660.

Dubois, D. F., Gilinsky, V., and Kivelson, M. G. (1962).—*Phys. Rev. Lett.* **8**, 419.

Gorman, D., and Montgomery, D. (1963).—*Phys. Rev.* **131**, 7.

Gross, E. P., and Krook, M. (1956).—*Phys. Rev.* **102**, 593.

Guernsey, R. L. (1962).—*Physics Fluids* **5**, 322.

Jackson, J. D. (1960).—*J. nucl. Energy* **1**, 171.

Lenard, A., and Bernstein, I. B. (1958).—*Phys. Rev.* **112**, 1456.

Ogasawara, M. (1963).—*J. phys. Soc. Japan* **18**, 1066.