HYDROMAGNETIC STABILITY OF THIN SELF-GRAVITATING DISKS AND SPIRAL STRUCTURE

By R. J. Hosking*

[*Manuscript received January* 21, 1969]

Summary

The hydromagnetic stability of infinitesimally thin, current-carrying, differentially rotating disks is considered. The perturbation theory described is first order (linear) throughout. It is shown that azimuthal electric current produces instability for all (radial) wavelengths. The case of radial equilibrium electric current reduces to the non-hydromagnetic problem treated by Lin and Shu (1964), although hydromagnetic effects are expected for a disk of finite thickness or to second order due to azimuthal perturbation current.

I. Introduction

Calculations for a hydromagnetic analogue of the gas-dynamical stability problem considered by both Lin and Shu (1964) and Toomre (1964) are described in this paper. A thin, differentially rotating disk is assumed to carry electric current which produces a magnetic field. It is shown that the resultant magnetic body force in the plane of the disk may significantly influence its stability.

Since galactic spiral structure was discovered (Lord Rosse 1850), its explanation has been one of the outstanding problems of astrophysics. The stability of differentially rotating disk systems is relevant to possible theory for the spiral pattern (Lin and Shu 1964; Toomre 1964; Goldreich and Lynden-Bell 1965; Julian and Toomre 1966; Lin 1966).

Observations seriously limit the topology of the magnetic field in our Galaxy. The optical polarization measurements of Hall (1949) and Hiltner (1949, 1951, 1956) indicate a large-scale average field parallel to the plane of the Galaxy (Davis and Greenstein 1951). Both the radio observations (Morris and Berge 1964) and the polarization of starlight (Smith 1956; Behr 1959) indicate that the local lines of force lie parallel to the direction of galactic longitude $l^{\mathrm{II}} = 70°\pm20°$, which agrees with the direction of the spiral arm determined from the distribution of interstellar gas and O star associations (Weaver 1953; van de Hulst, Muller, and Oort 1954; Westerhout 1957). This feature of the galactic magnetic field is illustrated in figures of Kaplan (1966), which compare interstellar polarizations with the line of sight perpendicular and parallel to the spiral arm. Morris and Berge (1964) also point out that the Faraday rotation indicates a reversal of the field across the plane of the Galaxy. With reference to the calculations reported in the present paper, it would appear that a radial equilibrium current is appropriate to the case of our Galaxy.

* School of Physical Sciences, Flinders University of South Australia, Bedford Park, S.A. 5042.

The assumption that any magnetic field is due to currents in the disk enables one to view the differentially rotating disk as an isolated system. Although the alternative open model of the galactic field removes the difficulty of explaining the fields of galaxies (and replaces it by the problem of an intergalactic field), the apparent requirement that the extragalactic field lines be somewhere attached implies that the magnetic field must be rapidly attenuated by the differentially rotating disk and hence plays a major role (see, for example, Piddington 1964, 1966, 1967). In the present paper, Ohm's law is taken to be (considering a "frozen-in" field)

$$\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B} = 0$$

for the hydromagnetic component (the "gas") in the disk, but the local origin of the magnetic field permits one to ignore such effects. Numerical calculations reported in Section III indicate that the magnetic field becomes small at intergalactic distances.

The calculations described in this paper apply to the dynamics of the gas over the disk as a whole, except that the perturbations considered are restricted to within the plane of the disk in the manner of Lin and Shu (1964). These calculations are therefore complementary to considerations of the stability of an individual spiral arm (Chandrasekhar and Fermi 1953; Simon 1958; Amano 1964; Setti 1965) and the universal Rayleigh–Taylor instability (Parker 1966).

Finally, it may be worth noting at this point that a gravitational "response" factor $\mathscr{R}$ for the gas is included (Section III); a value of $\mathscr{R} < 1$ implies that the gravitation acting on the gas is not entirely self-gravitation but includes a contribution from the non-hydromagnetic ("stellar") disk component. Theory for the response of the stellar component with stellar dispersion included has been given by Lin (1966), so that the present study may lead to an improved general theory for the possible total gravitational response of disk galaxies. Since turbulence has been ignored in this paper, the hydromagnetic theory given for the gas applies to systems in which the magnetic energy exceeds the energy of interstellar gas turbulence. Lin and Shu (1966) have considered the opposite situation by representing the effect of turbulence as similar to the effect of dispersion on the stellar component.

II. Hydromagnetic Equations

A cylindrical system of coordinates (r, θ, z) is adopted and both mass distribution and electric current are restricted to the plane $z = 0$ throughout the motion. As shown in Appendix I, the equation of continuity, the equation of motion, and Ohm's law (valid for $z = 0$) are

$$\frac{\partial\sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma v_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sigma v_\theta\right) = 0\,, \tag{1}$$

$$\sigma\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\,.\,\nabla\boldsymbol{v}\right) = \sigma\nabla\phi+c^{-1}\boldsymbol{J}\times\boldsymbol{B}\,, \tag{2}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times\boldsymbol{v}\times\boldsymbol{B}\,, \tag{3}$$

where $\sigma(r,\theta,t)$ is the surface mass density of the gas, $\boldsymbol{v}(r,\theta,0,t) = (v_r, v_\theta, 0)$ is its material velocity, $\phi(r,\theta,0,t)$ is the negative of the gravitational potential, $\boldsymbol{J}(r,\theta,t) =$

$(J_r, J_\theta, 0)$ is the surface current density, and $\boldsymbol{B}(r,\theta,0,t)$ is the magnetic field. In addition, throughout all space one has the electromagnetic equations

$$\nabla . \boldsymbol{B} = 0 \tag{4}$$

and

$$\nabla \times \boldsymbol{B} = (4\pi/c)\boldsymbol{J}\delta(z), \tag{5}$$

together with Poisson's equation

$$\nabla^2\phi = -4\pi G\sigma^{\text{tot}}\,\delta(z), \tag{6}$$

where $\delta(z)$ is the Dirac delta function, G is Newton's universal constant of gravitation,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{6a}$$

and σ^{tot} is the total surface mass density (both gas and stellar components).

It is convenient to introduce the vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \text{where} \qquad \nabla . \boldsymbol{A} = 0,$$

so that equation (5) may be written

$$\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2} = -(4\pi/c)J_r\,\delta(z), \tag{7a}$$

$$\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} = -(4\pi/c)J_\theta\,\delta(z), \tag{7b}$$

$$\nabla^2 A_z = 0. \tag{7c}$$

In the following two sections, these equations are used in discussion of the cases of azimuthal equilibrium current and radial equilibrium current respectively.

III. Azimuthal Equilibrium Current—Poloidal Field

The situation in which the equilibrium current is azimuthal is now considered. One may imagine a series of concentric current-carrying rings. In this case, one expects both radial and axial magnetic field components, whereas in the complementary case of radial current discussed in the following section the magnetic field is azimuthal. Accordingly, the initial state of equilibrium is described by $\sigma = \sigma_0(r)$, $v_r = 0$, $v_\theta = r\Omega(r)$, $J_r = 0$, $J_\theta = J_{0\theta}(r)$, $\phi = \phi_0(r,z)$, and $\boldsymbol{B} = \big(B_{0r}(r,z), 0, B_{0z}(r,z)\big)$.

Reference to equations (4) and (5) shows that the equilibrium magnetic field must satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_{0r}\right) + \frac{\partial B_{0z}}{\partial z} = 0, \tag{8}$$

$$\frac{\partial B_{0r}}{\partial z} - \frac{\partial B_{0z}}{\partial r} = (4\pi/c)J_{0\theta}(r)\,\delta(z). \tag{9}$$

The elimination of B_{0r} between (8) and (9) yields

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)B_{0z} = -\frac{4\pi}{c}\frac{1}{r}\frac{\partial}{\partial r}\left(rJ_{0\theta}(r)\right)\delta(z),$$

which has as solution

$$B_{0z}(r,z) = 4\int_0^\infty \mathrm{d}\rho\,\Big((\rho+r)^2+|z|^2\Big)^{-\frac{1}{2}}\frac{1}{c}\frac{\partial}{\partial\rho}\Big(\rho J_{0\theta}(\rho)\Big)\,\mathrm{K}\Big\{\Big(\frac{4\rho r}{(\rho+r)^2+|z|^2}\Big)^{\frac{1}{2}}\Big\}, \tag{10}$$

where K denotes the complete elliptic integral of the first kind. This integration is given in Appendix II. A convenient dimensionless function with which to characterize the equilibrium is

$$\mu(r,z) = 2\pi J_{0\theta}(r)/cB_{0z}(r,z)\,. \tag{11}$$

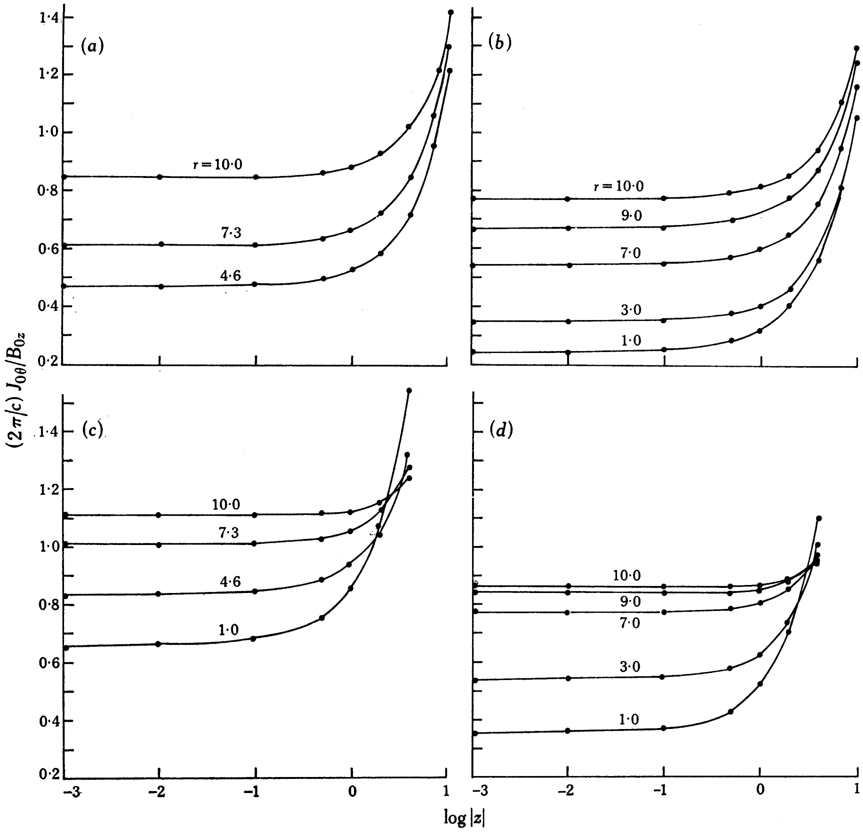

Fig. 1.—Spatial variation of the ratio of the equilibrium azimuthal disk electric current to the generated axial field for uniform current distribution ((*a*) and (*b*)) and when the current varies inversely exponentially with radius ((*c*) and (*d*)):

(*a*) $a = 1,\ b = 10,\ J_{0\theta} = \text{const.}$; (*b*) $a = 0,\ b = 10,\ J_{0\theta} = \text{const.}$;
(*c*) $a = 1,\ b = 10,\ J_{0\theta} = C\exp(-r/b)$; (*d*) $a = 0,\ b = 10,\ J_{0\theta} = C\exp(-r/b)$.

In Figure 1, $\mu(r,z)$ is plotted against $\log|z|$ for the cases

$$\begin{aligned} J_{0\theta}(r) &= \text{const.} > 0, && a < r < b,\\ &= 0, && \text{otherwise}, \end{aligned}$$

and

$$\begin{aligned} J_{0\theta}(r) &= \text{const.}\exp(-r/b), && a < r < b,\\ &= 0, && \text{otherwise}, \end{aligned}$$

using equation (A22), Appendix II. The limiting values $\mu(r, 0+)$ have been confirmed from equation (A24). The form of B_{0r} follows from equation (10); it is sufficient, however, to note that $B_{0r}(r, 0) = 0$.

The equilibrium equation of motion requires that

$$-r\Omega^2 = (\partial\phi_0/\partial r)_{z=0} + J_{0\theta}(r)\, B_{0z}(r, 0)/c\sigma_0 \tag{12}$$

and

$$0 = (\partial\phi_0/\partial z)_{z=0}. \tag{13}$$

From equation (6) one has

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\phi_0 = -4\pi G\sigma_0^{\text{tot}}(r)\,\delta(z),$$

so that from Appendix II

$$\phi_0(r, z) = 4G\int_0^\infty \mathrm{d}\rho\, \frac{\rho}{\{(\rho+r)^2 + |z|^2\}^{\frac{1}{2}}}\, \sigma_0^{\text{tot}}(\rho)\, \mathrm{K}\left\{\left(\frac{4\rho r}{(\rho+r)^2 + |z|^2}\right)^{\frac{1}{2}}\right\}.$$

In particular, equation (13) is satisfied and equation (12) relates $\Omega(r)$, $\sigma_0^{\text{tot}}(r)$, and $J_{0\theta}(r)$.

The instability of the disk to motions in its plane is now considered. Distinguishing perturbation quantities by the subscript 1, it may be first observed that if

$$\sigma_1(r, \theta, t; \alpha) = \sigma_1(\alpha)\, \mathrm{Re}\{\mathrm{H}_m(\alpha r)\}\exp\{\mathrm{i}(\omega t - m\theta)\},$$

then the solution of the Poisson equation

$$\nabla^2\phi_1 = -(4\pi G/\mathscr{R})\sigma_1\,\delta(z)$$

is

$$\phi_1(r, \theta, z, t; \alpha) = (2\pi G/\mathscr{R}\alpha)\sigma_1(r, \theta, t; \alpha)\exp(-\alpha\,|z|)$$

and for $\alpha r \gg 1$

$$(\partial\phi_1/\partial r)_{z=0} = \pm 2\pi \mathrm{i} G\sigma_1/\mathscr{R}, \qquad (\partial\phi_1/\partial\theta)_{z=0} = O(\sigma_1/\alpha\mathscr{R}) \tag{14}$$

(cf. Appendix II), where $\mathscr{R} \equiv \sigma_1/\sigma_1^{\text{tot}}$ defines the gas fraction of the total self-gravitating mass. Similarly, if

$$J_1(r, \theta, t; \alpha) = J_1(\alpha)\, \mathrm{Re}\{\mathrm{H}_m(\alpha r)\}\exp\{\mathrm{i}(\omega t - m\theta)\},$$

then the solution of the perturbation equation corresponding to equations (7) is

$$A_1(r, \theta, z, t; \alpha) = (2\pi/\alpha c)J_1(r, \theta, t; \alpha)\exp(-\alpha\,|z|)$$

(cf. Appendix III), so that

$$B_{1r} = (2\pi/c)\mathrm{sgn}(z)\, J_{1\theta}\exp(-\alpha\,|z|), \tag{15a}$$

$$B_{1\theta} = -(2\pi/c)\mathrm{sgn}(z)\, J_{1r}\exp(-\alpha\,|z|), \tag{15b}$$

$$B_{1z} = \frac{2\pi}{\alpha c}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rJ_{1\theta}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(J_{1r}\right)\right\}\exp(-\alpha\,|z|). \tag{15c}$$

In this paper, the perturbation problem is solved in a "local approximation" characterized by eigensolutions approximated on $z = 0$ by

$$f_1(r,\theta,t;\alpha) = f_1(\alpha)\,\mathrm{Re}\{\mathrm{H}_m(\alpha r)\}\exp\{\mathrm{i}(\omega t - m\theta)\}, \qquad \alpha R \gg 1,$$
$$\sim \exp\{\mathrm{i}(\omega t - m\theta \pm \alpha r)\},$$

where R is the reference radius. In this local approximation, the system of simultaneous equations derived from equations (1), (2), (3), (14), (15), and $\nabla . \boldsymbol{J}_1 = 0$ reduces to

$$\mathrm{i}(\omega - m\Omega)\sigma_1 = -\sigma_0\,\mathrm{D}v_{1r}, \tag{16a}$$

$$\mathrm{i}(\omega - m\Omega)v_{1r} - 2\Omega v_{1\theta} = \pm\frac{2\pi\mathrm{i}}{\mathscr{R}}G\sigma_1 + \frac{B_{0z}}{c\sigma_0}J_{1\theta} + \frac{J_{0\theta}}{c\sigma_0}B_{1z}, \tag{16b}$$

$$\mathrm{i}(\omega - m\Omega)v_{1\theta} + (\kappa^2/2\Omega)v_{1r} = -(B_{0z}/c\sigma_0)J_{1r}, \tag{16c}$$

$$\mathrm{i}(\omega - m\Omega)B_{1z} = -B_{0z}\,\mathrm{D}v_{1r}, \tag{16d}$$

$$B_{1z} = (2\pi/\alpha c)\mathrm{D}J_{1\theta}, \tag{16e}$$

$$\mathrm{D}J_{1r} - (\mathrm{i}m/R)J_{1\theta} = 0, \tag{16f}$$

where

$$\kappa^2 \equiv \left(4\Omega^2\{1 + (r/2\Omega)\mathrm{d}\Omega/\mathrm{d}r\}\right)_{r=R},$$

$\mathrm{D} \equiv \mathrm{d}/\mathrm{d}r \sim \pm\mathrm{i}\alpha$, and it has been assumed that $v_{1r} \sim v_{1\theta}$. Elimination between equations (16) yields

$$\left(\pm\frac{2\pi\mathrm{i}}{\mathscr{R}}G\sigma_0 + \frac{J_{0\theta}B_{0z}}{c\sigma_0}\right)\mathrm{D}^2v_{1r} + \left(\kappa^2 - (\omega - m\Omega)^2 + \frac{B_{0z}^2}{2\pi\sigma_0}\alpha\right)\mathrm{D}v_{1r} = 0. \tag{17}$$

Setting $\mathrm{D} = \pm\mathrm{i}\alpha$ in equation (17), one has that equations (16) define a self-consistent problem provided that

$$\omega_{\mathrm{i}}^2 - (\omega_{\mathrm{r}} - m\Omega)^2 + \kappa^2 - (2\pi G\sigma_0/\mathscr{R})\alpha + (B_{0z}^2/2\pi\sigma_0)\alpha = 0 \tag{18}$$

and

$$\pm(J_{0\theta}B_{0z}/c\sigma_0)\alpha - 2\omega_{\mathrm{i}}(\omega_{\mathrm{r}} - m\Omega) = 0, \tag{19}$$

where ω_{r} and ω_{i} are the real and imaginary parts of the eigenfrequency.

In the corresponding gas-dynamic calculation ($B_{0z} \equiv 0$), from equation (18) it follows that

$$\kappa^2 + \omega_{\mathrm{i}}^2 - (\omega_{\mathrm{r}} - m\Omega)^2 > 0,$$

which is inequality (13) of Lin and Shu (1964). From equation (19), either

$$\omega_{\mathrm{r}} - m\Omega = 0 \qquad \text{or} \qquad \omega_{\mathrm{i}} = 0 \quad \text{(neutral stability)},$$

when either

$$\omega_{\mathrm{i}}^2 = (2\pi G\sigma_0/\mathscr{R})\alpha - \kappa^2 \quad \text{or} \quad (\omega_{\mathrm{r}} - m\Omega)^2 = \kappa^2 - (2\pi G\sigma_0/\mathscr{R})\alpha$$

respectively. There is instability only for sufficiently small wavelength (sufficiently large α). The case $m = 0$ was first considered by Toomre (1964).

When $B_{0z} \neq 0$, one has immediately

$$\left(\kappa^2+\omega_i^2-(\omega_r-m\Omega)^2\right)\left(\frac{2\pi G\sigma_0}{\mathscr{R}}-\frac{B_{0z}^2}{2\pi\sigma_0}\right)^{-1}>0 \tag{20}$$

from equation (18) and

$$\omega_i \neq 0\,, \qquad \omega_r - m\Omega \neq 0$$

from equation (19). The growth rate of the instability is the negative root of

$$2\omega_i^2=\left(\frac{2\pi G\sigma_0}{\mathscr{R}}\alpha-\kappa^2-\frac{B_{0z}^2}{2\pi\sigma_0}\alpha\right)+\left\{\left(\frac{2\pi G\sigma_0}{\mathscr{R}}\alpha-\kappa^2-\frac{B_{0z}^2}{2\pi\sigma_0}\alpha\right)^2+\left(\frac{J_{0\theta}B_{0z}}{c\sigma_0}\alpha\right)^2\right\}^{\frac{1}{2}}, \tag{21}$$

which exists for *all* values of the radial wave number α. In particular, there is instability at wavelengths that are neutrally stable according to gas-dynamic theory.

After noting that

$$|J_{0\theta}B_{0z}/c\sigma_0| \div |B_{0z}^2/2\pi\sigma_0| = \mu\,,$$

where the ratio $\mu \equiv \mu(R,0) = O(1)$, one may consider some limiting cases of the hydromagnetic calculation:

(i) when

$$(2\pi G\sigma_0/\mathscr{R})\alpha-\kappa^2-(B_{0z}^2/2\pi\sigma_0)\alpha \simeq 0\,,$$

the instability has growth rate

$$-\omega_i \simeq (J_{0\theta}B_{0z}/2c\sigma_0)^{\frac{1}{2}};$$

(ii) when the impressed gravitational force is large relative to the magnetic (Lorentz) force such that

$$\frac{2\pi G\sigma_0}{\mathscr{R}}\alpha-\kappa^2-\frac{B_{0z}^2}{2\pi\sigma_0}\alpha \gg \left|\frac{J_{0\theta}B_{0z}}{c\sigma_0}\alpha\right|, \tag{22}$$

one has

$$\omega_i^2 \simeq \frac{2\pi G\sigma_0}{\mathscr{R}}-\kappa^2-\frac{B_{0z}^2}{2\pi\sigma_0}\alpha$$

and

$$\omega_r-m\Omega \simeq \left(\frac{J_{0\theta}B_{0z}}{2c\sigma_0}\alpha\right)\left(\frac{2\pi G\sigma_0}{\mathscr{R}}\alpha-\kappa^2-\frac{B_{0z}^2}{2\pi\sigma_0}\alpha\right)^{-\frac{1}{2}};$$

(iii) when the rotational force is large relative to the magnetic force such that

$$\kappa^2-\frac{2\pi G\sigma_0}{\mathscr{R}}\alpha+\frac{B_{0z}^2}{2\pi\sigma_0}\alpha \gg \left|\frac{J_{0\theta}B_{0z}}{c\sigma_0}\alpha\right|, \tag{23}$$

one has

$$-\omega_i \simeq \left(\frac{J_{0\theta}B_{0z}}{2c\sigma_0}\alpha\right)\left(\kappa^2-\frac{2\pi G\sigma_0}{\mathscr{R}}\alpha+\frac{B_{0z}^2}{2\pi\sigma_0}\alpha\right)^{-\frac{1}{2}}$$

and

$$(\omega_r-m\Omega)^2 \simeq \kappa^2-(2\pi G\sigma_0/\mathscr{R})\alpha+(B_{0z}^2/2\pi\sigma_0)\alpha\,.$$

It is noteworthy, however, that when

$$B_{0z} \gtrsim (4\pi^2 G\sigma_0^2/\mathscr{R})^{\frac{1}{2}} \tag{24}$$

inequality (22) is contravened. Further, if

$$B_{0z} \gg (4\pi^2 G\sigma_0^2/\mathscr{R})^{\frac{1}{2}},$$

none of the above three limiting cases are appropriate and

$$\omega_{\mathrm{i}}^2 \simeq (B_{0z}^2/4\pi\sigma_0)\alpha\{(1+\mu^2)^{\frac{1}{2}}-1\},$$

$$(\omega_{\mathrm{r}}-m\Omega)^2 \simeq (B_{0z}^2/4\pi\sigma_0)\alpha\{(1+\mu^2)^{\frac{1}{2}}+1\}.$$

Certain aspects of these results may be inferred from reference to equations (16) and physical argument. The essential hydromagnetic aspect of the present problem is represented by the radial Lorentz force (cf. equation (16b)). One may first note that

$$\left|\frac{J_{0\theta}}{c\sigma_0}B_{1z}\right| \div \left|\frac{B_{0z}}{c\sigma_0}J_{1\theta}\right| = \left|\pm\frac{2\pi\mathrm{i}}{c}\frac{J_{0\theta}}{B_{0z}}\right| = \mu$$

and that

$$\left|\frac{B_{0z}}{c\sigma_0}J_{1\theta}\right| \gtrsim \left|\pm\frac{2\pi\mathrm{i}}{\mathscr{R}}G\sigma_1\right|$$

requires inequality (24). When

$$B_{0z} \ll (4\pi^2 G\sigma_0^2/\mathscr{R})^{\frac{1}{2}}$$

the dominant radial force is gravitation; inequality (22) is valid in this case and the growth rate differs only slightly from the value predicted by gas-dynamic theory. Secondly, neither contribution to the radial Lorentz force is in phase with the radial gravitational force. Consequently, one might anticipate both a growth rate lower than that predicted by gas-dynamic theory when gravitation is dominant and also a destabilization at wave numbers for which the gas-dynamic theory predicts neutral stability when it is not.

In summary, a poloidal magnetic field can provide a destabilizing mechanism associated with a shift in phase (alteration of ω_{r}). This result is similar to the effect of a magnetic field on thermal convection, where a state of neutral stability becomes overstable when a magnetic field is introduced (Chandrasekhar 1961).

With respect to the relevance of the present hydromagnetic calculation to the dynamics of disk galaxies, three remarks may be made. In the first place, the magnetically responsive interstellar gas has a spiral pattern which differs in phase from that of the stars by an amount determined by the magnitude of the equilibrium magnetic field (cf. equations (17) and (18)). Secondly, inequality (24) defines when hydromagnetic effects must be considered. Finally, reference to inequality (20) indicates that

$$\kappa^2+\omega_{\mathrm{i}}^2-(\omega_{\mathrm{r}}-m\Omega)^2 \lesseqgtr 0$$

according as

$$B_{0z} \gtrless (4\pi^2 G\sigma_0^2/\mathscr{R})^{\frac{1}{2}}.$$

For the parameters $\mathscr{R} = 0\cdot 1$, $\sigma_0 = 10^{-3}\ \mathrm{g\,cm^{-3}}$, the critical magnitude of the magnetic field defined by the last two remarks is about 5 μG. (Note, however, that the precise value for $\mathscr{R}$ at any reference radius depends both on the relative response *maxima* of the gas and the stars and the phase difference mentioned in the first remark.)

IV. Radial Equilibrium Current—Toroidal Field

Suppose now that the initial state of equilibrium is described by $\sigma = \sigma_0(r)$, $v_r = 0$, $v_\theta = r\Omega(r)$, $J_r = J_{0r}(r)$, $J_\theta = 0$, $\phi = \phi_0(r,z)$, and $\boldsymbol{B} = (0, B_{0\theta}(r,z), 0)$. Since the divergence of equation (5) gives

$$\mathrm{d}(rJ_{0r})/\mathrm{d}r = 0\,,$$

one has

$$J_{0r} = M/r\,,$$

where the constant M may be considered to be the strength of a current source located at $z = 0$. From the viewpoint of the application to galactic forms, such a radial current may be supposed to be supplied to the disk-like part of the galaxy from the central bulge and returned via the halo.

Reference to equation (5) shows that the equilibrium magnetic field must satisfy

$$-\partial B_{0\theta}/\partial z = (4\pi/c)J_{0r}(r)\,\delta(z)\,,$$

$$r^{-1}\,\partial(rB_{0\theta})/\partial r = 0\,,$$

whence

$$B_{0\theta}(r,z) = -(2\pi/c)J_{0r}(r)\,\mathrm{sgn}\,z\,. \tag{25}$$

Note that the topology of the magnetic field described by equation (25) is consistent with the observations for our Galaxy.

From equations (25) and (15) one has that

$$B_{0\theta} = B_{1r} = B_{1\theta} = 0 \qquad \text{on} \qquad z = 0;$$

the nontrivial linear perturbation equation derived from equation (3) is therefore

$$\mathrm{i}(\omega - m\Omega)B_{1z} = 0 \qquad \text{on} \qquad z = 0\,.$$

It follows that one has to solve equation (1) and

$$\partial\boldsymbol{v}/\partial t + \boldsymbol{v}\,.\,\nabla\boldsymbol{v} = \nabla\phi$$

on $z = 0$, together with equation (6). The problem reduces to the gas-dynamic one treated by Lin and Shu (1964).

It must be emphasized that this conclusion applies strictly on the plane $z = 0$ only, and then only to first order. For a real galactic disk of finite thickness, hydromagnetic effects may be significant for $z \neq 0$ (when $B_{0\theta} \neq 0$). Moreover, to second order $B_{1z} \neq 0$ and there is a nonzero Lorentz force on $z = 0$. The perturbation theory described in this paper is first order (linear).

V. Conclusions

The stability of thin, differentially rotating disks which initially carry either azimuthal or radial current has been considered. Significant azimuthal current produces a poloidal magnetic field, whereas significant radial current produces a toroidal magnetic field with a null at the centre of the current layer. Both the radial and azimuthal magnetic field components change rapidly across a thin current layer.

Indeed, in the mathematical description adopted in which the disk is considered infinitesimally thin, the radial and azimuthal magnetic fields have an appropriate discontinuity; only the axial magnetic field is significant for motions within the plane of the disk.

The systems are subject to instabilities that can produce density distributions of spiral foım, as first demonstrated in gas-dynamic theory by Lin and Shu (1964).

By means of a "local approximation" related to that used by Lin and Shu (1964), it has been shown that an equilibrium azimuthal electric current can provide a destabilizing mechanism associated with a shift in phase (Section III). The character of the magnetogravitational instability differs significantly from its gravitational counterpart for magnetic fields greater than about $(4\pi^2G\sigma_0^2/\mathscr{R})^{\frac{1}{2}}$ gauss, which is of the order of 5 μG for the parameters of the Galaxy.

For our Galaxy, however, the equilibrium current appropriate to the observed topology of the magnetic field would appear to be radial. In the first-order (linear) theory considered in this paper, in Section IV it is shown that on $z = 0$ this case reduces to the purely gas-dynamic discussion (Lin and Shu 1964; Toomre 1964; Lin 1966). Any hydromagnetic modification requires finite disk thickness or nonlinear theory.

The gravitational response of a possible stellar component has been allowed for through the "response" factor $\mathscr{R}$. Lin (1966) has provided theory for the stellar component in which the effect of stellar dispersion is considered.

The radical differences between the behaviour of the two systems with significant poloidal and toroidal field respectively may prove important in galactic dynamics.

VI. Acknowledgments

The author is grateful to Professor C. C. Lin for much helpful comment, particularly relating to the relevance of the present work to current theory of galactic spiral structure. The hospitality of the Department of Theoretical Physics at the Institute of Advanced Studies, Australian National University, also helped to complete this paper.

VII. References

Abramowitz, M., and Stegun, I. A. (1964).—"Handbook of Mathematical Functions." (National Bureau of Standards: Washington.)

Amano, T. (1964).—*Prog. theor. Phys., Kyoto* **31**, 742.

Behr, A. (1959).—*Nachr. Akad. Wiss. Göttingen* **185**.

Chandrasekhar, S. (1961).—"Hydrodynamic and Hydromagnetic Stability." (Clarendon Press: Oxford.)

Chandrasekhar, S., and Fermi, E. (1953).—*Astrophys. J.* **118**, 116.

Davis, L., and Greenstein, J. (1951).—*Astrophys. J.* **114**, 206.

Erdelyi, A. (1954).—"Tables of Integral Transforms." Vol. 2, p. 183. (McGraw-Hill: New York.)

Goldreich, P., and Lynden-Bell, D. (1965).—*Mon. Not. R. astr. Soc.* **130**, 125.

Gubler, E. (1897).—*Math. Annln* **48**, 37.

Hall, J. S. (1949).—*Science* **109**, 166.

Hiltner, W. A. (1949).—*Astrophys. J.* **109**, 471.

Hiltner, W. A. (1951).—*Astrophys. J.* **114**, 241.

Hiltner, W. A. (1956).—*Astrophys. J. Suppl. Ser.* **2**, 389.

van de Hulst, H. C., Muller, C. A., and Oort, J. H. (1954).—*Bull. astr. Insts Neth.* **12**, 117.

Julian, W. H., and Toomre, A. (1966).—*Astrophys. J.* **146**, 810.

Kaplan, S. A. (1966).—"Interstellar Gas Dynamics." (Pergamon Press: Oxford.)
Lin, C. C. (1966).—*J. Soc. ind. appl. Math.* **14,** 876.
Lin, C. C., and Shu, F. H. (1964).—*Astrophys. J.* **140,** 646.
Lin, C. C., and Shu, F. H. (1966).—*Proc. natn. Acad. Sci. U.S.A.* **55,** 229.
Morris, D., and Berge, G. L. (1964).—*Astrophys. J.* **139,** 1388.
Parker, E. N. (1966).—*Astrophys. J.* **145,** 811.
Piddington, J. H. (1964).—*Mon. Not. R. astr. Soc.* **128,** 345.
Piddington, J. H. (1966).—*Mon. Not. R. astr. Soc.* **133,** 163.
Piddington, J. H. (1967).—*Mon. Not. R. astr. Soc.* **136,** 293.
Rosse, Lord (1850).—*Phil. Trans. R. Soc.* p. 499.
Setti, G. (1965).—*Bull. astr. Insts Neth.* **18,** 51.
Simon, R. (1958).—*Astrophys. J.* **128,** 375.
Smith, E. van P. (1956).—*Astrophys. J.* **124,** 43.
Sneddon, I. A. (1951).—"Fourier Transforms." p. 53. (McGraw-Hill: New York.)
Toomre, A. (1964).—*Astrophys. J.* **139,** 1217.
Weaver, H. F. (1953).—*Astrophys. J.* **58,** 177.
Westerhout, G. (1957).—*Bull. astr. Insts Neth.* **13,** 201.

Appendix I

Hydromagnetic Equations for Infinitesimally Thin Disk

The appropriate hydromagnetic equations for the interstellar gas are derived here. Both mass and electric current distributions are considered restricted to the plane $z = 0$. Accordingly, the mass and current densities may be written as

$$\rho(r,\theta,z,t) = \sigma(r,\theta,t)\,\delta(z) \qquad \text{and} \qquad \boldsymbol{j}(r,\theta,z,t) = \boldsymbol{J}(r,\theta,t)\,\delta(z)$$

respectively, where $\sigma(r,\theta,t)$ denotes the surface mass density and $\boldsymbol{J}(r,\theta,t)$ denotes the surface current density.

The three-dimensional equation of continuity is

$$\partial\rho/\partial t + \nabla.\rho\boldsymbol{v} = 0 \tag{A1}$$

or

$$\left\{\frac{\partial\sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma v_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sigma v_\theta\right)+\frac{\partial}{\partial z}\left(\sigma v_z\right)\right\}\delta(z)+\sigma v_z\,\delta'(z) = 0\,, \tag{A2}$$

so that

$$\frac{\partial\sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\Big(r\sigma v_r(r,\theta,0,t)\Big)+\frac{1}{r}\frac{\partial}{\partial\theta}\Big(\sigma v_\theta(r,\theta,0,t)\Big) = 0\,, \qquad v_z(r,\theta,0,t) = 0 \tag{A3}$$

with the help of an appropriate choice of trial functions for the distributions $\delta(z)$ and $\delta'(z)$ and the property

$$f(z)\,\delta(z) = f(0)\,\delta(z)\,. \tag{A4}$$

In passing, one may note that in like manner

$$\nabla.\boldsymbol{j} = 0 \tag{A5}$$

yields

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(rJ_r\Big)+\frac{1}{r}\frac{\partial}{\partial\theta}\Big(J_\theta\Big) = 0\,, \qquad J_z = 0\,. \tag{A6}$$

The three-dimensional (pressureless) equation of motion is

$$\rho(\partial \boldsymbol{v}/\partial t + \boldsymbol{v}.\nabla \boldsymbol{v}) = \rho\, \nabla\phi + c^{-1}\boldsymbol{j}\times\boldsymbol{B} \tag{A7}$$

or

$$\sigma(\partial \boldsymbol{v}/\partial t + \boldsymbol{v}.\nabla \boldsymbol{v})\delta(z) = \sigma\, \nabla\phi\, \delta(z) + c^{-1}\boldsymbol{J}\times\boldsymbol{B}\,\delta(z). \tag{A8}$$

Using (A4) and (A3), one therefore has

$$\sigma(\partial \boldsymbol{v}/\partial t + \boldsymbol{v}.\nabla \boldsymbol{v}) = \sigma\,(\nabla\phi)_{z=0} + c^{-1}\boldsymbol{J}\times\boldsymbol{B}(r,\theta,0,t), \tag{A9}$$

where $\boldsymbol{v}$ now denotes $\boldsymbol{v}(r,\theta,0,t)$.

Finally, one has Ohm's law

$$\boldsymbol{E}(r,\theta,0,t) + c^{-1}\boldsymbol{v}(r,\theta,0,t)\times\boldsymbol{B}(r,\theta,0,t) = 0, \tag{A10}$$

or equivalently

$$\{\boldsymbol{E}(r,\theta,z,t) + c^{-1}\boldsymbol{v}(r,\theta,z,t)\times\boldsymbol{B}(r,\theta,z,t)\}\delta(z) = 0. \tag{A11}$$

The curl of equation (A11) yields

$$(\partial\boldsymbol{B}/\partial t + \nabla\times\boldsymbol{v}\times\boldsymbol{B})\delta(z) + \{-(\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B})_\theta\,\hat{\boldsymbol{r}} + (\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B})_r\,\hat{\boldsymbol{\theta}}\}\delta'(z) = 0, \tag{A12}$$

whence

$$\partial\boldsymbol{B}/\partial t + \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = 0 \qquad \text{on} \qquad z = 0. \tag{A13}$$

Appendix II

Solution of Poisson's Equation

The boundary value problem which recurs throughout this paper is of the form

$$\nabla^2 V(r,\theta,z) = -4\pi\sigma(r,\theta)\,\delta(z), \tag{A14}$$

where ∇^2 is given by equation (6a),

$$V(r,\theta,0-) = V(r,\theta,0+), \tag{A15}$$

and

$$V\to 0 \qquad \text{as} \qquad (r^2+z^2)^{\frac{1}{2}}\to\infty. \tag{A16}$$

Since the problem is linear, one may Fourier analyse:

$$V(r,\theta,z) = \sum_{m=0}^{\infty} \mathrm{e}^{-im\theta}\int_0^\infty V_m(r,z;\alpha)\,\mathrm{d}\alpha$$

$$\sigma(r,\theta) = \sum_{m=0}^{\infty} \mathrm{e}^{-im\theta}\int_0^\infty \sigma_m(r;\alpha)\,\mathrm{d}\alpha$$

and consider

$$\frac{\partial^2 V_m}{\partial r^2} + \frac{1}{r}\frac{\partial V_m}{\partial r} - \frac{m^2}{r^2}V_m + \frac{\partial^2 V_m}{\partial z^2} = -4\pi\sigma_m\,\delta(z) \tag{A14a}$$

subject to

$$V_m(r,0-;\alpha) = V_m(r,0+;\alpha) \tag{A15a}$$

and

$$V_m\to 0 \qquad \text{as} \qquad (r^2+z^2)^{\frac{1}{2}}\to\infty. \tag{A16a}$$

One has

$$V_m(r, z; \alpha) = A(\alpha)\, \mathrm{J}_m(\alpha r) \exp(-\alpha\,|\,z\,|)$$

$$\equiv A(\alpha)\, \mathrm{Re}(\mathrm{H}_m(\alpha r)) \exp(-\alpha\,|\,z\,|),$$

provided that

$$2\alpha A(\alpha)\, \mathrm{J}_m(\alpha r) = 4\pi\sigma_m(r; \alpha),$$

where J_m and H_m denote respectively the mth order Bessel function of the first kind and the mth order Hankel function. Thus, if one writes

$$\sigma_m(r; \alpha) = S(\alpha)\, \mathrm{J}_m(\alpha r),$$

it follows that

$$A(\alpha) = (2\pi/\alpha) S(\alpha)$$

and

$$V_m(r, z; \alpha) = (2\pi/\alpha)\sigma_m(r; \alpha) \exp(-\alpha\,|\,z\,|). \tag{A17}$$

Parenthetically, we may note from equation (A17) that

$$(\partial(\mathrm{e}^{-\mathrm{i}m\theta}\, V_m)/\partial\theta)_{z=0} = O(\sigma_m/\alpha), \tag{A18a}$$

$$(\partial(\mathrm{e}^{-\mathrm{i}m\theta}\, V_m)/\partial z)_{z=0} = 0, \tag{A18b}$$

together with

$$(\partial(\mathrm{e}^{-\mathrm{i}m\theta}\, V_m)/\partial r)_{z=0} = \pm 2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}m\theta}\, \sigma_m \tag{A18c}$$

for $\alpha r \gg 1$, where we have used

$$\mathrm{H}_m(\alpha r) \sim (2/\pi\alpha r)^{\frac{1}{2}} \exp\{\pm \mathrm{i}(\alpha r - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi)\}$$

with $+$ and $-$ corresponding to $\mathrm{H}_m^{(1)}$ and $\mathrm{H}_m^{(2)}$ respectively.

The complete solution may be written

$$V(r, \theta, z) = \sum_{m=0}^{\infty} \exp(-\mathrm{i}m\theta) \int_0^{\infty} (2\pi/\alpha) S(\alpha)\, \mathrm{J}_m(\alpha r) \exp(-\alpha\,|\,z\,|)\, \mathrm{d}\alpha. \tag{A19}$$

If one defines

$$\sigma_m(r) = \int_0^{\infty} \sigma_m(r; \alpha)\, \mathrm{d}\alpha = \int_0^{\infty} S(\alpha)\, \mathrm{J}_m(\alpha r)\, \mathrm{d}\alpha,$$

by the Hankel inversion theorem (Sneddon 1951)

$$S(\alpha)/\alpha = \int_0^{\infty} r\sigma_m(r)\, \mathrm{J}_m(\alpha r)\, \mathrm{d}r$$

so that

$$V(r, \theta, z) = 2\pi \sum_{m=0}^{\infty} \exp(-\mathrm{i}m\theta) \int_0^{\infty} \mathrm{d}\alpha \exp(-\alpha\,|\,z\,|)\, \mathrm{J}_m(\alpha r) \int_0^{\infty} \mathrm{d}\rho\; \rho\sigma_m(\rho)\, \mathrm{J}_m(\alpha\rho)$$

or, interchanging the order of integration,

$$V(r, \theta, z) = 2\pi \sum_{m=0}^{\infty} \exp(-\mathrm{i}m\theta) \int_0^{\infty} \mathrm{d}\rho\; \rho\sigma_m(\rho) \int_0^{\infty} \mathrm{d}\alpha \exp(-\alpha\,|\,z\,|)\, \mathrm{J}_m(\rho\alpha)\, \mathrm{J}_m(r\alpha). \tag{A20}$$

Reference to Erdelyi (1954) gives (provided $z \neq 0$)

$$V(r,\theta,z) = 2\sum_{m=0}^{\infty} \exp(-im\theta) \int_0^\infty \mathrm{d}\rho \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \sigma_m(\rho)\, \mathrm{Q}_{m-\frac{1}{2}}\left(\frac{\rho^2+r^2+|z|^2}{2\rho r}\right), \tag{A21}$$

where $\mathrm{Q}_{m-\frac{1}{2}}$ is the $m-\frac{1}{2}$ order associated Legendre polynomial. When there is no θ dependence ($m = 0$ only), the solution reduces to (cf. Abramowitz and Stegun 1964)

$$V(r,z) = 4\int_0^\infty \mathrm{d}\rho \frac{\rho}{\{(\rho+r)^2+|z|^2\}^{\frac{1}{2}}} \sigma(\rho)\, \mathrm{K}\left\{\left(\frac{4\rho r}{(\rho+r)^2+|z|^2}\right)^{\frac{1}{2}}\right\}, \tag{A22}$$

where K denotes the complete elliptic integral of the first kind.

The exceptional value $z = 0$ corresponds to the potential in the plane of the disk distribution. Gubler (1897) has shown that

$$\int_0^\infty \mathrm{J}_m(\rho\alpha)\,\mathrm{J}_m(r\alpha)\,\mathrm{d}\alpha = \frac{r^m\,\Gamma(m+\frac{1}{2})}{\rho^{m+1}\,\Gamma(m+1)\,\Gamma(\frac{1}{2})}\,{}_2\mathrm{F}_1(m+\tfrac{1}{2},\tfrac{1}{2};m+1;r^2/\rho^2),$$

where $r < \rho$ and Γ and ${}_2\mathrm{F}_1$ denote the gamma and hypergeometric functions respectively,

$$= \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{1}{(\rho^2-r^2)^{\frac{1}{2}}} \mathrm{P}_{-\frac{1}{2}}^{-m}\left(\frac{\rho^2+r^2}{\rho^2-r^2}\right),$$

where P denotes the Legendre function. Consequently,

$$V(r,\theta,0) = 2\pi\sum_{m=0}^{\infty} \exp(-im\theta) \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\left\{\int_0^r \mathrm{d}\rho\, \rho\sigma_m(\rho) \frac{1}{(r^2-\rho^2)^{\frac{1}{2}}} \mathrm{P}_{-\frac{1}{2}}^{-m}\left(\frac{r^2+\rho^2}{r^2-\rho^2}\right)\right.$$
$$\left. + \int_r^\infty \mathrm{d}\rho\, \rho\sigma_m(\rho) \frac{1}{(\rho^2-r^2)^{\frac{1}{2}}} \mathrm{P}_{-\frac{1}{2}}^{-m}\left(\frac{\rho^2+r^2}{\rho^2-r^2}\right)\right\}. \tag{A23}$$

The physical interpretation of the discontinuity in the integrand at $\rho = r$ is that all disk elements except that at $\rho = r$ contribute to the potential $V(r,\theta,0)$. When there is no θ dependence ($m = 0$ only), the solution reduces to

$$V(r,0) = 4\left(\int_0^r \mathrm{d}\rho\, (\rho/r)\sigma(\rho)\,\mathrm{K}(\rho/r) + \int_r^\infty \mathrm{d}\rho\, \sigma(\rho)\,\mathrm{K}(r/\rho)\right), \tag{A24}$$

since (Abramowitz and Stegun 1964)

$$\mathrm{P}_{-\frac{1}{2}}\left(\frac{\rho^2+r^2}{\rho^2-r^2}\right) = \frac{2}{\pi}\frac{(\rho^2-r^2)^{\frac{1}{2}}}{\rho}\mathrm{K}\left(\frac{r}{\rho}\right), \qquad r < \rho.$$

Appendix III

Perturbation Magnetic Vector Potential

The perturbation magnetic vector potential A_1 is defined by

$$\boldsymbol{B}_1 = \nabla\times\boldsymbol{A}_1. \tag{A25}$$

Equation (5) gives the perturbation equation

$$\nabla\times\nabla\times\boldsymbol{A}_1 = (4\pi/c)\boldsymbol{J}_1\,\delta(z) \tag{A26}$$

or

$$\left(\nabla^2-\frac{1}{r^2}\right)A_{1r}-\frac{2}{r^2}\frac{\partial A_{1\theta}}{\partial\theta} = -(4\pi/c)J_{1r}(r,\theta)\,\delta(z)\,, \tag{A27a}$$

$$\left(\nabla^2-\frac{1}{r^2}\right)A_{1\theta}+\frac{2}{r^2}\frac{\partial A_{1r}}{\partial\theta} = -(4\pi/c)J_{1\theta}(r,\theta)\,\delta(z)\,, \tag{A27b}$$

$$\nabla^2 A_{1z} = 0\,, \tag{A27c}$$

where ∇^2 is given by equation (6a) and

$$\nabla . \boldsymbol{A}_1 = 0\,. \tag{A28}$$

In the nonrelativistic approximation, equation (A28) corresponds to the Lorentz gauge condition

$$\nabla . \boldsymbol{A}+c^{-1}\partial\chi/\partial t = 0\,, \tag{A29}$$

where χ denotes the electric scalar potential. This follows since

$$\frac{1}{c}\frac{\partial\chi}{\partial t}\sim\frac{1}{c}\frac{\chi}{T}\sim\frac{1}{c}\frac{L\,|\,\boldsymbol{E}\,|}{T}\sim\frac{1}{c^2}\frac{LV\,|\,\boldsymbol{B}\,|}{T}\sim\frac{V^2}{c^2}\,|\,\boldsymbol{B}\,|\,,$$

where L, T, and V denote characteristic length, time, and velocity respectively; since $V \ll c$, one has approximately

$$\nabla . \boldsymbol{A} = 0\,. \tag{A29a}$$

As indicated in Appendix II, one may Fourier analyse:

$$J_1(r,\theta\,;\alpha) = \boldsymbol{J}_1(\alpha)\,\mathrm{Re}\big(\mathrm{H}_m(\alpha r)\big)\exp(-im\theta)\,, \tag{A30a}$$

$$A_1(r,\theta,z\,;\alpha) = \boldsymbol{A}_1(\alpha)\,\mathrm{Re}\big(\mathrm{H}_m(\alpha r)\big)\exp(-\alpha\,|\,z\,|-im\theta)\,. \tag{A30b}$$

Substitution into (A27) gives

$$-2\alpha\delta(z)\,A_{1r}(r,\theta,z\,;\alpha)-(1/r^2)A_{1r}(r,\theta,z\,;\alpha)+(2im/r^2)A_{1\theta}(r,\theta,z\,;\alpha)$$
$$= -(4\pi/c)J_{1r}(r,\theta\,;\alpha)\,\delta(z)\,,$$

$$-2\alpha\delta(z)\,A_{1\theta}(r,\theta,z\,;\alpha)-(1/r^2)A_{1\theta}(r,\theta,z\,;\alpha)-(2im/r^2)A_{1r}(r,\theta,z\,;\alpha)$$
$$= -(4\pi/c)J_{1\theta}(r,\theta\,;\alpha)\,\delta(z)\,,$$

$$-2\alpha\delta(z)\,A_{1z}(r,\theta,z\,;\alpha) = 0\,,$$

whence

$$A_{1r}(r,\theta,0\,;\alpha) = (2\pi/\alpha c)J_{1r}(r,\theta\,;\alpha)\,,$$

$$A_{1\theta}(r,\theta,0\,;\alpha) = (2\pi/\alpha c)J_{1\theta}(r,\theta\,;\alpha)\,,$$

$$A_{1z}(r,\theta,0\,;\alpha) = 0\,.$$

Consequently,

$$A_{1r}(r,\theta,z\,;\alpha) = (2\pi/\alpha c)J_{1r}(r,\theta\,;\alpha)\exp(-\alpha\,|\,z\,|)\,, \tag{A31a}$$

$$A_{1\theta}(r,\theta,z\,;\alpha) = (2\pi/\alpha c)J_{1\theta}(r,\theta\,;\alpha)\exp(-\alpha\,|\,z\,|)\,, \tag{A31b}$$

$$A_{1z}(r,\theta,z\,;\alpha) = 0\,. \tag{A31c}$$